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PICARD BOUNDARY VALUE PROBLEMS FOR SECOND ORDER NONLINEAR FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. Sufficient conditions for the existence of solutions of the Picard boundary value problem for the second order nonlinear integro-differential equation are established. We allow G to grow linearly, superlinearly or sublinearly in our obtained results, see Theorem 2.1 and Theorem 2.2. Examples are presented to illustrate the efficiency of our theorems.

1. Introduction

Recently, many papers have discussed the existence of solutions or positive solutions of two-pint boundary value problems for second order differential equations, one may see the text books [1,2] and references therein.

In this paper, we study the solvability of Picard boundary value problems for second order of functional integro-differential equations

$$\begin{cases} x''(t) + lx'(t) + kx(t) + G(t, \int_0^\pi h(t,s)x(s)ds, x(t), x(t-\tau_1(t)), \\ \qquad\qquad \ldots, x(t-\tau_m(t))) = p(t), \quad t \in (0,\pi), \\ x(0) = x(\pi) = 0, \\ x(t) = \phi(t), \quad t \in [-\tau, 0], \\ x(t) = \psi(t), \quad t \in [\pi, \pi+\delta], \end{cases} \tag{1}$$

where $l, k \in R$, $G : [0,\pi] \times R^{m+2} \to R$ is a continuous function, $p \in C^0[0,\pi]$, $\tau_i : [0,\pi] \to R$ is continuous differentiable on $[0,\pi]$, and $\tau_i'(t) < 1$ for all $t \in [0,\pi]$ with its inverse function being denoted μ_i for $i = 1, \cdots, m$, $h(\cdot,\cdot) : [0,\pi] \times [0,\pi] \to R^+$ is continuous, $\psi : [\pi, \pi+\delta] \to R$ and $\phi : [-\tau, 0] \to R$ are

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continuous functions with $\phi(0) = \psi(\pi) = 0$, τ and δ are defined by

$$\tau = \max\{\max_{t\in[0,\pi]}\{\tau_i(t)\} : \quad i = 1, \cdots, m\},$$
$$\delta = -\min\{\min_{t\in[0,\pi]}\{\tau_i(t)\} : \quad i = 1, \cdots, m\}.$$

This study is similar to the recent papers in which solvability of Picard boundary value problems for second order differential equations are studied. First, we find that BVP

$$\begin{cases} x''(t) + lx'(t) + x(t) = \sin t, & t \in (0,\pi), \\ x(0) = x(\pi) = 0 \end{cases} \tag{2}$$

has no solution, see Remark 2 in Section 3. BVP(2) is a special case of BVP(1) when $k = 1$, $G(t, x, y) \equiv 0$ and $p(t) = \sin t$.

In [6], Kuo studied the boundary value problem

$$\begin{cases} u''(t) + k^2u + g(x,u) = h(x), & x \in (0,\pi), \\ u(0) = u(\pi) = 0 \end{cases} \tag{3}$$

under the assumption:

(H$_1$) there is a constant $r_0 > 0$ and nonnegative functions p and b such that $||p||_{L^1} < 2k(k+1)\tan\frac{\pi}{2(k+1)}$, and for a.e $x \in (0,\pi)$ and $|u| \geq r_0$ $|g(x,u)| \leq p(x)|u| + b(x)$, holds, where $h \in L^1(0,\pi)$ and $g : (0,\pi)\times R \to R$ is a Caratheodory function (Condition (H_1) admits g to grow linearly about u.)

In [8,22], BVP(3) was also studied under the assumptions that (H_1) holds and $k = 1$ and $\int_0^\pi h(x)\sin x dx = 0$. In [7,21], BVP(3) was studied under the assumptions that $||p||_{L^1} \leq 2k$, and a Landesman-Lazer condition

$$\int_0^\pi h(x)v(x) < \int_{v>0} g_+(x)v(x)dx + \int_{v<0} g_-(x)v(x)dx$$

holds, where

$$g_+(x) = \lim_{u\to+\infty}\inf g(x,u), g_-(x) = \lim_{u\to-\infty}\sup g(x,u)$$

and

$$v(x) = \alpha \sin kx \text{ for } \alpha \in R \setminus \{0\}.$$

In [20], Han studied BVP(3) when $k = 1$ under the following assumptions (G) and one of (G_1) and (G_2) (Condition (G) allows g to grow linearly:

