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SOME UNIQUENESS RESULTS ON MEROMORPHIC FUNCTIONS SHARING TWO OR THREE SETS

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Abstract. Using the notions of weighted and pseudo sharing of sets, we prove some uniqueness theorems on meromorphic functions that share two or three sets. The results in this paper improve and supplement many known results.

1. Introduction and main results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [11]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying

$$S(r,h) = o\left(T(r,h)\right) \quad (r \longrightarrow \infty, r \notin E).$$

Let f and g be two non-constant meromorphic functions and let a be a complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM (see [23]).

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. Denote by $\overline{E}_f(S)$ the reduced form of $E_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM (see [10]).

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Let m be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_{m)}(a;f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. If for some $a \in \mathbb{C} \cup \{\infty\}$, $E_{\infty)}(a;f)$=$E_{\infty)}(a;g)$ we say that f, g share the value a CM. For a set S of distinct elements of $\mathbb{C}$ we define $E_{m)}(S,f) = \bigcup_{a \in S} E_{m)}(a,f)$. The condition $E_{m)}(S,f) = E_{m)}(S,g)$ obviously implies $E_{j)}(S,f) = E_{j)}(S,g)$ for all $1 \leq j \leq m$.

The uniqueness problem for entire and meromorphic functions sharing sets of distinct elements instead of values was initiated by a famous question of F. Gross in [10]. In 1976 he posed the following question:

Question A [10] Can one find two finite sets S_j $(j = 1,2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1,2$ must be identical ?

For meromorphic functions it is natural to ask the following question:

Question B [24] Can one find three finite sets S_j $(j = 1,2,3)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1,2,3$ must be identical ?

Gradually the research on Question A corresponding to meromorphic function as well as Question B gained pace and today it has become one of the most prominent branches of the uniqueness theory. Among a number of situations, depending on the nature and the number of shared sets, the uniqueness of two meromorphic functions was studied by many authors. {cf.[1]-[9], [13], [16]-[18], [19]-[21], [27]-[28]}.

In [13] I. Lahiri proved the following result which dealt with Question B.

Theorem A. *Let $S_1 = \{z : z^n + az^{n-1} + b = 0\}$ and $S_2 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n (≥ 8) is an integer. If f and g are two non-constant meromorphic functions having no simple pole such that $E_f(S_i) = E_g(S_i)$ for $i = 1,2$ then $f \equiv g$.*

In 2003, Fang and Lahiri [9] obtained the following result which improved Theorem A by reducing the cardinality of the range set S_1.

Theorem B. *Let $S_1 = \{z : z^n + az^{n-1} + b = 0\}$ and $S_2 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n (≥ 7) is an integer. If f and g are two non-constant meromorphic functions having no simple pole such that $E_f(S_i) = E_g(S_i)$ for $i = 1,2$ then $f \equiv g$.*

To state the subsequent results we require the following definition which is the gradation of sharing of values known as weighted sharing which measure how close a shared value is to being shared IM or to being shared CM.

Definition 1.1. [14, 15] *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.*

Definition 1.1 implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m ($\leq k$) if and only if it is an a-point of g with multiplicity m ($\leq k$) and z_0 is an a-point of f with multiplicity m ($> k$) if and only if it is an a-point of g with multiplicity n ($> k$), where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [14] *Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S,k)$ the set $E_f(S,k) = \bigcup_{a \in S} E_k(a;f)$.*

Remark 1.1. From Definition 1.1 we have $E_f(S) = E_f(S,\infty)$ and $\overline{E}_f(S) = E_f(S,0)$.

Recently the first author [4] proved the following result which improved Theorem B by relaxing the nature of sharing the set S_1.

Theorem C. *Let S_1 and S_2 be defined as in Theorem B. If for two non-constant meromorphic functions f and g, $\Theta(\infty; f) > \frac{1}{2}$, $\Theta(\infty; g) > \frac{1}{2}$ and $E_{3)}(S_1, f) = E_{3)}(S_1, g)$, $E_f(S_2,\infty) = E_g(S_2,\infty)$ then $f \equiv g$.*

Since two meromorphic functions f and g having no simple poles imply $\Theta(\infty, f) \geq \frac{1}{2}$ and $\Theta(\infty, g) \geq \frac{1}{2}$ so it will be better to concentrate the attention on the relaxation of sharing both the range sets in Theorem B. In this paper, we will deal with this problem and improve Theorem B by relaxing the nature of sharing of both the range sets.

In 1997, Fang and Xu proved the following result which dealt with Question B.

Theorem D. [8] *Let $S_1 = \{z : z^3 - z^2 - 1 = 0\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$. Suppose that f and g are two non-constant meromorphic functions satisfying $\Theta(\infty; f) > \frac{1}{2}$ and $\Theta(\infty; g) > \frac{1}{2}$. If $E_f(S_i,\infty) = E_g(S_i,\infty)$ for $i = 1, 2, 3$ then $f \equiv g$.*

Afterwards Qiu and Fang proved the following result which improved Theorem D.

Theorem E. [21] *Let $n \geq 3$ be a positive integer $S_1 = \{z : z^n - z^{n-1} - 1 = 0\}$, $S_2 = \{0\}$ and and $S_3 = \{\infty\}$. Let f and g be two non-constant meromorphic functions whose poles are of multiplicities at least 2. If $E_f(S_i, \infty) = E_g(S_i, \infty)$ for $i = 1, 2, 3$ then $f \equiv g$.*

In 2004, Lahiri and Banerjee investigated the situation for $\Theta(\infty, f) \leq \frac{1}{2}$ and $\Theta(\infty, g) \leq \frac{1}{2}$ and got the following result which improved and supplemented Theorem D.

Theorem F. [17] *Let $S_1 = \{z : z^n + az^{n-1} + b = 0\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n (≥ 4) is an integer. If for two non-constant meromorphic functions f and g $E_f(S_i, \infty) = E_g(S_i, \infty)$ for $i = 1, 2, 3$ and $\Theta(\infty; f) + \Theta(\infty; g) > 0$ then $f \equiv g$.*

In 2004 Yi and Lin [27] proved the following theorem.

Theorem G. [27] *Let S_1, S_2 and S_3, be given as in Theorem F. If for two non-constant meromorphic functions f and g, $E_f(S_i, \infty) = E_g(S_i, \infty)$ for $i = 1, 3$, $E_f(S_2, 0) = E_g(S_2, 0)$ and $\Theta(\infty; f) > 0$ then $f \equiv g$.*

In 2007, the first author [1] relaxed the nature of sharing the set S_1 in Theorem F by using the idea of weighted sharing. In this paper, we will prove the following seven theorems by new methods that are different from those in [1]. These results also improve Theorems E-G.

