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PROFINITE COMPLETIONS AND CONTINUOUS EXTENSIONS OF MORPHISMS BETWEEN FREE GROUPS AND THEIRS PROFINITE COMPLETIONS

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Dedicated to Professor Harry Miller on the occasion of his 70th birthday

Abstract. It is known that every morphism $\varphi : F \to F'$ between free groups of pseudovariety **V** of finite groups is uniformly continuous when both groups are equipped with their respective pro-**V** topologies. In this paper we prove that this morphism can be uniquely extended to a continuous morphism between their pro-**V** completions $\hat{\varphi} : \hat{F} \to \hat{F'}$?

1. Preliminaries

Throughout this paper we use the following definitions and assertions.

Definition 1.1. ([7]) *Let $\mathcal{C}$ be a formation of finite groups (i.e. nonempty class of finite groups closed under quotients and finite subdirect products), and let* G *be a group that belongs to $\mathcal{C}$. If the nonempty collection $\mathcal{N}_{\mathcal{C}}(\mathrm{G}) = \{\mathrm{N} \lhd_f \mathrm{G} \mid \mathrm{G}/\mathrm{N} \in \mathcal{C}\}$, is filtered, then the corresponding topology on* G *is called a* full pro-$\mathcal{C}$ topology *of* G. *In particular, the pro-$\mathcal{C}$ topology of* G *is* Hausdorff *if and only if $\bigcap_{\mathrm{N}\in\mathcal{N}_{\mathcal{C}}(\mathrm{G})} \mathrm{N} = 1$. A group* G *which satisfies this condition is called* residually $\mathcal{C}$. *The* Pro-$\mathcal{C}$ completion *of* G *with respect to pro-$\mathcal{C}$ topology is $\mathcal{K}_{\mathcal{C}}(\mathrm{G}) = \varprojlim_{\mathrm{N}\in\mathcal{N}_{\mathcal{C}}(\mathrm{G})} \mathrm{G}/\mathrm{N}$ and usually is denoted by $\mathrm{G}_{\hat{\mathrm{C}}}$.*

Note that $\mathcal{K}_{\mathcal{C}}(\mathrm{G})$ is a profinite group, and there is a natural continuous homomorphism $\iota = \iota_{\mathcal{N}_{\mathcal{C}}(\mathrm{G})} : \mathrm{G} \to \mathcal{K}_{\mathcal{C}}(\mathrm{G})$, induced by the epimorphisms $\mathrm{G} \to \mathrm{G}/\mathrm{N}$ ($\mathrm{N} \in \mathcal{N}_{\mathcal{C}}(\mathrm{G})$), defined by $\iota(g) = (g\mathrm{N})_{\mathrm{N}\in\mathcal{N}_{\mathcal{C}}(\mathrm{G})}$, for each $g \in \mathrm{G}$. The map ι is *injective* if and only if $\bigcap_{\mathrm{N}\in\mathcal{N}_{\mathcal{C}}(\mathrm{G})} \mathrm{N} = 1$. Moreover, whenever $\theta : \mathrm{G} \to \mathrm{H}$ is a continuous homomorphism, there exists a uniquely continuous homomorphism $\hat{\theta} : \mathcal{K}_{\mathcal{C}}(\mathrm{G}) \to \mathrm{H}$ such that $\theta = \hat{\theta}\iota$.

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The notation $\hat{G}$ is usually used for the *profinite completion* of G, (i.e. for the completion $G_{\hat{C}}$ where $\mathcal{C}$ is the formation of all finite groups) and the respective topology is called the *profinite topology.*

Proposition 1.2. ([7]) *Let $\mathcal{C}$ be a formation of finite groups and let* G *be a residually $\mathcal{C}$ group. Identify* G *with its image in its pro-$\mathcal{C}$ completion* $G_{\hat{C}}$. *Let $\bar{X}$ denote the closure in* $G_{\hat{C}}$ *of a subset X of* G. *Then*

- (i) *Let $\phi : \{N \mid N \leq_o G\} \to \{U \mid U \leq_o G_{\hat{C}}\}$ be the mapping that assigns to each open subgroup* H *of* G *its closure $\bar{H}$ in* $G_{\hat{C}}$. *Then ϕ is a one-to-one correspondence between the set of all open subgroups* H *in the pro-$\mathcal{C}$ topology of* G *and the set of all open subgroups of* $G_{\hat{C}}$.
- (ii) *The map ϕ sends normal subgroups to normal subgroups.*
- (iii) *The topology of* $G_{\hat{C}}$ *induces on* G *its full pro-$\mathcal{C}$ topology.*

