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ON THE COMPLETION OF PRO-$\mathcal{P}$ GROUPS

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Dedicated to professor Harry Miller on the occasion of his 70th birthday

ABSTRACT. We examine the class of finite paragraded groups which we will denote by $\mathcal{P}$. After observing that the class $\mathcal{P}$ is closed with respect to subgroups and direct products ([5]), we define the notion of a pro-$\mathcal{P}$ group and consider the completion of such a group.

1. Introduction

Let G be an abstract group and I a non-empty filter base of normal subgroups of finite index. We say that a subset of G is open if it can be written as a union of cosets Kg of subgroups $K \in I$. Then G is a topological group.

Definition 1. *The completion of G with respect to I consists of a profinite group $\hat{G}$ and a continuous homomorphism $j : G \to \hat{G}$ with the following property: whenever $\theta : G \to H$ is a continuous homomorphism to a finite group H there is a unique continuous homomorphism $\hat{\theta} : \hat{G} \to H$ such that $\theta = \hat{\theta} j$.*

The next theorem, proven in [8], shows that it is not difficult to find such a profinite group.

Theorem 2. *Let $\hat{G} = s\varprojlim_I G/K$ and let j be the map $g \to (Kg)$ from G to $\hat{G}$. The pair $(\hat{G}, j)$ has the properties of the completion of G with respect to I.*

It is useful to write down explicitly what $\hat{G} = s\varprojlim_I G/K$ is. $\hat{G} = s\varprojlim_I G/K$ is an inverse limit (X, φ_K), where

$$X = \{\ \overline{x} \in \prod_{K \in I} G/K \mid q_{K_1 K_2} p_{K_2}(\overline{x}) = p_{K_1}(\overline{x})\ (K_1 \leq' K_2)\ \}$$

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and $\varphi_K = p_K|_X$ $(K \in I)$, where p_K is the projection map from $\prod_{K\in I} G/K$ to G/K and where $\leq'$ is the relation defined by $K_1 \leq' K_2$ if and only if K_2 is a subgroup of K_1 and $q_{K_1K_2} : G/K_2 \to G/K_1$ the map defined by $q_{K_1K_2}(K_2 g) = K_1 g$ $(g \in G)$. Also, we will use the following result ([8]).

Theorem 3. *Let $(\hat{G}, j)$ be the completion of G with respect to I. Then* $\ker j = \bigcap_{K\in I} K$.

Now, let $\mathcal{C}$ be a class of finite groups closed with respect to subgroups and direct products. The *pro-$\mathcal{C}$ completion* of G is the completion of G with respect to the family of normal subgroups K such that $G/K \in \mathcal{C}$.

In [2], [3], [4] and [5] the notion of paragraded group is introduced. Since it is not so familiar, let us right down the definition of paragraduation.

Definition 4. *The map $\pi : \Delta \to \mathrm{Sg}(G)$, $\pi(\delta) = G_\delta$ $(\delta \in \Delta)$, of the partially ordered set $(\Delta, <)$, which is from bellow complete semi-lattice and from beyond inductively ordered, to the set $\mathrm{Sg}(G)$ of subgroups of the group G, is called paragraduation if it satisfies the following six-axiom system:*

i) $\pi(0) = G_0 = \{e\}$, *where* $0 = \inf \Delta$; $\delta < \delta' \Rightarrow G_\delta \subseteq G_{\delta'}$;

   Remark 5. $H = \bigcup_{\delta\in\Delta} G_\delta$ is called the homogenous part of G with respect to π.

ii) $\theta \subseteq \Delta \Rightarrow \bigcap_{\delta\in\theta} G_\delta = G_{\inf\theta}$;
iii) *If $x, y \in H$ and $yx = zxy$, then $z \in H$ and $\delta(z) \leq \inf(\delta(x), \delta(y))$;*
iv) *The homogeneous part H is a generating set of G;*
v) *Let $A \subseteq H$ be a subset such that for all $x, y \in A$ there exists an upper bound for $\delta(x), \delta(y)$. Then there exists an upper bound for all $\delta(x)$, $x \in A$;*
vi) *G is generated by H with the set of H-internal and left commutation relations (see* [5]*).*

The group is called paragraded if it has a paragraduation.

The following two theorems due to Krasner and Vuković ([5]) state that the category of paragraded groups is closed for homogenous subgroups and direct products.

Theorem 6. *Let G be a paragraded group with homogenous part H. If K is a homogenous subgroup of G, i.e. K is generated by $K \cap H$, then K is also paragraded.*

Theorem 7. *The direct product of paragraded groups is a paragraded group.*

2. The main result

Now, since the category of paragraded groups is closed with respect to homogenous subgroups and direct products, we may consider the class of finite paragraded groups which we will denote simply by $\mathcal{P}$.

Definition 8. *We call a group F a* $\mathcal{P}$-group *if it is finite paragraded and we call G a* pro-$\mathcal{P}$ group *if it is an inverse limit of $\mathcal{P}$-groups.*

We know that subgroups $\pi(\delta)$ are normal subgroups of G ([5]).

Lemma 9. *The set $\mathcal{F} = \{ \pi(\delta) \mid \delta \in \Delta \}$ constitutes a filter base.*

Proof. Let us observe that the subset $\{ \delta_1, \delta_2 \} \subseteq \Delta$. By the third axiom of paragraduation (see Definition 4), $\inf\{ \delta_1, \delta_2 \}$ exists, denote it by δ_3. Hence $\delta_3 < \delta_1$ and $\delta_3 < \delta_2$, which, by the first axiom of paragraduation (see Definition 4) implies that $\pi(\delta_3) \subseteq \pi(\delta_1)$ and $\pi(\delta_3) \subseteq \pi(\delta_2)$. Hence $\pi(\delta_3) \subseteq \pi(\delta_1) \cap \pi(\delta_2)$ and so there exists a normal subgroup of G which is contained in $\pi(\delta_1) \cap \pi(\delta_2)$. □

All subgroups $\pi(\delta)$ are homogenous normal subgroups of G, hence all factor groups $G/\pi(\delta)$ are paragraded as is proved in [5]. Hence, we may consider the inverse limit of $G/\pi(\delta)$ with respect to $I = \{ \pi(\delta) \mid \delta \in \Delta \}$. If we consider the pro-$\mathcal{P}$ completion $\hat{G}$ of G with respect to such a family I, we get an interesting result.

Theorem 10. *The group G can be injected into its completion $\hat{G}$ with respect to I.*

Proof. For $\hat{G}$ we may take $s\varprojlim_I G/\pi(\delta)$. Observe the map $j : G \to \hat{G}$ defined by $j(g) = (\pi(\delta)g)$ $(g \in G)$. Then, by Theorem 3, $\ker j = \bigcap_{\pi(\delta) \in I} \pi(\delta)$, hence $\ker j = \bigcap_{\delta \in \Delta} \pi(\delta) = \pi(\inf \Delta) = \pi(0) = \{e\}$, according to axioms i) and ii) of paragraduation. □

Thus, we may consider G as a subgroup of $\hat{G}$.

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