Theorem 1.1. *Let S_1 and S_2 be defined as in Theorem B. If f and g are two non-constant meromorphic functions having no simple poles such that $E_{3)}(S_1, f) = E_{3)}(S_1, g)$, $E_f(S_2, 3) = E_g(S_2, 3)$ then $f \equiv g$.*

Theorem 1.2. *Let S_1 and S_2, be defined as in Theorem B. If f and g are two non-constant meromorphic functions having no simple poles such that $E_{4)}(S_1, f) = E_{4)}(S_1, g)$, $E_f(S_2, 1) = E_g(S_2, 1)$ then $f \equiv g$.*

Theorem 1.3. *Let S_1, S_2 and S_3 be defined as in Theorem F and n (≥ 3) is an integer. If f and g are two non-constant meromorphic functions having no simple poles such that $E_{10)}(S_1, f) = E_{10)}(S_1, g)$, $E_f(S_2, 0) = E_g(S_2, 0)$ and $E_f(S_3, \infty) = E_g(S_3, \infty)$ then $f \equiv g$.*

Theorem 1.4. *Let S_1, S_2 and S_3 be defined as in Theorem F and n (≥ 3) is an integer. If f and g are two non-constant meromorphic functions having no simple poles such that $E_{9)}(S_1, f) = E_{9)}(S_1, g)$, $E_f(S_2, 1) = E_g(S_2, 1)$ and $E_f(S_3, \infty) = E_g(S_3, \infty)$ then $f \equiv g$.*

Theorem 1.5. *Let S_1, S_2 and S_3 be defined as in Theorem F and n (≥ 3) is an integer. If f and g are two non-constant meromorphic functions having no simple poles such that $E_{8)}(S_1, f) = E_{8)}(S_1, g)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$ and $E_f(S_3, \infty) = E_g(S_3, \infty)$ then $f \equiv g$.*

Theorem 1.6. *Let S_1, S_2 and S_3 be defined as in Theorem F. If for two non-constant meromorphic functions f and g $E_{6)}(S_1, f) = E_{6)}(S_1, g)$, $E_f(S_2, 0) = E_g(S_2, 0)$ and $E_f(S_3, \infty) = E_g(S_3, \infty)$ and $\Theta(\infty; f)+\Theta(\infty; g) > 0$ then $f \equiv g$.*

Theorem 1.7. *Let S_1, S_2 and S_3 be defined as in Theorem F. If for two non-constant meromorphic functions f and g $E_{5)}(S_1, f) = E_{5)}(S_1, g)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$ and $E_f(S_3, \infty) = E_g(S_3, \infty)$ and $\Theta(\infty; f)+\Theta(\infty; g) > 0$ then $f \equiv g$.*

The following example shows that the condition $\Theta(\infty; f) + \Theta(\infty; g) > 0$ is sharp in Theorems 1.6-1.7.

Example 1.1. Let

$$g = -a\frac{e^{(n-1)z} - 1}{e^{nz} - 1}, \quad f(z) = e^z g(z)$$

and $S_i^{'}s$ be as in Theorem 1.1. Then $E_f(S_i, \infty) = E_g(S_i, \infty)$ for $i = 1, 2, 3$ because $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$ and $f \equiv e^z g$. Also $\Theta(\infty; f)+\Theta(\infty; g) = 0$ and $f \not\equiv g$.

Next we introduce some notations which are used throughout this paper.

Definition 1.3. [12] *For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of the simple a points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ the counting function of those a points of f whose multiplicities are not greater(less) than m, where each a point is counted according to its multiplicity. We denote by $N(r, a; f \mid< m)$, $(N(r, a; f \mid> m))$ the counting function of those a-points of f whose multiplicities are less (greater) than m, where each point is counted according to its multiplicity. We denote by $\overline{N}(r, a; f \mid\leq m)$, $\overline{N}(r, a; f \mid\geq m)$, $\overline{N}(r, a; f \mid< m)$ and $\overline{N}(r, a; f \mid> m)$ the reduced forms of $N(r, a; f \mid\leq m)$, $N(r, a; f \mid\geq m)$, $N(r, a; f \mid< m)$ and $N(r, a; f \mid> m)$ respectively.*

Definition 1.4. *We denote by $\overline{N}(r, a; f \mid= k)$ the reduced counting function of those a-points of f whose multiplicities are exactly k, where $k \geq 2$ is an integer.*

Definition 1.5. *Let f and g be two non-constant meromorphic functions such that f and g share the value a IM where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ $(\overline{N}_L(r, a; g))$ the counting function of those a-points of f and g where $p > q$ $(q > p)$, each point in these counting functions is counted only once.*

Definition 1.6. *Let f and g be two non-constant meromorphic functions and m be a positive integer such that $E_{m)}(a;f) = E_{m)}(a;g)$ where $a \in \mathbb{C}\cup\{\infty\}$. Let z_0 be an a-point of f with multiplicity $p > 0$, an a-point of g with multiplicity $q > 0$. We denote by $\overline{N}_L^{m)}(r,a;f)$ ($\overline{N}_L^{m)}(r,a;g)$) the counting function of those a-points of f and g where $p > q$ ($q > p$), each a-point is counted only once.*

Definition 1.7. [15] *We define $N_2(r,a;f)$ by $N_2(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2)$.*

Definition 1.8. *Let m be a positive integer and for $a \in \mathbb{C}$, $E_{m)}(a;f) = E_{m)}(a;g)$. Let z_0 be a zero of $f-a$ of multiplicity p and a zero of $g-a$ of multiplicity q. We denote by $\overline{N}_{f\geq m+1}(r,a;f \mid g \neq a)$ ($\overline{N}_{g\geq m+1}(r,a;g \mid f \neq a)$) the reduced counting functions of those a-points of f and g for which $p \geq m+1$ and $q = 0$ ($q \geq m+1$ and $p = 0$).*

Definition 1.9. [15] *Let f, g share $(a,0)$. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Remark 1.2. From Definition 1.5 and Definition 1.9 we have $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

Definition 1.10. *For $E_{m)}(1;f) = E_{m)}(1;g)$, let z_0 be a zero of $f-1$ with multiplicity $p(\geq 0)$ and a zero of $g-1$ with multiplicity $q(\geq 0)$. We denote by $\overline{N}_{\otimes}(r,1;f,g)$ the reduced counting function of those common 1 points of f and g with $p \neq q$.*

Remark 1.3. From Definition 1.6 and Definition 1.10 we have

$$\begin{aligned}\overline{N}_{\otimes}(r,1;f,g) = \overline{N}_L^{m)}(r,1;f) + \overline{N}_L^{m)}(r,1;g) + \overline{N}_{f\geq m+1}(r,1;f \mid g \neq 1)\\ + \overline{N}_{g\geq m+1}(r,1;g \mid f \neq 1).\end{aligned}$$