(G) $\lim_{|u|\to\infty}\frac{g(x,u)}{u} = \gamma(x)$, where $\gamma \in L^1(0,\pi)$ and the convergence is uniform for a.e. $x \in (0,\pi)$;

(G$_1$) for every $a(x) \in L^1(0,\pi)$ with $-1 \leq a(x) \leq \gamma(x)$, BVP

$$u''(x) + u(x) + a(x)u(x) = 0, \ \ x \in (0,\pi), \quad u(0) = u(\pi) = 0$$

has only trivial solution in $H_0^1(0,\pi)$;

$(\boldsymbol{G_2})$ BVP

$$u''(x) + u(x) + a(x)u(x) = 0, \quad x \in (0, \pi), \quad u(0) = u(\pi) = 0$$

has no sign-changing solution for all $a(x) \in L^1(0, \pi)$ satisfying $0 \le a(x) \le \gamma(x)$.

In [10], the authors investigated BVP(3) assuming that $k = 1$ and

$(\boldsymbol{H_2})$ there are constants $a, \lambda, \mu \ge 0$, $\lambda\mu < 1$, $b \in L^1(0, \pi)$ such that for a.e. $x \in (0, \pi)$ and all $u \ge 0$ $g(x, u) \le a|u|^\lambda + b(x)$ holds and for a.e. $x \in (0, \pi)$ and all $u \le 0$ $g(x, u) \ge -a|u|^\mu - b(x)$ holds and other assumptions ((H_2) allows g to grow superlinearly in u in one of the directions $u \to \infty$ and $u \to -\infty$, and grow sublinearly in the other.)

Concerning the solvability of BVP(3), based upon above mentioned papers, the assumptions imposed on g are either Landesman-Lazer conditions or growth conditions, there has been no paper concerned with the solvability of BVP(1) when G grows superlinearly or sublinearly.

In [9], the solvability of BVP

$$\begin{cases} x'''(t) + k^2 x'(t) + g(x, x') = p(t), & t \in (0, \pi), \\ x'(0) = x'(\pi) = x(\eta) = 0 \end{cases} \tag{4}$$

was studied, where $0 < \eta < \pi$, g is bounded and continuous. BVP(4) when $k = 1$ was also studied in [11] by Nagle and Pothoven under the condition that g is bounded on one side.

In [12], Gupta studied the existence of solutions to boundary value problems similar to (4) of the type

$$\begin{cases} x'''(t) + \pi^2 x'(t) + g(t, x, x', x'') = p(t), & t \in (0, \pi), \\ x'(0) = x'(1) = x(\eta) = 0, \ 0 \le \eta \le 1 \end{cases} \tag{5}$$

under the conditions that

$$\int_0^1 p(t) \sin \pi t dt = 0,$$

$$g(t, u, v, w)v \ge 0, \quad \text{for } t \in [0, 1], \ u, v, w \in R,$$

and

$$\lim_{|v| \to \infty} \frac{g(t, u, v, w)}{v} < 3\pi^2 \text{ uniformly for } (t, u, v) \in [0, 1] \times R^2.$$

Take the transformation $x(t) = \int_\eta^t y(s) ds$, BVP(4) is equivalent to

$$\begin{cases} y''(t) + k^2 y(t) + g\left(\int_\eta^t y(r) dr, y(t)\right) = p(t), & t \in (0, \pi), \\ y(0) = y(\pi), \end{cases} \tag{6}$$

which is a special case of BVP(1).

Boundary value problem for the fourth-order differential equation

$$\begin{cases} y''''(t) = f\left(t, y(t), y''(t)\right), & t \in (0, \pi), \\ y(0) = y(\pi) = y''(0) = y''(\pi) = 0 \end{cases} \tag{7}$$

has extensively applications and was studied by many authors, see [1,2,13-16] and references therein. Let $y''(t) = x(t)$, then it becomes

$$\begin{cases} x''(t) = f\left(t, \int_0^\pi G(t,s)x(s)ds, x(t)\right), & t \in (0, \pi), \\ x(0) = x(\pi) = 0, \end{cases} \tag{$*$}$$

which is a special case of BVP(1), where $G(t,s)$ is Green's function for the problem $y''(t) = 0, y(0) = y(\pi) = 0$. Obtaining solutions of problem (7) is equivalent to obtaining solutions of the boundary value problem $(*)$ for second order integro-differential equations.

