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MULTIVARIATE FRACTIONAL TAYLOR'S FORMULA REVISITED

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Abstract. This is a continuation of [2]. Here is established a multivariate fractional Taylor's formula via the Caputo fractional derivative. The fractional remainder is expressed as a composition of two Riemann-Liouville fractional integrals.

We estimate the remainder.

1. Background

We start with

Definition 1. ([4]) *Let $\nu \geq 0$, the operator J_a^ν, defined on $L_1(a,b)$ by*

$$J_a^\nu f(x) := \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t)\, dt \tag{1}$$

for $a \leq x \leq b$, is called the Riemann-Liouville fractional integral operator of order ν. For $\nu = 0$, we set $J_a^0 := I$, the identity operator. Here Γ stands for the gamma function.

By Theorem 2.1 of [4], p 13, $J_a^\nu f(x)$, $\nu > 0$, exists for almost all $x \in [a,b]$ and $J_a^\nu f \in L_1(a,b)$, where $f \in L_1(a,b)$.

Here $AC^n([a,b])$ is the space of functions with absolutely continuous $(n-1)$-st derivative.

We need to mention

Definition 2. ([4],[3]) *Let $\nu \geq 0$, $n := \lceil \nu \rceil$, $\lceil \cdot \rceil$ is ceiling of the number, $f \in AC^n([a,b])$. We call Caputo fractional derivative*

$$D_{*a}^\nu f(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{2}$$

$\forall x \in [a,b]$.

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The above function $D_{*a}^{\nu} f(x)$ exists almost everywhere for $x \in [a, b]$.

If $\nu \in \mathbb{N}$, then $D_{*a}^{\nu} f = f^{(\nu)}$ the ordinary derivative, also it is $D_{*a}^{0} f = f$.

We need

Theorem 3. (Taylor expansion for Caputo derivatives, [4], p.40) *Assume* $\nu \geq 0$, $n = \lceil \nu \rceil$, *and* $f \in AC^n([a, b])$.

Then

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} D_{*a}^{\nu} f(t)\, dt, \tag{3}$$

$\forall x \in [a, b]$.

2. Results

We establish analogs of Theorem 3 to the multivariate case. We make

Remark 4. Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \geq 2$; $z := (z_1, \ldots, z_k)$, $x_0 := (x_{01}, \ldots, x_{0k}) \in Q$. Let $f \in C^n(Q)$, $n \in \mathbb{N}$.

Set

$$g_z(t) := f(x_0 + t(z - x_0)),$$

$$0 \leq t \leq 1;\ \ g_z(0) = f(x_0),\ \ g_z(1) = f(z). \tag{4}$$

Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i}\right)^j f\right](x_0 + t(z - x_0)), \tag{5}$$

$j = 0, 1, 2, \ldots, n$, and

$$g_z^{(n)}(0) = \left[\left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i}\right)^n f\right](x_0). \tag{6}$$

If all $f_\alpha(x_0) := \frac{\partial^\alpha f}{\partial x^\alpha}(x_0) = 0$, $\alpha := (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, k$; $|\alpha| := \sum_{i=1}^k \alpha_i =: l$, then $g_z^{(l)}(0) = 0$, where $l \in \{0, 1, \ldots, n\}$. We quote that

$$g_z'(t) = \sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial f}{\partial x_i}(x_0 + t(z - x_0)). \tag{7}$$

When $f \in C^2(Q)$, $Q \subseteq \mathbb{R}^2$, we have

$$\begin{aligned} g_z''(t) &= (z_1 - x_{01})^2 \frac{\partial^2 f}{\partial x_1^2}(x_0 + t(z - x_0)) \\ &+2(z_1 - x_{01})(z_2 - x_{02}) \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_0 + t(z - x_0)) \\ &+ (z_2 - x_{02})^2 \frac{\partial^2 f}{\partial x_2^2}(x_0 + t(z - x_0)), \end{aligned} \tag{8}$$

etc. Clearly here $g_z \in C^n([0,1])$, hence $g_z \in AC^n([0,1])$.