Definition 1.11. [18] *Let $a,b \in \mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f \mid\ g = b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.*

Definition 1.12. [18] *Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C}\cup\{\infty\}$. We denote by $N(r,a; f \mid g \neq b_1, b_2, \ldots, b_q)$ the counting function of those a-points of f, counted according to their multiplicities, which are not the b_i-points of g for $i = 1,2,\ldots,q$.*

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined as follows.

$$F=\frac{f^{n-1}(f+a)}{-b}, G=\frac{g^{n-1}(g+a)}{-b}. \tag{2.1}$$

Henceforth we will denote respectively by H, Φ and V the following three functions

$$H=(\frac{F^{''}}{F^{'}}-\frac{2F^{'}}{F-1})-(\frac{G^{''}}{G^{'}}-\frac{2G^{'}}{G-1}),$$

$$\Phi=\frac{F^{'}}{F-1}-\frac{G^{'}}{G-1}$$

and

$$V=(\frac{F^{'}}{F-1}-\frac{F'}{F})-(\frac{G^{'}}{G-1}-\frac{G'}{G})=\frac{F'}{F(F-1)}-\frac{G'}{G(G-1)}.$$

Lemma 2.1. [20] *For $E_{m)}(1;F)=E_{m)}(1;G)$ and $H\not\equiv 0$ then*

$$N(r,1;F\mid=1)=N(r,1;G\mid=1)\leq N(r,H)+S(r,F)+S(r,G).$$

Lemma 2.2. *Let S_1, S_2 and S_3 be defined as in Theorem F and F, G be given by* (2.1)*. If for two non-constant meromorphic functions f and g, $E_{m)}(S_1,f)=E_{m)}(S_1,g)$, $E_f(S_2,0)=E_g(S_2,0)$, $E_f(S_3,0)=E_g(S_3,0)$ and $H\not\equiv 0$ then*

$$\begin{aligned}N(r,H)\leq\ &\overline{N}_*(r,0,f,g)+\overline{N}(r,0;f+a\mid\geq 2)+\overline{N}(r,0;g+a\mid\geq 2)\\&+\overline{N}_{\otimes}(r,1;F,G)+\overline{N}_*(r,\infty;f,g)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'}),\end{aligned}$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{'})$ is similarly defined.

Proof. Since $E_{m)}(S_1,f)=E_{m)}(S_1,g)$, it follows that $E_{m)}(1;F)=E_{m)}(1;G)$. We can easily verify that possible poles of H occur at (i) those zeros of f and g whose multiplicities are distinct from the multiplicities of the corresponding zeros of g and f respectively, (ii) multiple zeros of $f+a$ and $g+a$, (iii) those poles of f and g whose multiplicities are distinct from the multiplicities of the corresponding poles of g and f respectively, (iv) those 1-points of F and G with different multiplicities, (v) zeros of $F^{'}$ which are not the zeros of $F(F-1)$, (v) zeros of $G^{'}$ which are not the zeros of $G(G-1)$.

Since H has only simple poles, the lemma follows from Remark 1.3 and from the above explanations. This proves the lemma. □

Lemma 2.3. ([17], Lemma 4) *If two non-constant meromorphic functions F and G share $(1,0)$, $(\infty,0)$ and $H \not\equiv 0$ then*

$$N(r,H) \leq \overline{N}(r,0;F \mid\geq 2) + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}_*(r,\infty;F,G) + \overline{N}_0(r,0;F^{'}) + \overline{N}_0(r,0;G^{'}).$$

Lemma 2.4. [22] *Let f be a non-constant meromorphic function and $P(f)$= $a_0+a_1f+a_2f^2+\cdots+a_nf^n$, where $a_0,a_1,a_2\ldots,a_n$ are constants and $a_n \neq 0$. Then $T(r,P(f)) = nT(r,f)+O(1)$.*

Lemma 2.5. [2] *Let F and G be given by (2.1). If f, g share $(0,0)$ and 0 is not a Picard exceptional value of f and g. Then $\Phi \equiv 0$ implies $F \equiv G$.*

Lemma 2.6. *Let F and G be given by (2.1), $n \geq 3$ an integer and $\Phi \not\equiv 0$. If $E_{m)}(1,F) = E_{m)}(1,G)$ and f, g share $(0,p)$, (∞,k), where $0 \leq p < \infty$ then*

$$[(n-1)p+n-2]\,\overline{N}(r,0;f \mid\geq p+1) \leq \overline{N}_{\otimes}(r,1;F,G) + \overline{N}_*(r,\infty;F,G) + S(r,f) + S(r,g).$$

Proof. Suppose 0 is an e.v.P. (Picard exceptional value) of f and g then the lemma follows immediately.

Next suppose 0 is not an e.v.P. of f and g. Let z_0 is a zero of f with multiplicity q and a zero of g with multiplicity r. From (2.1) we know that z_0 is a zero of F with multiplicity $(n-1)q$ and a zero of G with multiplicity $(n-1)r$. We note that F and G have no zero of multiplicity t where $(n-1)p < t < (n-1)(p+1)$. So from the definition of Φ it is clear that z_0 is a zero of Φ with multiplicity at least $(n-1)(p+1)-1$. So we have

$$\begin{aligned}
&[(n-1)p+n-2]\overline{N}(r,0;f \mid\geq p+1)\\
&= [(n-1)p+n-2]\overline{N}(r,0;g \mid\geq p+1)\\
&= [(n-1)p+n-2]\overline{N}\,(r,0;F \mid\geq (n-1)(p+1))\\
&\leq N(r,0;\Phi)\\
&\leq N(r,\infty;\Phi) + S(r,f) + S(r,g)\\
&\leq \overline{N}_{\otimes}(r,1;F,G) + \overline{N}_*(r,\infty;F,G) + S(r,f) + S(r,g).
\end{aligned}$$

The lemma follows from above. □

Lemma 2.7. [2] *Let F and G be given by (2.1) and f, g share $(\infty,0)$ and ∞ is not a Picard exceptional value of f and g. Then $V \equiv 0$ implies $F \equiv G$.*

Lemma 2.8. *Let F, G be given by* (2.1) *and $V \not\equiv 0$. If f, g share* $(0,0)$, (∞, k), *where* $0 \le k < \infty$, *and* $E_{m)}(1,F) = E_{m)}(1,G)$, *then the poles of F and G are the zeros of V and*

$$\begin{aligned}(n-1)\; &N(r,\infty; f \mid = 1) + (2n-1)\overline{N}(r,\infty; f \mid = 2) + \dots \\ &+ (nk-1)\overline{N}(r,\infty; f \mid = k) + (nk+n-1)\overline{N}(r,\infty; f \mid \ge k+1) \\ &\le \overline{N}_*(r,0; f,g) + \overline{N}(r,0; f+a) + \overline{N}(r,0; g+a) \\ &\qquad + \overline{N}_{\otimes}(r,1; F,G) + S(r,f) + S(r,g).\end{aligned}$$

Proof. Suppose ∞ is an e.v.P. of f and g then the lemma follows immediately.