The present work is also motivated by the fact that a boundary value problem models various dynamical systems, see [13]. BVP(1) describes a vast spectrum of nonlinear phenomena such as gas diffusion through porous media, nonlinear diffusion generated by nonlinear sources, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, adiabatic tubular reactor processes, as well as concentration in chemical or biological problems; see [13,18,19] and references therein for examples. It is important to study the solvability of BVP(1).

The purpose of this paper is to establish existence results for solutions of BVP(1). We allow G to grow linearly, superlinearly or sublinearly, which is weaker or different from conditions in known results. To compare with the conditions of known theorems mentioned above, i.e. (H_3) and (H_4) in next section to (H_1), $(G), (G_1), (G_2)$ and (H_2) in this section, the assumptions in the results in this paper are even new when $G(t, x_0, y_0, x_1, \ldots, x_m)$ becomes the special form $G(t, x_1)$, for example problem (3), $G(t, x_0, y_0)$, for example problem (6) and problem $(*)$. On the other hand, the results in this paper show us that they are un-improvable, see Remarks in Section 3.

The outline of the paper is as follows. In Section 2, we prove the main theorems in this paper. Examples and remarks will be given in Section 3.

2. Main Results

In this section, we establish existence results for solutions of BVP(1).

Lemma 2.1. [3] *Let X and Y be Banach spaces. Suppose $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero with $KerL = \{0\}$, $N : X \to Y$ is $L-$compact on each open bounded subset of X. If $0 \in \Omega \subset X$ is an open bounded subset and $Lx \neq \lambda Nx$ for all $x \in D(L) \cap \partial\Omega$ and $\lambda \in [0, 1]$, then there exists at least one $x \in \Omega$ such that $Lx = Nx$.*

Choose $X = C^0[-\tau, \pi + \delta]$, $Y = C^0[-\tau, \pi + \delta]$. Let X be endowed with the norm $||x|| = \sup_{t\in[-\tau,\pi+\delta]} |x(t|$ for $x \in X$ and $||y|| = \sup_{t\in[-\tau,\pi+\delta]} |x(t|$ for $y \in Y$, then X and Y are Banach spaces. Let

$$\mathrm{Dom}L = \left\{ x \in X : x', x'' \in C^[0, \pi] \text{ with } x(0) = x(\pi) = 0 \right\}.$$

Define the linear operator $L : \mathrm{Dom}\, L \cap X \to Y$ by

$$Lx(t) = \begin{cases} x''(t) + lx'(t) \text{ for } t \in [0, \pi], \\ x(t) \text{ for } t \in [-\tau, 0], \\ x(t) \text{ for } t \in [\pi, \pi + \delta], \end{cases} \qquad x \in \mathrm{Dom}\, L.$$

Define the nonlinear operator $N : X \to Y$ by

$$Nx(t) = \begin{cases} -kx(t) - G(t, \int_0^\pi h(t, s)x(s)ds, x(t),\ x(t - \tau_1(t)), \\ \qquad\qquad \dots, x(t - \tau_m(t))) + p(t),\ \ t \in [0, \pi], \\ \phi(t),\ \ t \in [-\tau, 0], \\ \psi(t),\ \ t \in [\pi, \pi + \delta] \end{cases}$$

for $x \in X$. It is easy to prove that

(i) $x(t)$ is a solution of BVP(1) if and only if x is a solution of the operator equation $Lx = Nx$ in $\mathrm{Dom}\, L$ and

(ii) $\mathrm{Ker}L = \{x(t) \equiv 0,\ \ t \in [-\tau, \pi + \delta]\}$;

(iii) L is a Fredholm operator of index zero, N is L-compact on any open bounded subset of X.