Let now $\nu > 0$ with $\lceil \nu \rceil = n$.

By applying (3) for g_z we obtain

$$g_z(1) = \sum_{l=0}^{n-1} \frac{g_z^{(l)}(0)}{l!} + \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} D_{*0}^{\nu} g_z(t)\, dt. \tag{9}$$

Here we observe by (2) that

$$D_{*0}^{\nu} g_z(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-s)^{n-\nu-1} g_z^{(n)}(s)\, ds. \tag{10}$$

Let us consider the case of $0 < \nu \leq 1$, i.e. $n = 1$. Then

$$\begin{aligned} D_{*0}^{\nu} g_z(t) &= \frac{1}{\Gamma(1-\nu)} \int_0^t (t-s)^{-\nu} g_z'(s)\, ds \\ &\overset{(7)}{=} \frac{1}{\Gamma(1-\nu)} \int_0^t (t-s)^{-\nu} \left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial f}{\partial x_i}(x_0 + s(z - x_0)) \right) ds \\ &= \frac{1}{\Gamma(1-\nu)} \left(\sum_{i=1}^{k} (z_i - x_{0i}) \int_0^t (t-s)^{-\nu} \frac{\partial f}{\partial x_i}(x_0 + s(z - x_0))\, ds \right) \\ &= \sum_{i=1}^{k} (z_i - x_{0i}) J_0^{1-\nu} \left(\frac{\partial f}{\partial x_i}(x_0 + t(z - x_0)) \right). \end{aligned} \tag{11}$$

That is

$$D_{*0}^{\nu} g_z(t) = \sum_{i=1}^{k} (z_i - x_{0i}) J_0^{1-\nu} \left(\frac{\partial f}{\partial x_i}(x_0 + t(z - x_0)) \right), \tag{12}$$

for all $t \in [0,1]$.

Consequently by (9) and (12) we obtain

$$f(z) = f(x_0) + \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} \times \left(\sum_{i=1}^{k} (z_i - x_{0i}) J_0^{1-\nu} \left(\frac{\partial f}{\partial x_i} \left(x_0 + t(z - x_0) \right) \right) \right) dt$$

$$= f(x_0) + \sum_{i=1}^{k} (z_i - x_{0i}) \times \left[\frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} J_0^{1-\nu} \left(\frac{\partial f}{\partial x_i} (x_0 + t(z - x_0)) \right) dt \right]. \tag{13}$$

Based on the last comments we present the following basic multivariate fractional fundamental theorem.

Theorem 5. *Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \geq 2$; $z := (z_1, \ldots, z_k)$, $x_0 := (x_{01}, \ldots, x_{0k}) \in Q$, $f \in C^1(Q)$, $0 < \nu \leq 1$. Then*

$$f(z) = f(x_0) + \sum_{i=1}^{k} (z_i - x_{0i}) \left[\frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} J_0^{1-\nu} \left(\frac{\partial f}{\partial x_i} (x_0 + t(z - x_0)) \right) dt \right]. \tag{14}$$

We make

Remark 6. This is a continuation of Remark 4.

Let now $1 < \nu \leq 2$, i.e. $n = 2$, $f \in C^2(Q)$, $Q \subseteq \mathbb{R}^2$.
Then

$$D_{*0}^{\nu} g_z(t) = \frac{1}{\Gamma(2-\nu)} \int_0^t (t-s)^{1-\nu} g_z''(s)\, ds$$

$$\overset{(8)}{=} \frac{1}{\Gamma(2-\nu)} \int_0^t (t-s)^{1-\nu} \left[(z_1 - x_{01})^2 \frac{\partial^2 f}{\partial x_1^2} (x_0 + s(z - x_0)) + 2(z_1 - x_{01})(z_2 - x_{02}) \frac{\partial^2 f}{\partial x_1 \partial x_2} (x_0 + s(z - x_0)) + (z_2 - x_{02})^2 \frac{\partial^2 f}{\partial x_2^2} (x_0 + s(z - x_0)) \right] ds$$