Next suppose ∞ is not an e.v.P. of f and g. Since f, g share (∞, k), it follows that F, G share (∞, nk) and so a pole of F with multiplicity $p(\ge nk+1)$ is a pole of G with multiplicity $r(\ge nk+1)$ and vice versa. We note that F and G have no pole of multiplicity q where $nk < q < nk+n$. Also any common pole of F and G of multiplicity $p \le nk$ is a zero of V of multiplicity $\ge p-1$. Using *Lemma* 2.4 we get from the definition of V

$$\begin{aligned}&(n-1)\overline{N}(r,\infty; f) \\ &\le (2n-1)\overline{N}(r,\infty; f) - n\overline{N}(r,\infty; f \mid = 1) \\ &\le (n-1)\; N(r,\infty; f \mid = 1) + (2n-1)\overline{N}(r,\infty; f \mid = 2) + \dots \\ &\qquad + (nk-1)\overline{N}(r,\infty; f \mid = k) + (nk+n-1)\overline{N}(r,\infty; f \mid \ge k+1) \\ &\le N(r,0; V) \le N(r,\infty; V) + S(r,f) + S(r,g) \\ &\le \overline{N}_*(r,0; f,g) + \overline{N}(r,0; f+a) \\ &\qquad + \overline{N}(r,0; g+a) + \overline{N}_{\otimes}(r,1; F,G) + S(r,f) + S(r,g),\end{aligned}$$

which yields the conclusion of Lemma 2.8. □

Proceeding as in the proof of Lemma 2.8, we get the following result.

Lemma 2.9. *Let F, G be given by* (2.1) *and $V \not\equiv 0$. If f, g share* (∞, k), *where* $0 \le k < \infty$ *and F, G share* $(1, m)$ *then the poles of F and G are the zeros of V and*

$$\begin{aligned}(n-1)\; &N(r,\infty; f \mid = 1) + (2n-1)\overline{N}(r,\infty; f \mid = 2) + \dots \\ &+ (nk-1)\overline{N}(r,\infty; f \mid = k) + (nk+n-1)\overline{N}(r,\infty; f \mid \ge k+1) \\ &\le \overline{N}(r,0; f) + \overline{N}(r,0; g) + \overline{N}(r,0; f+a) + \overline{N}(r,0; g+a) \\ &\qquad + \overline{N}_{\otimes}(r,1; F,G) + S(r,f) + S(r,g).\end{aligned}$$

Lemma 2.10. *Let f and g be two meromorphic meromorphic functions such that $E_{m)}(1;f)=E_{m)}(1;g)$, where $1\leq m<\infty$. Then*

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)+\left(\frac{m}{2}-\frac{1}{2}\right)\{\overline{N}_{f\geq m+1}(r,1;f\mid g\neq 1)$$
$$+\overline{N}_{g\geq m+1}(r,1;g\mid f\neq 1)\}+\left(m-\frac{1}{2}\right)\left\{\overline{N}_L^{m)}(r,1;f)+\overline{N}_L^{m)}(r,1;g)\right\}$$
$$\leq\frac{1}{2}\left[N(r,1;f)+N(r,1;g)\right]$$

Proof. Since $E_{m)}(1;f)=E_{m)}(1;g)$, we note that common zeros of $f-1$ and $g-1$ up-to multiplicity m are the same. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. If $p=m+1$ the possible values of q are as follows (i) $q=m+1$ (ii) $q\geq m+2$ (iii) $q=0$. Similarly when $p=m+2$ the possible values of q are (i) $q=m+1$ (ii) $q=m+2$ (iii) $q\geq m+3$ (iv) $q=0$. If $p\geq m+3$ we can similarly find the possible values of q. Now the lemma follows from above explanation. □

Lemma 2.11. *Let F, G be given by (2.1) and $H\not\equiv 0$. If $E_{m)}(1;F)=E_{m)}(1;G)$, f, g share (∞,k), $(0,p)$, where $1\leq m<\infty$. Then*

$$\left(\frac{n}{2}-1\right)\{T(r,f)+T(r,g)\}\leq\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)$$
$$+\overline{N}(r,\infty;g)+\overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)$$
$$-\left(\frac{m}{2}-\frac{3}{2}\right)\left\{\overline{N}_{F\geq m+1}(r,1;F\mid G\neq 1)\right.$$
$$\left.+\overline{N}_{G\geq m+1}(r,1;G\mid F\neq 1)\right\}-\left(m-\frac{3}{2}\right)$$
$$\left\{\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)\right\}+S(r,f)+S(r,g).$$

Proof. By the second fundamental theorem we get

$$T(r,F)+T(r,G)\leq\overline{N}(r,1;F)+\overline{N}(r,0;F)+\overline{N}(r,\infty;F)$$
$$+\overline{N}(r,1;G)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)$$
$$-N_0(r,0;F^{'})-N_0(r,0;G^{'})+S(r,F)+S(r,G).\quad(2.2)$$

Using Lemmas 2.1, 2.2, 2.4 and 2.10 we see that

$$\overline{N}(r,1;F)+\overline{N}(r,1;G)$$
$$\leq\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+N(r,1;F\mid=1)$$
$$-\left(\frac{m}{2}-\frac{1}{2}\right)\left\{\overline{N}_{F\geq m+1}(r,1;F\mid G\neq 1)\right.$$

$$+\ \overline{N}_{G\geq m+1}(r,1;G\mid F\neq 1)\Big\} - \Big(m-\frac{1}{2}\Big)\Big\{\,\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)\Big\}$$
$$\leq \frac{n}{2}\{T(r,f)+T(r,g)\}+\overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)$$
$$+\overline{N}(r,0;f+a\mid\geq 2)+\overline{N}(r,0;g+a\mid\geq 2)+\overline{N}_\otimes(r,1;F,G)$$
$$-\Big(\frac{m}{2}-\frac{1}{2}\Big)\Big\{\,\overline{N}_{F\geq m+1}(r,1;F\mid g\neq 1)$$
$$+\overline{N}_{G\geq m+1}(r,1;G\mid F\neq 1)\Big\}-\Big(m-\frac{1}{2}\Big)\Big\{\,\overline{N}_L^{m)}(r,1;F)$$
$$+\overline{N}_L^{m)}(r,1;G)\Big\}+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g)$$
$$\leq \frac{n}{2}\{T(r,f)+T(r,g)\}+\overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)$$
$$+\overline{N}(r,0;f+a\mid\geq 2)+\overline{N}(r,0;g+a\mid\geq 2)-\Big(\frac{m}{2}-\frac{3}{2}\Big)$$
$$\Big\{\,\overline{N}_{F\geq m+1}(r,1;F\mid G\neq 1)+\overline{N}_{G\geq m+1}(r,1;G\mid F\neq 1)\Big\}$$
$$-\Big(m-\frac{3}{2}\Big)\Big\{\,\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)\Big\}+\overline{N}_0(r,0;F')$$
$$+\overline{N}_0(r,0;G')+S(r,f)+S(r,g). \tag{2.3}$$