Suppose that $\boldsymbol{(H_3)}$ there are continuous functions g and h such that

$$G(t, x_0, y_0, x_1, \dots, x_m) = g(t, x_0, y_0, x_1, \dots, x_m) + h(t, x_0, y_0, x_1, \dots, x_m),$$

and there are numbers $\beta > 0$ and $\theta \ge 1$ such that

$$g(t, x_0, y_0, x_1, \dots, x_m)y_0 \le -\beta|y_0|^{\theta+1},$$

and there are continuous functions $h_1, h_2,\ g_i,\ q_1, q_2, p_i, e$ such that

$$|h(t, x_0, y_0, x_1, \dots, x_m)| \le h_1(t, x_0) + h_2(t, y_0) + \sum_{i=1}^{m} g_i(t, x_i) + e(t)$$

with

$$\lim_{x\to\infty} \frac{|h_1(t, x)|}{|x|^\theta} = q_1(t) \text{ uniformly for } t \in [0, \pi],$$

$$\lim_{x\to\infty} \frac{|h_2(t, x)|}{|x|^\theta} = q_2(t) \text{ uniformly for } t \in [0, \pi],$$

and

$$\lim_{x\to\infty} \frac{|g_i(t, x)|}{|x|^\theta} = p_i(t) \text{ uniformly for } t \in [0, \pi],\ \ i = 1, \dots, m;$$

(H_4) there are continuous functions h_1, h_2, g_i, and functions $q_1, q_2, p_i, e \in L^1(0,\pi)$, and number $0<\theta<1$ such that

$$|G(t,x_0,y_0,x_1,\dots,x_m)| \le h_1(t,x_0)+h_2(t,y_0)+\sum_{i=1}^m g_i(t,x_i)+e(t)$$

and

$$\lim_{x\to\infty}\frac{|h_1(t,x)|}{|x|^\theta}=q_1(t) \text{ uniformly for } t\in[0,\pi],$$

$$\lim_{x\to\infty}\frac{|h_2(t,x)|}{|x|^\theta}=q_2(t) \text{ uniformly for } t\in[0,\pi],$$

and

$$\lim_{x\to\infty}\frac{|g_i(t,x)|}{|x|^\theta}=p_i(t) \text{ uniformly for } t\in[0,\pi],\quad i=1,\dots,m$$

hold.

Theorem 2.1. *Let* $\lambda_i=\max_{t\in[0,\pi]}\left|\frac{1}{1-\tau_i'(\mu_i(t))}\right|$, $i=1,\dots,m$. *Suppose that* (H_3) *holds. Then BVP(1) has at least one solution provided*

$$k+||q_1||\max_{(t,s)\in[0,\pi]^2}h(t,s)\pi+||q_2||+\sum_{i=1}^m||p_i||\lambda_i<\beta \text{ if } \theta=1, \tag{8}$$

$$||q_1||\max_{(t,s)\in[0,\pi]^2}h(t,s)\pi^\theta+||q_2||+\sum_{i=1}^m||p_i||\lambda_i^\theta<\beta \text{ if } \theta>1. \tag{9}$$

Proof. To apply Lemma 2.1, we should define an open bounded subset Ω of X such that conditions Lemma 2.1 hold. Let

$$\Omega_1=\{x\in \operatorname{Dom} L,\ Lx=\lambda Nx \text{ for some } \lambda\in(0,1)\}.$$

We prove Ω_1 is bounded. For $y\in\Omega_1$, we get

$$y''(t)+ly'(t)=\lambda\Big[-ky(t)-G\Big(t,\int_0^\pi h(t,s)y(s)ds, y(t), y(t-\tau_1(t)), \dots, y(t-\tau_m(t))\Big)+p(t)\Big] \tag{10}$$

and

$$y(t)=\lambda\phi(t), t\in[-\tau,0],$$
$$y(t)=\lambda\psi(t), t\in[\pi,\pi+\delta].$$

Hence

$$\begin{aligned}\int_0^\pi [y''(t) + ly'(t)]y(t)dt \\ = \lambda \int_0^\pi \Big[- ky(t) - G\Big(t, \int_0^\pi h(t,s)y(s)ds, y(t), y(t-\tau_1(t)), \\ \dots, y(t-\tau_m(t))\Big) + p(t)\Big] y(t)dt.\end{aligned}$$

Since $y(0) = y(\pi) = 0$, integrating (10) from 0 to π, we get

$$\begin{aligned}&-\int_0^\pi [y'(t)]^2 dt = -k\lambda \int_0^\pi y^2(t)dt + \lambda \int_0^\pi p(t)y(t)dt \\ &-\lambda \int_0^\pi G\Big(t, \int_0^\pi h(t,s)y(s)ds, y(t), y(t-\tau_1(t)), \dots, y(t-\tau_m(t))\Big) y(t)dt.\end{aligned}$$

So

$$\begin{aligned}&k\int_0^\pi y^2(t)dt - \int_0^\pi p(t)y(t)dt \\ &+\int_0^\pi G\Big(t, \int_0^\pi h(t,s)y(s)ds, y(t), y(t-\tau_1(t)), \dots, y(t-\tau_m(t))\Big) y(t)dt \ge 0.\end{aligned}$$

We will prove that there is a constant $B > 0$ such that $||x|| \le B$. We divide this into two steps.