$$= (z_1 - x_{01})^2 \left[\frac{1}{\Gamma(2-\nu)} \int_0^t (t-s)^{1-\nu} \frac{\partial^2 f}{\partial x_1^2} (x_0 + s(z - x_0))\, ds \right] + 2(z_1 - x_{01})(z_2 - x_{02})$$

$$\times\left[\frac{1}{\Gamma(2-\nu)}\int_0^t (t-s)^{1-\nu}\,\frac{\partial^2 f}{\partial x_1\partial x_2}(x_0+s(z-x_0))\,ds\right]$$
$$+(z_2-x_{02})^2\left(\frac{1}{\Gamma(2-\nu)}\int_0^t (t-s)^{1-\nu}\,\frac{\partial^2 f}{\partial x_2^2}(x_0+s(z-x_0))\,ds\right)$$
$$=(z_1-x_{01})^2\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_1^2}(x_0+t(z-x_0))\right)\right)$$
$$+2(z_1-x_{01})(z_2-x_{02})\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_1\partial x_2}(x_0+t(z-x_0))\right)\right)$$
$$+(z_2-x_{02})^2\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_2^2}(x_0+t(z-x_0))\right)\right). \tag{15}$$

That is we get

$$D_{*0}^{\nu} g_z(t)=(z_1-x_{01})^2\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_1^2}(x_0+t(z-x_0))\right)\right)$$
$$+2(z_1-x_{01})(z_2-x_{02})\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_1\partial x_2}(x_0+t(z-x_0))\right)\right)$$
$$+(z_2-x_{02})^2\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_2^2}(x_0+t(z-x_0))\right)\right),\quad 0\le t\le 1. \tag{16}$$

Thus by (6), (7), (9) and (16) we obtain

$$f(z)=f(x_0)+(z_1-x_{01})\frac{\partial f}{\partial x_1}(x_0)+(z_2-x_{02})\frac{\partial f}{\partial x_2}(x_0)$$
$$+\frac{1}{\Gamma(\nu)}\int_0^1 (1-t)^{\nu-1}\left[(z_1-x_{01})^2\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_1^2}(x_0+t(z-x_0))\right)\right)\right.$$
$$+2(z_1-x_{01})(z_2-x_{02})\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_1\partial x_2}(x_0+t(z-x_0))\right)\right)$$
$$\left.+(z_2-x_{02})^2\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_2^2}(x_0+t(z-x_0))\right)\right)\right]dt \tag{17}$$

$$=f(x_0)+(z_1-x_{01})\frac{\partial f}{\partial x_1}(x_0)+(z_2-x_{02})\frac{\partial f}{\partial x_2}(x_0)$$
$$+(z_1-x_{01})^2\left(\frac{1}{\Gamma(\nu)}\int_0^1 (1-t)^{\nu-1}\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_1^2}(x_0+t(z-x_0))\right)\right)dt\right)$$
$$+2(z_1-x_{01})(z_2-x_{02})\left(\frac{1}{\Gamma(\nu)}\int_0^1 (1-t)^{\nu-1}\right.$$
$$\left.\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_1\partial x_2}(x_0+t(z-x_0))\right)\right)dt\right)$$

$$+\left(z_2-x_{02}\right)^2\left(\frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_2^2}\left(x_0+t\left(z-x_0\right)\right)\right)\right)dt\right). \tag{18}$$

We have established the following fractional bivariate Taylor formula.