Using (2.3) in (2.2) the lemma follows. □

Lemma 2.12. *Let F, G be given by* (2.1) *and $H\not\equiv 0$. If $E_{m)}(1;F)=E_{m)}(1;G)$, f, g share (∞,k), where $1\leq m<\infty$. Then*

$$\Big(\frac{n}{2}-1\Big)\{T(r,f)+T(r,g)\}\leq 2\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+2\overline{N}(r,0;g)$$
$$+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f,g)-\Big(\frac{m}{2}-\frac{3}{2}\Big)\Big\{\,\overline{N}_{F\geq m+1}(r,1;F\mid G\neq 1)$$
$$+\overline{N}_{G\geq m+1}(r,1;G\mid F\neq 1)\Big\}-\Big(m-\frac{3}{2}\Big)$$
$$\Big\{\,\overline{N}_L^{m)}(r,1;F)+\overline{N}_L^{m)}(r,1;G)\Big\}+S(r,f)+S(r,g).$$

Proof. Using Lemmas 2.1, 2.3, 2.4 and 2.10 and using the same procedure as that in the proof of Lemma 2.11 we see that

$$\overline{N}(r,1;F)+\overline{N}(r,1;G)$$
$$\leq \frac{n}{2}\{T(r,f)+T(r,g)\}+\overline{N}(r,0;f)+\overline{N}(r,0;f+a\mid\geq 2)+\overline{N}(r,0;g)$$
$$+\overline{N}(r,0;g+a\mid\geq 2)+\overline{N}_*(r,\infty;f,g)+\overline{N}_\otimes(r,1;F,G)$$
$$-\Big(\frac{m}{2}-\frac{1}{2}\Big)\Big\{\,\overline{N}_{F\geq m+1}(r,1;F\mid g\neq 1)$$

$$
\begin{aligned}
&+\overline{N}_{G\geq m+1}(r,1;G\mid F\neq 1)\Big\}-\left(m-\frac{1}{2}\right)\Big\{\,\overline{N}_{L}^{m)}(r,1;F)\\
&+\overline{N}_{L}^{m)}(r,1;G)\Big\}+\overline{N}_{0}(r,0;F^{'})+\overline{N}_{0}(r,0;G^{'})+S(r,f)+S(r,g)\\
\leq\ &\frac{n}{2}\left\{T(r,f)+T(r,g)\right\}+\overline{N}(r,0;f)+\overline{N}(r,0;f+a\mid\geq 2)+\overline{N}(r,0;g)\\
&+\overline{N}(r,0;g+a\mid\geq 2)+\overline{N}_{*}(r,\infty;f,g)-\left(\frac{m}{2}-\frac{3}{2}\right)\\
&\Big\{\,\overline{N}_{F\geq m+1}(r,1;F\mid G\neq 1)+\overline{N}_{G\geq m+1}(r,1;G\mid F\neq 1)\Big\}\\
&-\left(m-\frac{3}{2}\right)\Big\{\,\overline{N}_{L}^{m)}(r,1;F)+\overline{N}_{L}^{m)}(r,1;G)\Big\}+\overline{N}_{0}(r,0;F^{'})\\
&+\overline{N}_{0}(r,0;G^{'})+S(r,f)+S(r,g). \qquad (2.4)
\end{aligned}
$$

Using (2.4) in (2.2) the lemma follows. □

Lemma 2.13. [5] *Let* $F=\frac{f^{n-1}(f+a)}{-b}$, $G=\frac{g^{n-1}(g+a)}{-b}$, *where* $n(\geq 7)$ *is an integer. If* $H\equiv 0$ *then* $f^{n-1}(f+a)g^{n-1}(g+a)\equiv b^{2}$ *or* $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$.

Lemma 2.14. ([16], Lemma 5) *If* f, g *share* $(\infty,0)$ *then for* $n(\geq 2)$

$$f^{n-1}(f+a)g^{n-1}(g+a)\not\equiv b^{2},$$

where a,b are finite nonzero constants.

Lemma 2.15. ([17], Lemma 9) *Let* f, g *be two meromorphic functions such that* $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n-1}$, *where* $n(\geq 4)$ *is an integer. Then* $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$ *implies* $f\equiv g$, *a is a nonzero finite constant.*

Lemma 2.16. *If* $N(r,0;f^{'}\mid f\neq 0)$ *denotes the counting function of those zeros of* $f^{'}$ *which are not the zeros of* f, *where a zero of* $f^{'}$ *is counted according to its multiplicity, then*

$$N(r,0;f^{'}\mid f\neq 0)\leq\overline{N}(r,\infty;f)+\overline{N}(r,0;f)+S(r,f).$$

Proof. By the first fundamental theorem and the Milloux theorem ([see [11], Theorem 3.1]) we get

$$
\begin{aligned}
N\left(r,0;f^{'}\mid f\neq 0\right)&\leq N\left(r,0;\frac{f^{'}}{f}\right)\\
&\leq N\left(r,\infty;\frac{f^{'}}{f}\right)+m\left(r,\frac{f^{'}}{f}\right)+O(1)\\
&\leq\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f).
\end{aligned}
$$

□

Lemma 2.17. *Let F, G be given by (2.1). If $E_{m)}(1;F) = E_{m)}(1;G)$, f, g share $(\infty;0)$ and $\omega_1, \omega_2 \ldots \omega_n$ are the distinct roots of the equation $z^n + az^{n-1} + b = 0$ and $n \geq 3$. Then*

$$\overline{N}_{\otimes}(r,1;F,G) \leq \frac{1}{m}\Big[\overline{N}(r,0;f) + \overline{N}(r,0;g) + 2\overline{N}(r,\infty;f) - N_{\oplus}(r,0;f^{'}) - N_{\oplus}(r,0;g^{'})\Big] + S(r,f) + S(r,g),$$

where $N_{\oplus}(r,0;f^{'}) = N(r,0;f^{'} \mid f \neq 0, \omega_1, \omega_2 \ldots \omega_n)$ and $N_{\oplus}(r,0;g^{'})$ is similarly defined.