Lemma 1.3. ([7]) *Let $\mathcal{C}$ be a variety (respectively, a formation closed under taking normal subgroups) of finite groups. Assume that* $K \leq G$ *(respectively,* $K \lhd G$*), and let $\imath : K \to G$ denote the inclusion map. Then $\imath_{\hat{C}} : K_{\hat{C}} \to G_{\hat{C}}$ is injective if and only if the pro-$\mathcal{C}$ topology of* G *induces on* K *its full pro-$\mathcal{C}$ topology.*

We will continue our examination of *pseudovariety* of finite groups **V**, i.e. on the class of finite groups closed under subgroups, homomorphic images, and finite direct products. In this case, the pro-**V** topology on a group G is defined as the initial topology which makes all morphisms from G into elements of **V** continuous, and the normal subgroups H of G such that $G/H \in \mathbf{V}$ form a basis of neighborhoods of 1. By definition, the pro-**V** topology on a group G is Hausdorff if and only if G is residually **V**.

If H is a subgroup of G, we denote by $Cl_{\mathbf{V}}(H)$ or simply $Cl(H)$ its topological closure in the pro-**V** topology of G, and by $\overline{H}$ its topological closure in the completion $\hat{G}$.

When $G = F(A)$ is a free group over a nonempty finite set (alphabet) A, we denote the pro-**V** completion of F(A) by $\hat{G} = \hat{F}_{\mathbf{V}}(A)$. Moreover, $\hat{F}_{\mathbf{V}}(A)$ is the *free pro-***V** *group over* A.

We note that every morphism $\varphi : F(A) \to F(B)$ between free groups of pseudovariety **V** is uniformly continuous when both groups are equipped with their respective pro-**V** topologies.

Now, we ask if it is possible to uniquely extend this morphism to a continuous morphism between their pro-**V** completions $\hat{\varphi} : \hat{F}_{\mathbf{V}}(A) \to \hat{F}_{\mathbf{V}}(B)$, respectively if Proposition 1.2 and Lemma 1.3 is satisfied in the case of pseudovariety of finite groups **V** ?

2. Continuous extension morphisms between free groups to their profinite completions

It is known that a finitely generated subgroup H of a free group F(B) has a *property of coincidence* if the pro-$\mathbf{V}$ topology on H coincides with the topology on H induced by the pro-$\mathbf{V}$ topology on F(B).

Lemma 2.1. *Let* $\varphi : \mathrm{F(A)} \to \mathrm{F(B)}$ *be a morphism between finitely generated free groups and let* $\hat{\varphi} : \hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{A}) \to \hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{B})$ *be the continuous extension of* φ *between their pro-*$\mathbf{V}$ *completions. If the range of* φ *is* $\mathrm{H} = \varphi(F(A))$*, then the range of* $\hat{\varphi}$ *is* $\bar{\mathrm{H}}$.

Proof. It is trivial to see that $\mathrm{H} \subseteq \hat{\varphi}(\hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{A}))$ and by continuity, $\hat{\varphi}(\hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{A})) = \hat{\varphi}(\overline{\mathrm{F(A)}}) \subseteq \overline{\varphi(\mathrm{F(A)})} = \bar{\mathrm{H}}$. As the free group $\hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{A})$ is compact and $\hat{\varphi}$ is a continuous morphism, we can conclude that the group $\hat{\varphi}(\hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{A}))$ is closed and so $\hat{\varphi}(\hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{A})) = \bar{\mathrm{H}}$. □

Applying Lemma 1.3 on free groups, we obtain

Lemma 2.2. *Let H be a finitely generated subgroup of a free group* F(B) *and let* $\iota : \mathrm{H} \to \mathrm{F(B)}$ *be the natural injection of* H *into* F(B)*. Then the continuous extension of* ι *between their pro-*$\mathbf{V}$ *completions,* $\hat{\iota} : \hat{\mathrm{H}} \to \hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{B})$ *is injective if and only if the subgroup* H *has the property of coincidence.*

By Lemma 2.1 the range of morphism $\hat{\iota}$ is $\bar{\mathrm{H}}$ and the closure $\bar{\mathrm{H}}$ is homeomorphic to the completion $\hat{\mathrm{H}}$.

It is also known that pseudovariety $\mathbf{V}$ is *closed under extension* whenever $1 \to G_1 \to G_2 \to G_3 \to 1$ is a short exact sequence of finite groups such that if $G_1, G_3 \in \mathbf{V}$, then $G_2 \in \mathbf{V}$.

The next two theorems give us the answer to the posed question.