Theorem 7. *Let $f \in C^2(Q)$, $Q \subseteq \mathbb{R}^2$ compact and convex, $z := (z_1, z_2)$, $x_0 := (x_{01}, x_{02}) \in Q$, and $1 < \nu \leq 2$. Then*

$$\begin{aligned}
1)\quad f(z_1,z_2) &= f(x_{01},x_{02}) + (z_1-x_{01})\frac{\partial f}{\partial x_1}(x_{01},x_{02}) \\
&+ (z_2-x_{02})\frac{\partial f}{\partial x_2}(x_{01},x_{02}) + (z_1-x_{01})^2 \\
&\left(\frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_1^2}\left(x_0+t\left(z-x_0\right)\right)\right)\right)dt\right) \\
&+2(z_1-x_{01})(z_2-x_{02})\left(\frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}\right. \\
&\left.\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_1\partial x_2}\left(x_0+t\left(z-x_0\right)\right)\right)\right)dt\right) \\
&+\left(z_2-x_{02}\right)^2\left(\frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_2^2}\left(x_0+t\left(z-x_0\right)\right)\right)\right)dt\right).
\end{aligned} \tag{19}$$

Additionally assume that

$$f(x_0) = \frac{\partial f}{\partial x_1}(x_0) = \frac{\partial f}{\partial x_2}(x_0) = 0,$$

then

$$\begin{aligned}
2)\quad f(z_1,z_2) &= (z_1-x_{01})^2\left(\frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}\right. \\
&\left.\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_1^2}\left(x_0+t\left(z-x_0\right)\right)\right)\right)dt\right. \\
&+2(z_1-x_{01})(z_2-x_{02})\left(\frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}\right. \\
&\left.\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_1\partial x_2}\left(x_0+t\left(z-x_0\right)\right)\right)\right)dt\right) \\
&\left(z_2-x_{02}\right)^2\left(\frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}\left(J_0^{2-\nu}\left(\frac{\partial^2 f}{\partial x_2^2}\left(x_0+t\left(z-x_0\right)\right)\right)\right)dt\right).
\end{aligned} \tag{20}$$

We make

Remark 8. This is another continuation of Remark 4.

Let $\nu > 0$, $n = \lceil \nu \rceil$, $f \in C^n(Q)$. By (4), (6) and (9) we get

$$f(z) = f(x_0) + \sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial f}{\partial x_i}(x_0)$$

$$+ \sum_{l=2}^{n-1} \frac{\left[\left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^l f \right](x_0)}{l!} + \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} D_{*0}^{\nu} g_z(t)\, dt. \tag{21}$$

But we have

$$D_{*0}^{\nu} g_z(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-s)^{n-\nu-1} g_z^{(n)}(s)\, ds$$

$$= \frac{1}{\Gamma(n-\nu)} \int_0^t (t-s)^{n-\nu-1} \left[\left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right] (x_0 + s(z - x_0))\, ds$$

$$= J_0^{n-\nu} \left\{ \left[\left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right] (x_0 + t(z - x_0)) \right\}. \tag{22}$$

We have proved the following general multivariate fractional Taylor formula.

Theorem 9. *Let $\nu > 0$, $n = \lceil \nu \rceil$, $f \in C^n(Q)$, where Q is a compact and convex subset of $\mathbb{R}^k$, $k \geq 2$; $z := (z_1, \ldots, z_k)$, $x_0 := (x_{01}, \ldots, x_{0k}) \in Q$. Then*

$$1) \quad f(z) = f(x_0) + \sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial f(x_0)}{\partial x_i}$$

$$+ \sum_{l=2}^{n-1} \frac{\left[\left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^l f \right](x_0)}{l!} + \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1}$$

$$\left[J_0^{n-\nu} \left\{ \left[\left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right] (x_0 + t(z - x_0)) \right\} \right] dt. \tag{23}$$

Additionally assume that $f_\alpha(x_0) = 0$, $\alpha := (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, k$; $|\alpha| := \sum_{i=1}^{k} \alpha_i =: r$, $r = 0, \ldots, n-1$, then

$$2) \quad f(z) = \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1}$$

$$\left[J_0^{n-\nu}\left\{\left[\left(\sum_{i=1}^{k}(z_i-x_{0i})\frac{\partial}{\partial x_i}\right)^n f\right](x_0+t(z-x_0))\right\}\right]dt=:R_\nu. \quad (24)$$

We continue with

Remark 10. Here we estimate the remainder of (23), which the same as R_ν of (24).