Proof. In view of Definition 1.3 and Remark 1.3 and Lemma 2.16 we note that

$$\begin{aligned}
&\overline{N}_{\otimes}(r,1;F,G) \leq \overline{N}(r,1;F \mid\geq m+1) + \overline{N}(r,1;F \mid\geq m+1)\\
&\leq \frac{1}{m}\big[N(r,1;F) - \overline{N}(r,1;F) + N(r,1;G) - \overline{N}(r,1;G)\big]\\
&\leq \frac{1}{m}\left[\sum_{j=1}^{n}\big(N(r,\omega_j;f) - \overline{N}(r,\omega_j;f)\big) + \sum_{j=1}^{n}\big(N(r,\omega_j;g) - \overline{N}(r,\omega_j;g)\big)\right]\\
&\leq \frac{1}{m}\Big(N(r,0;f^{'} \mid f \neq 0) - N_{\oplus}(r,0;f^{'}) + N(r,0;g^{'} \mid g \neq 0) - N_{\oplus}(r,0;g^{'})\Big)\\
&\leq \frac{1}{m}\Big[\overline{N}(r,0;f) + \overline{N}(r,0;g) + 2\overline{N}(r,\infty;f) - N_{\oplus}(r,0;f^{'}) - N_{\oplus}(r,0;g^{'})\Big]\\
&\qquad + S(r,f) + S(r,g).
\end{aligned}$$

□

Lemma 2.18. *Let F, G be given by (2.1) and $V \not\equiv 0$. If $E_{m)}(1;F) = E_{m)}(1;G)$, f, g having no simple poles such that they share $(\infty,1)$, then*

$$[m(2n-1)-2]\,\overline{N}(r,\infty;f \mid\geq 2) \leq (m+2)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g).$$

Proof. Using Lemma 2.17, we obtain from Lemma 2.9 with $k = 1$ that

$$\begin{aligned}
(2n-1)\overline{N}(r,\infty;f \mid\geq 2) &\leq 2T(r,f) + 2T(r,g) + \frac{1}{m}\{\overline{N}(r,0;f) + \overline{N}(r,0;g)\}\\
&\quad + \frac{2}{m}\overline{N}(r,\infty;f) + S(r,f) + S(r,g)\\
&\leq \frac{m+2}{m}\{T(r,f)+T(r,g)\} + \frac{2}{m}\overline{N}(r,\infty;f) + S(r,f) + S(r,g).
\end{aligned}$$

Now the lemma follows. □

Lemma 2.19. *Let F and G be given by (2.1), $n \geq 3$ an integer and $\Phi \not\equiv 0$. If $E_{m)}(1;F) = E_{m)}(1;G)$, f, g share $(0,p)$, (∞,∞), where $0 \leq p < \infty$ then*

$$\{m(n-2)-2\}\, \overline{N}(r,0;f) \leq 2\, \overline{N}(r,\infty;f) + S(r,f) + S(r,g).$$

Proof. Since f, g share $(0,p)$ it follows that they share $(0,0)$. So using Lemma 2.6 for $p=0$, Lemma 2.17 and noting that f, g share (∞,∞) we see that

$$\begin{aligned}(n-2)\, \overline{N}(r,0;f) &\leq \overline{N}_{\otimes}(r,1;F,G) + S(r,f) + S(r,g)\\ &\leq \frac{2}{m}[\overline{N}(r,0;f) + \overline{N}(r,\infty;f)] + S(r,f) + S(r,g),\end{aligned}$$

from which the lemma follows. □

Lemma 2.20. ([26], Lemma 6) *If $H \equiv 0$, then F, G share $(1,\infty)$. If further F, G share $(\infty,0)$ then F, G share (∞,∞).*

Lemma 2.21. ([17], Lemma 3) *Let f, g be two non-constant meromorphic functions sharing $(0,\infty)$, (∞,∞) and $\Theta(\infty;f) + \Theta(\infty;g) > 0$. Then $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$ implies $f \equiv g$, where n (≥ 2) is an integer and a is a nonzero complex number.*

Lemma 2.22. [25] *Let F, G be two non-constant meromorphic functions sharing $(1,\infty)$ and (∞,∞). If*

$$N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,\infty;F) < \lambda T_1(r) + S_1(r),$$

where $\lambda < 1$ and $T_1(r) = \max\{T(r,F), T(r,G)\}$ and $S_1(r) = o(T_1(r))$, $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure, then $F \equiv G$ or $FG \equiv 1$.

Lemma 2.23. *Let F, G be given by (2.1) and $n \geq 3$. If $E_{m)}(1;F) = E_{m)}(1;G)$, f, g are two non-constant meromorphic functions having no simple poles such that they share $(0,0)$, $(\infty,1)$, where $1 \leq m < \infty$ and $H \equiv 0$. Then $f \equiv g$.*

Proof. Since f and g share $(\infty,1)$, we get from $H \equiv 0$ and Lemma 2.20 that F and G share $(1,\infty)$ and (∞,∞). Thus f and g share (∞,∞). Assume that $F \not\equiv G$. Then from Lemma 2.5 and Lemma 2.6 we have

$$\overline{N}(r,0;f) = \overline{N}(r,0;g) = S(r).$$

Again from Lemma 2.7 and the supposition $F \not\equiv G$ we see that $V \not\equiv 0$ or ∞ is a Picard exceptional value of f and g. Combining the condition $n \geq 3$

and the condition that f and g have no simple poles we get from Lemma 2.8

$$\begin{aligned}\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g) &= \overline{N}(r,\infty;f\mid\geq 2)+\overline{N}(r,\infty;g\mid\geq 2)\\ &\leq \frac{2}{2n-1}N(r,0;V)\\ &= \frac{2}{2n-1}N(r,\infty;V)+S(r)\\ &\leq \frac{2}{2n-1}\{\overline{N}(r,-a;f)+\overline{N}(r,-a;g)\}T(r)+S(r)\\ &\leq \frac{4}{2n-1}T(r)+S(r).\end{aligned}$$

Therefore we see that

$$\begin{aligned}&N_2(r,0;F)+N_2(r,0;G)+2\overline{N}(r,\infty;F)\\ &\leq 2\overline{N}(r,0;f)+2\overline{N}(r,0;g)+N_2(r,0;f+a)+N_2(r,0;g+a)+2\overline{N}(r,\infty;f)\\ &\leq N_2(r,0;f+a)+N_2(r,0;g+a)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r)\end{aligned}\tag{2.5}$$

Using Lemma 2.4 we obtain

$$T_1(r)=n\ \max\{T(r,f),T(r,g)\}+O(1)=n\,T(r)+O(1).\tag{2.6}$$

So again using Lemma 2.4 we get from (2.5) and (2.6)

$$N_2(r,0;F)+N_2(r,0;G)+2\overline{N}(r,\infty;F)\leq \frac{2+\frac{4}{2n-1}}{n}\,T_1(r)+S_1(r).$$