Theorem 2.3. *Let* $\mathbf{V}$ *be a pseudovariety of groups such that free groups are residually* $\mathbf{V}$ *and let* $\varphi : \mathrm{F(A)} \to \mathrm{F(B)}$ *be a injective morphism between finitely generated free groups and* $\mathrm{H} = \varphi(F(A))$*. Then the continuous extension of* φ*,* $\hat{\varphi} : \hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{A}) \to \hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{B})$ *is injective if and only if* H *has a property of coincidence.*

Proof. Let us assume that $\rho : \mathrm{F(A)} \to \mathrm{H}$ is a restriction of the morphism φ. Now, we can define $\hat{\rho} : \hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{A}) \to \mathrm{H}$ as a continuous extension of ρ. Since ρ is an isomorphism, we can conclude that $\hat{\rho}$ is a homeomorphism. As in Lemma 2.2 we can choose the natural injection $\iota : \mathrm{H} \to \mathrm{F(B)}$ and its continuous extension between theirs pro-$\mathbf{V}$ completions $\hat{\iota} : \hat{\mathrm{H}} \to \hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{B})$. In this case $\varphi = \iota \circ \rho$, as well as $\hat{\varphi} = \hat{\iota} \circ \hat{\rho}$. As the mapping $\hat{\rho}$ is a homeomorphism, $\hat{\varphi}$ is injective if and only if $\hat{\iota}$ is injective too, or by Lemma 2.2, if and only if H has the property of coincidence, as we wanted to prove. □

Theorem 2.4. *Let* $\mathbf{V}$ *be a pseudovariety of groups closed under extension such that free groups are residually* $\mathbf{V}$*. Let* $\varphi : \mathrm{F(A)} \to \mathrm{F(B)}$ *be an injective morphism between finitely generated free groups and* $\mathrm{H} = \varphi(F(A))$*. Then the continuous extension of* φ*,* $\hat{\varphi} : \hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{A}) \to \hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{B})$ *is injective if and only if* H *and its closure* $Cl(\mathrm{H})$ *have the same rank.*

Proof. Since, by Lemma 2.1, the range of extension $\hat{\varphi}$ is the closure $\bar{\mathrm{H}}$ and since φ is an injective mapping between two compact free groups, than the homeomorphism onto its image is also injective and $\bar{\mathrm{H}}$ is homeomorphic to the free pro-$\mathbf{V}$ group for which $rank|A| = rank(\mathrm{H})$. By Proposition 1.2, if pseudovariety of groups $\mathbf{V}$ is closed under extension, then every finitely generated and closed subgroup of the free group F(A) has the property of coincidence. Thus $Cl(\mathrm{H})$ also has the property of coincidence. Now, applying Lemma 2.2 to $Cl(\mathrm{H})$, we conclude that $\overline{Cl(\mathrm{H})}$ is homeomorphic to the free pro-$\mathbf{V}$ group of $rank(Cl(\mathrm{H}))$. Since $\bar{\mathrm{H}} = \overline{Cl(\mathrm{H})}$, we have proved that $rank(\mathrm{H}) = rank(Cl(\mathrm{H}))$.

Conversely, let us assume that H and closure $Cl(\mathrm{H})$ have the same rank. As we have shown, if pseudovariety of groups $\mathbf{V}$ is closed under extension, then every finitely generated, closed subgroup of a free group has the property of coincidence, and so pro-$\mathbf{V}$ topology on $Cl(\mathrm{H})$ coincides with the topology on $Cl(\mathrm{H})$ induced by the pro-$\mathbf{V}$ topology on the free group. Therefore H is dense in the pro-$\mathbf{V}$ topology on $Cl(\mathrm{H})$.

Finally, it remains to prove the claim: if H is a finitely generated subgroup of the free group F(A) which is dense in the pro-$\mathbf{V}$ topology of F(A) and if $rank(\mathrm{H}) = rank(\mathrm{F(A)})$, then the subgroup H has the property of coincidence. Indeed, when H and F(A) have the same rank, we may consider an injective endomorphism $\rho : \mathrm{F(A)} \to \mathrm{H}$. If $\hat{\rho} : \hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{A}) \to \hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{A})$ is the continuous extension of ρ, then by Lemma 2.1, we can conclude that $\hat{\rho}$ is a surjective endomorphism of $\hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{A})$. Since every continuous surjective endomorphism of a finitely generated profinite group is injective [1, Prop. 15.3], $\hat{\rho}$ is injective too. Theorem 2.3, implies that the subgroup H has the property of coincidence.

Hence, by using this claim, we can conclude that the pro-$\mathbf{V}$ topology on the subgroup H coincides with the topology on H induced by the pro-$\mathbf{V}$ topology on $Cl(\mathrm{H})$. By Theorem 2.3, the continuous extension of φ, $\hat{\varphi} : \hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{A}) \to \hat{\mathrm{F}}_{\mathbf{V}}(\mathrm{B})$ is injective. □

It is known that for a profinite topology, every finitely generated subgroup of the free group is closed [7]. Applying this to Theorem 2.4 we obtain the following result

Corollary 2.5. *Every injective morphism between free groups of finite rank can be extended to a injective continuous morphism between their profinite completions.*

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