The function

$$G_\nu(t):=J_0^{n-\nu}\left\{\left[\left(\sum_{i=1}^{k}(z_i-x_{0i})\frac{\partial}{\partial x_i}\right)^n f\right](x_0+t(z-x_0))\right\},\ \ t\in[0,1], \quad (25)$$

which appears in R_ν is continuous, see Proposition 114 of [1] and $R_\nu\in\mathbb{R}$.

Similarly the remainder of (14) exists and the same holds for all the integrals of (19), they are all real numbers.

So we can write

$$R_\nu=\frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}G_\nu(t)\,dt,\ \ \nu>0. \quad (26)$$

When $\nu\geq1$ we get

$$\begin{aligned}|R_\nu|&\leq\frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}|G_\nu(t)|\,dt\\&\leq\frac{1}{\Gamma(\nu)}\int_0^1|G_\nu(t)|\,dt=\frac{1}{\Gamma(\nu)}\|G_\nu\|_{L_1(0,1)},\end{aligned}$$

i.e.

$$|R_\nu|\leq\frac{1}{\Gamma(\nu)}\|G_\nu\|_{L_1(0,1)}. \quad (27)$$

Also for $p,\ q>1:\frac{1}{p}+\frac{1}{q}=1$ and with $p(\nu-1)+1>0$ for $\nu>0$, we obtain

$$\begin{aligned}|R_\nu|&\leq\frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}|G_\nu(t)|\,dt\\&\leq\frac{1}{\Gamma(\nu)}\left(\int_0^1\left((1-t)^{\nu-1}\right)^p dt\right)^{1/p}\left(\int_0^1|G_\nu(t)|^q\,dt\right)^{1/q}\\&=\frac{1}{\Gamma(\nu)}\frac{1}{(p(\nu-1)+1)^{1/p}}\|G_\nu\|_{L_q([0,1])}.\end{aligned}$$

I.e.

$$|R_\nu|\leq\frac{1}{\Gamma(\nu)}\frac{1}{(p(\nu-1)+1)^{1/p}}\|G_\nu\|_{L_q([0,1])}. \quad (28)$$

In case of $p = q = 2$ and $\nu > 1/2$ we have

$$|R_\nu| \leq \frac{1}{\Gamma(\nu)} \frac{1}{\sqrt{2\nu - 1}} \|G_\nu\|_{L_2([0,1])}. \quad (29)$$

Finally we get that

$$|R_\nu| \leq \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} |G_\nu(t)| \, dt \leq \frac{\|G_\nu\|_\infty}{\Gamma(\nu+1)},$$

i.e.

$$|R_\nu| \leq \frac{\|G_\nu\|_\infty}{\Gamma(\nu+1)}, \quad \nu > 0. \quad (30)$$

We have established the following remainder estimate.

Theorem 11. *All here as in Theorem* 9. *Let* R_ν *be the remainder in* (23), *and* $G_\nu(t)$, $t \in [0,1]$ *as in* (25), $\nu \geq 1$.

Then

$$|R_\nu| \leq \min \left\{ \frac{\|G_\nu\|_{L_1([0,1])}}{\Gamma(\nu)}, \frac{\|G_\nu\|_{L_q([0,1])}}{\Gamma(\nu)(p(\nu-1)+1)^{1/p}}, \frac{\|G_\nu\|_{L_2([0,1])}}{\Gamma(\nu)\sqrt{2\nu-1}}, \frac{\|G_\nu\|_\infty}{\Gamma(\nu+1)} \right\}, \quad (31)$$

where $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

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