Since $\frac{2+\frac{4}{2n-1}}{n}<1$ for $n\geq 3$ by Lemma 2.22 we have $FG\equiv 1$, which is impossible by Lemma 2.14. Hence $F\equiv G$ i.e. $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$. This together with the assumption that f and g share $(0,0)$ implies that f and g share $(0,\infty)$. Combining Lemma 2.21 and the fact that f and g share (∞,∞), we get the conclusion of Lemma 2.23. □

Lemma 2.24. *Let F, G be given by* (2.1) *and $n\geq 4$. Also let $E_{m)}(1;F)=E_{m)}(1;G)$. If f, g share* $(0,0)$, (∞,k), *where $0\leq k<\infty$, $\Theta(\infty;f)+\Theta(\infty;g)>0$ and $H\equiv 0$. Then $f\equiv g$.*

Proof. Since $H\equiv 0$ we get from Lemma 2.20 F and G share $(1,\infty)$ and (∞,∞). We omit the proof since the proof can be found in Lemma 2.17 [3]. □

3. Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (2.1). Then $E_{3)}(1,F)=E_{3)}(1,G)$ and they share $(\infty,4n-1)$. Since f and g share $(\infty,3)$ no simple poles

clearly $\overline{N}_*(r,\infty;f,g)=\overline{N}(r,\infty;f\mid\geq 4)=\overline{N}(r,\infty;g\mid\geq 4)$. We consider the following cases.
Case 1. Let $H\not\equiv 0$. Then $F\not\equiv G$. If ∞ is an e.v.P. of f and g, from Lemma 2.12 we get

$$\begin{aligned}\left(\frac{n}{2}-1\right)\{T(r,f)+T(r,g)\} &\leq 2\overline{N}(r,0;f)+2\overline{N}(r,0;g)+S(r,f)+S(r,g)\\ &\leq 2\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),\end{aligned}$$

which implies $n\leq 6$. From this and the condition $n\geq 7$ we get a contradiction. Thus ∞ is not an e.v.P. of f and g. Then by Lemma 2.7 we get $V\not\equiv 0$. Noting that f and g have no simple poles and applying Lemma 2.9, Lemma 2.12 and Lemma 2.17 with $m=k=3$ we obtain

$$\begin{aligned}&\left(\frac{n}{2}-3\right)\{T(r,f)+T(r,g)\}\\ &\leq 2\overline{N}(r,\infty;f\mid\geq 2)+\overline{N}(r,\infty;f\mid\geq 4)+S(r,f)+S(r,g)\\ &\leq \frac{2}{2n-1}\{\,2T(r,f)+2T(r,g)+\overline{N}_{\otimes}(r,1;F,G)\}\\ &\qquad+\frac{1}{4n-1}\{\,2T(r,f)+2T(r,g)+\overline{N}_{\otimes}(r,1;F,G)\}+S(r,f)+S(r,g)\\ &\leq \left[\frac{14}{3(2n-1)}+\frac{7}{3(4n-1)}\right]\{T(r,f)+T(r,g)\}\\ &\qquad+\left[\frac{4}{3(2n-1)}+\frac{2}{3(4n-1)}\right]\overline{N}(r,\infty;f\mid\geq 2)+S(r,f)+S(r,g).\end{aligned} \tag{3.1}$$

Using Lemma 2.18 for $m=3$ in (3.1) we obtain

$$\begin{aligned}&\left[\frac{n}{2}-3-\left\{\frac{14}{3(2n-1)}+\frac{7}{3(4n-1)}+\frac{20}{3(6n-5)(2n-1)}\right.\right.\\ &\qquad\left.\left.+\frac{10}{3(6n-5)(4n-1)}\right\}\right]\\ &\qquad\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),\end{aligned}$$

which leads to a contradiction for $n\geq 7$.
Case 2. Let $H\equiv 0$. Now since by the given condition $\Theta(\infty;f)+\Theta(\infty;g)\geq 1>\frac{4}{n-1}$ the theorem follows from Lemmas 2.13, 2.14 and 2.15. □

Proof of Theorem 1.2. Let F, G be given by (2.1). Then $E_{4)}(1,F)=E_{4)}(1,G)$ and they share $(\infty,2n-1)$. We consider the following cases.
Case 1. Let $H\not\equiv 0$. Then $F\not\equiv G$. Proceeding as in the proof of *Theorem* 1.1 we see that ∞ is not an e.v.P. of f and g. Then by Lemma 2.7 we

get $V \not\equiv 0$. Noting that f and g have no simple poles using Lemma 2.9 in Lemma 2.12 with $m = 4$ and $k = 1$ we obtain

$$\begin{aligned}
&\left(\frac{n}{2} - 3\right) \{T(r, f) + T(r, g)\} \\
&\qquad \le 3\overline{N}(r, \infty; f \mid\ge 2) - \frac{1}{2}\overline{N}_{\otimes}(r, 1; F, G) + S(r, f) + S(r, g) \\
&\qquad \le \frac{3}{2n-1}\{\, 2T(r, f) + 2T(r, g)\} + S(r, f) + S(r, g). \qquad (3.2)
\end{aligned}$$

From (3.2) we have

$$\left[\frac{n}{2} - 3 - \frac{6}{(2n-1)}\right] \{T(r, f) + T(r, g)\} \le S(r, f) + S(r, g),$$

which is a contradiction for $n \ge 7$.
Case 2. Let $H \equiv 0$. Now since by the given condition $\Theta(\infty; f) + \Theta(\infty; g) \ge 1 > \frac{4}{n-1}$ the theorem follows from Lemmas 2.13, 2.14 and 2.15. □

Proof of Theorem 1.3. Let F, G be given by (2.1). Then $E_{10)}(1, F) = E_{10)}(1, G)$ and they share (∞, ∞). We consider the following cases.
Case 1. Let $H \not\equiv 0$. Then $F \not\equiv G$. Suppose 0, ∞ are not exceptional values Picard of f and g. Then by Lemma 2.5 and Lemma 2.7 we get $\Phi \not\equiv 0$ and $V \not\equiv 0$. Hence from Lemmas 2.6, 2.8, 2.11 and 2.19 with $k = \infty$, $p = 0$ and $m = 10$ we obtain

$$\begin{aligned}
&\left(\frac{n}{2} - 1\right) \{T(r, f) + T(r, g)\} \\
&\le 3\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) - \frac{7}{2}\overline{N}_{\otimes}(r, 1; F, G) + S(r, f) + S(r, g) \\
&\le \frac{3}{n-2}\overline{N}_{\otimes}(r, 1; F, G) + 2\overline{N}(r, \infty; f \mid\ge 2) - \frac{7}{2}\overline{N}_{\otimes}(r, 1; F, G) + S(r, f) + S(r, g) \\
&\le \frac{2}{2n-1}\big[T(r, f) + T(r, g) + \overline{N}(r, 0; f) \\
&\qquad + \overline{N}_{\otimes}(r, 1; F, G)\big] - \frac{1}{2}\,\overline{N}_{\otimes}(r, 1; F, G) + S(r, f) + S(r, g) \\
&\le \frac{2}{2n-1}\left[T(r, f) + T(r, g) + \frac{2}{10n-22}\overline{N}(r, \infty; f)\right] + S(r, f) + S(r, g) \\
&\le \frac{2}{2n-1}\left(1 + \frac{1}{10n-22}\right)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g). \qquad (3.3)
\end{aligned}$$