Step 1. Prove that there are constants $M > 0$ such that $\int_0^\pi |y(t)|^{\theta+1} dt \le M$. It follows from (H_3) that

$$\begin{aligned}&k\int_0^\pi y^2(t)dt \\ &+\int_0^\pi g\Big(t, \int_0^\pi h(t,s)y(s)ds, y(t), y(t-\tau_1(t)), \dots, y(t-\tau_m(t))\Big) y(t)dt \\ &+\int_0^\pi h\left(t, \int_0^\pi h(t,s)y(s)ds, y(t), y(t-\tau_1(t)), \dots, y(t-\tau_m(t))\right) y(t)dt \\ &\qquad - \int_0^\pi p(t)y(t)dt \ge 0.\end{aligned}$$

Hence (H_3) implies that

$$\beta\int_0^\pi |y(t)|^{\theta+1}dt \le k\int_0^\pi y^2(t)dt$$
$$+\int_0^\pi h\left(t,\int_0^\pi h(t,s)y(s)ds, y(t), y(t-\tau_1(t)),\dots,y(t-\tau_m(t))\right)y(t)dt$$
$$-\int_0^\pi p(t)y(t)dt \le k\int_0^\pi y^2(t)dt$$
$$+\int_0^\pi \left|h\left(t,\int_0^\pi h(t,s)y(s)ds, y(t), y(t-\tau_1(t)),\dots,y(t-\tau_m(t))\right)\right||y(t)|dt$$
$$+\int_0^\pi |p(t)||y(t)|dt \le k\int_0^\pi y^2(t)dt$$
$$+\int_0^\pi \left|h_1\left(t,\int_0^\pi h(t,s)y(s)ds\right)\right||y(t)|dt+\int_0^\pi |h_2(t,y(t))||y(t)|dt$$
$$+\sum_{i=1}^m\int_0^\pi |g_i(t,y(t-\tau_i(t))|y(t)|dt+\int_0^\pi(|e(t)|+|p(t)|)|y(t)|dt.$$

From (8) and (9), choosing $\epsilon>0$ so that

$$\beta-k-(||q_1||+\epsilon)\max_{(t,s)\in[0,\pi]^2}h(t,s)\pi-(||q_2||+\epsilon)-\sum_{i=1}^m(||p_i||_\infty+\epsilon)\lambda_i>0 \text{ for } \theta=1, \tag{11}$$

and

$$\beta-(||q_1||+\epsilon)\max_{(t,s)\in[0,\pi]^2}h(t,s)\pi^\theta-(||q_2||+\epsilon)-\sum_{i=0}^m(||p_i||_\infty+\epsilon)\lambda_i^\theta>0 \text{ for } \theta>1. \tag{12}$$

For such ϵ, there is $\delta>0$ so that

$$|g_i(t,x)|\le(p_i(t)+\epsilon)|x|^\theta \quad \text{for } |x|\ge\delta,\ t\in[0,\pi],\quad i=1,\cdots,m$$

and

$$h_1(t,x)\le(q_1(t)+\epsilon)|x|^\theta \quad \text{for } |x|\ge\delta,\ t\in[0,\pi],$$
$$h_2(t,x)\le(q_2(t)+\epsilon)|x|^\theta \quad \text{for } |x|\ge\delta,\ t\in[0,\pi].$$

Denote,

$$g_{\delta,i}=\max_{\substack{t\in[0,\pi]\\|y|\le\delta}}|g_i(t,y)|,$$

and

$$h_{\delta,1}=\max_{\substack{t\in[0,\pi]\\|y|\le\delta}}|h_1(t,y)|,\qquad h_{\delta,2}=\max_{\substack{t\in[0,\pi]\\|y|\le\delta}}|h_2(t,y)|,$$

and for $i = 1, \dots, m$ and

$$\Delta_{1,i} = \{t : \ t \in [0, \pi], \ |y(t - \tau_i(t))| \le \delta\},$$
$$\Delta_{2,i} = \{t : t \in [0, \pi], \ |y(t - \tau_i(t))| > \delta\},$$

and

$$\Delta_1' = \{t \in [0, \pi], \left| \int_0^\pi h(t,s)y(s)ds \right| \le \delta\},$$
$$\Delta_2' = \{t \in [0, \pi], \left| \int_0^\pi h(t,s)y(s)ds \right| > \delta\},$$
$$\Delta_1 = \{t \in [0, \pi], |y(t)| \le \delta\}, \quad \Delta_2 = \{t \in [0, \pi], |y(t)| > \delta\}.$$

Hence

$$\begin{aligned}
\beta \int_0^\omega |y(s)|^{\theta+1} ds &\le k \int_0^\pi y^2(t)dt \\
&+ \int_{\Delta_1'} \left| h_1\left(t, \int_0^\pi h(t,s)y(s)ds\right)\right| |y(t)|dt + \int_{\Delta_1} |h_2(t, y(t))||y(t)|dt \\
&+ \sum_{i=1}^m \int_{\Delta_{1,i}} |g_i(t, y(t - \tau_i(t))||y(t)|dt + \int_0^\pi (|e(t)| + |p(t)|)|y(t)|dt \\
&+ \sum_{i=1}^m \int_{\Delta_{2,i}} |g_i(t, y(t - \tau_i(t))||y(t)|dt + \int_{\Delta_2} |h_2(t, y(t))||y(t)|dt \\
&+ \int_{\Delta_2'} \left| h_1\left(t, \int_0^\pi h(t,s)y(s)ds\right)\right| |y(t)|dt \\
&\le k \int_0^\pi y^2(t)dt + \int_{\Delta_2'} \left| h_1\left(t, \int_0^\pi h(t,s)y(s)ds\right)\right| |y(t)|dt \\
+ \int_{\Delta_2} |h_2(t, y(t))||y(t)|dt &+ \sum_{i=1}^m \int_{\Delta_{2,i}} |g_i(t, y(t - \tau_i(t))|y(t)|dt + h_{\delta,1} \int_{\Delta_1'} |y(t)|dt \\
+ h_{\delta,2} \int_{\Delta_1} |y(t)|dt &+ \sum_{i=1}^m g_{\delta,i} \int_{\Delta_{1,i}} |y(t)|dt + (||e|| + ||p||) \int_0^\pi |y(t)|dt \le k \int_0^\pi y^2(t)dt \\
&+ \int_{\Delta_2'} \left| h_1\left(t, \int_0^\pi h(t,s)y(s)ds\right)\right| |y(t)|dt + \int_{\Delta_2} |h_2(t, y(t))||y(t)|dt \\
&+ \sum_{i=1}^m \int_{\Delta_{2,i}} |g_i(t, y(t - \tau_i(t))|y(t)|dt + \left(h_{\delta,2} + h_{\delta,1} + \sum_{i=1}^m g_{\delta,i} + ||e|| + ||p||\right) \\
\int_0^\pi |y(t)|dt &\le k \int_0^\pi y^2(t)dt + \int_{\Delta_2'} (q_1(t) + \epsilon) \left(\int_0^\pi h(t,s)|y(s)|ds\right)^\theta |y(t)|dt
\end{aligned}$$

$$+\int_{\Delta_2}(q_2(t)+\epsilon)|y(t)|^{\theta+1}dt+\sum_{i=1}^m\int_{\Delta_{1,i}}(p_i(t)+\epsilon)|y(t-\tau_i(t))|^\theta|y(t)|dt$$

$$+\left(h_{\delta,2}+h_{\delta,1}+\sum_{i=1}^m g_{\delta,i}+||e||+||p||\right)\int_0^\pi|y(t)|dt\le k\int_0^\pi y^2(t)dt+(||q_1||+\epsilon)$$

$$\max_{(t,s)\in[0,\pi]^2}h(t,s)\pi^\theta\int_0^\pi|y(s)|^{\theta+1}ds+(||q_2||+\epsilon)\int_0^\pi|y(t)|^{\theta+1}dt$$

$$+\sum_{i=1}^m(||p_i||_\infty+\epsilon)\left(\int_0^\pi|y(s-\tau_i(s))|^{\theta+1}ds\right)^{\frac{\theta}{\theta+1}}\left(\int_0^\pi||x(s)|^{\theta+1}ds\right)^{\frac{\theta}{\theta+1}}$$

$$+\left(h_{\delta,2}+h_{\delta,1}+\sum_{i=1}^m g_{\delta,i}+||e||+||p||\right)\int_0^\pi|y(t)|dt.$$