If 0, ∞ are Picard exceptional values of f and g, from Lemma 2.11 and $n \ge 3$ we get

$$T(r, f) + T(r, g) \le S(r, f) + S(r, g),$$

which is impossible. Thus at least one of 0 and ∞ is not a Picard exceptional value of f and g, and so (3.3) automatically holds. From (3.3) we see that

$$\left[\frac{n}{2}-1-\frac{2}{2n-1}\left(1+\frac{1}{2(5n-11)}\right)\right]\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g)$$

which is a contradiction for $n\geq 3$.
Case 2. Let $H\equiv 0$. The theorem follows from Lemma 2.23. □

Proof of Theorem 1.4. Let F, G be given by (2.1). Then $E_{9)}(1,F)=E_{9)}(1,G)$ and they share (∞,∞). We consider the following cases.
Case 1. Let $H\not\equiv 0$. Then $F\not\equiv G$. Suppose 0, ∞ are not exceptional values Picard of f and g. Then by Lemma 2.5 and Lemma 2.7 we get $\Phi\not\equiv 0$ and $V\not\equiv 0$. Hence from Lemmas 2.6, 2.8 and 2.11 with $k=\infty$, $p=1$ and $m=9$ we obtain

$$\begin{aligned}\left(\frac{n}{2}-1\right)\{T(r,f)+T(r,g)\} &\leq 2\overline{N}(r,0;f)+\overline{N}(r,0;f\mid\geq 2)+\overline{N}(r,\infty;f)\\ &\quad+\overline{N}(r,\infty;g)-3\,\overline{N}_{\otimes}(r,1;F,G)+S(r,f)+S(r,g)\\ &\leq \frac{2}{n-2}\overline{N}_{\otimes}(r,1;F,G)+\overline{N}(r,0;f\mid\geq 2)\\ &\quad+2\overline{N}(r,\infty;f\mid\geq 2)-3\,\overline{N}_{\otimes}(r,1;F,G)+S(r,f)+S(r,g)\\ &\leq \frac{2n+1}{2n-1}\overline{N}(r,0;f\mid\geq 2)+\frac{2}{2n-1}\left[T(r,f)+T(r,g)\right]\\ &\quad-\frac{2n-3}{2n-1}\overline{N}_{\otimes}(r,1;F,G)+S(r,f)+S(r,g)\\ &\leq \frac{2}{2n-1}\left[T(r,f)+T(r,g)\right]\\ &\quad+\left[\frac{2n+1}{(2n-1)(2n-3)}-\frac{2n-3}{2n-1}\right]\overline{N}_{\otimes}(r,1;F,G)+S(r,f)+S(r,g)\\ &\leq \frac{2}{2n-1}\left[T(r,f)+T(r,g)\right]+S(r,f)+S(r,g). \end{aligned}\tag{3.4}$$

Proceeding as in the proof of Theorem 1.3 we see that at least one of 0, ∞ is not a Picard exceptional value of f and g, and so (3.4) automatically holds. Clearly from (3.4) we can deduce a contradiction for $n\geq 3$.
Case 2. Let $H\equiv 0$. The theorem follows from Lemma 2.23. □

Proof of Theorem 1.5. Proceeding as in the proof of Theorem 1.4 we get the conclusion of Theorem 1.5. □

Proof of Theorem 1.6. Let F, G be given by (2.1). Then $E_{6)}(1,F)=E_{6)}(1,G)$ and they share (∞,∞). We consider the following cases.

Case 1. Let $H \not\equiv 0$. Then $F \not\equiv G$. If $\overline{N}(r,0;f) + \overline{N}(r,0;g) = S(r,f) + S(r,g)$, from Lemma 2.11 we get

$$\begin{aligned}\left(\frac{n}{2}-1\right)\{T(r,f)+T(r,g)\} &\le \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,f)+S(r,g)\\ &= T(r,f)+T(r,g)+S(r,f)+S(r,g),\end{aligned}$$

which implies $\frac{n}{2}-1 \le 1$. From this and $n \ge 4$ we get $n=4$ and $T(r,f) = \overline{N}(r,\infty;f)+O(1)$, $T(r,g) = \overline{N}(r,\infty;g)+O(1)$. From this we get

$$\Theta(\infty;f)+\Theta(\infty;g)=0,$$

which contradicts $\Theta(\infty;f)+\Theta(\infty;g) > 0$. Thus $\overline{N}(r,0;f)+\overline{N}(r,0;g) \ne S(r,f)+S(r,g)$. This together with Lemma 2.19 we see that 0 and ∞ are not Picard exceptional values of f and g. Then by Lemma 2.5 and Lemma 2.7 we get $\Phi \not\equiv 0$ and $V \not\equiv 0$. Hence from Lemmas 2.6, 2.8, 2.11, 2.17 and 2.19 with $k=\infty$, $p=0$ and $m=6$ we obtain

$$\begin{aligned}&\left(\frac{n}{2}-1\right)\{T(r,f)+T(r,g)\}\\ &\le 3\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)-\frac{3}{2}\overline{N}_{\otimes}(r,1;F,G)+S(r,f)+S(r,g)\\ &\le \frac{3}{n-2}\overline{N}_{\otimes}(r,1;F,G)+2\overline{N}(r,\infty;f)-\frac{3}{2}\overline{N}_{\otimes}(r,1;F,G)+S(r,f)+S(r,g)\\ &\le \frac{2}{n-1}\left[T(r,f)+T(r,g)+\overline{N}(r,0;f)+\overline{N}_{\otimes}(r,1;F,G)\right]+S(r,f)+S(r,g)\\ &\le \frac{2}{n-1}\left[T(r,f)+T(r,g)+\frac{4}{3}\overline{N}(r,0;f)+\frac{1}{3}\overline{N}(r,\infty;f)\right]+S(r,f)+S(r,g)\\ &\le \frac{2}{n-1}\left[\frac{7}{6}+\frac{4}{3(6n-14)}\right]\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \qquad (3.5)\end{aligned}$$

It is easy to verify that (3.5) gives a contradiction for $n \ge 4$.
Case 2. Let $H \equiv 0$. Now the theorem follows from Lemma 2.24. □

Proof of Theorem 1.7. Proceeding as in the proof of Theorem 1.6 we get the conclusion of Theorem 1.7. □

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