Since $\tau_i'(t)<1$, and μ_i is the inverse function of τ_i, we see

$$\int_0^\pi|y(t-\tau_i(t))|^{\theta+1}dt=\int_{s\in\{t-\tau_i(t),\ t\in[0,\pi]\sqcap[0,\pi]}\left(\frac{|y(s)|}{1-\tau_i'(\mu_i(s))}\right)^{\theta+1}ds$$

$$+\int_{s\in\{t-\tau_i(t),\ t\in[0,\pi]\sqcap[\pi,\pi+\delta]}\left(\frac{|y(s)|}{1-\tau_i'(\mu_i(s))}\right)^{\theta+1}ds$$

$$+\int_{s\in\{t-\tau_i(t),\ t\in[0,\pi]\sqcap[-\tau,0]}\left(\frac{|y(s)|}{1-\tau_i'(\mu_i(s))}\right)^{\theta+1}ds$$

$$\le\lambda_i^{\theta+1}\int_0^\pi|y(s)|^{\theta+1}ds+\int_\pi^{\pi+\delta}\left(\frac{|\psi(s)|}{1-\tau_i'(\mu_i(s))}\right)^{\theta+1}ds$$

$$+\int_{-\tau}^0\left(\frac{\phi(s)|}{1-\tau_i'(\mu_i(s))}\right)^{\theta+1}ds=:\lambda_i^{\theta+1}\int_0^\pi|x(s)|^{\theta+1}ds+Q_i.$$

Hence

$$\beta\int_0^1|x(s)|^{\theta+1}ds\le k\int_0^\pi y^2(t)dt+(||q_1||+\epsilon)$$

$$\cdot\max_{(t,s)\in[0,\pi]^2}h(t,s)\pi^\theta\int_0^\pi|y(s)|^{\theta+1}ds+(||q_2||+\epsilon)\int_0^\pi|y(t)|^{\theta+1}dt$$

$$+\sum_{i=1}^m(||p_i||_\infty+\epsilon)\left(\lambda_i^{\theta+1}\int_0^\pi|y(s)|^{\theta+1}ds+Q_i\right)^{\frac{\theta}{\theta+1}}\left(\int_0^\pi||x(s)|^{\theta+1}ds\right)^{\frac{\theta}{\theta+1}}$$

$$+\left(h_{\delta,2}+h_{\delta,1}+\sum_{i=1}^m g_{\delta,i}+||e||+||p||\right)\pi^\theta\left(\int_0^\pi|y(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}.$$

Since $\lim_{x\to 0^+} \frac{(1+x)^y-1}{(1+y)x} = \frac{y}{1+y} < 1$ for $y > 0$, then there is $\sigma > 0$ such that $(1+x)^y \le 1 + (1+y)x$ for $0 \le x \le \sigma$. Let

$$I_1 = \left\{ i \in \{1, \ldots, m\} : \int_0^\pi |y(t)|^{\theta+1} dt \le \frac{Q_i}{\sigma \lambda_i^{\theta+1}} \right\},$$

$$I_2 = \left\{ i \in \{1, \ldots, m\} : \int_0^\pi |y(t)|^{\theta+1} dt > \frac{Q_i}{\sigma \lambda_i^{\theta+1}} \right\}.$$

Then

$$\begin{aligned}
\beta \int_0^1 |x(s)|^{\theta+1} ds &\le k \int_0^\pi y^2(t)dt + (||q_1|| + \epsilon) \\
&\cdot \max_{(t,s)\in[0,\pi]^2} h(t,s)\pi^\theta \int_0^\pi |y(s)|^{\theta+1} ds + (||q_2|| + \epsilon) \int_0^\pi |y(t)|^{\theta+1} dt \\
&+ \sum_{i\in I_1} (||p_i||_\infty + \epsilon) \left(\frac{Q_i}{\sigma} + Q_i \right)^{\frac{\theta}{\theta+1}} \left(\int_0^\pi |y(s)|^{\theta+1} ds \right)^{\frac{\theta}{\theta+1}} \\
&+ \sum_{i\in I_2} \lambda_i^\theta (||p_i||_\infty + \epsilon) \left(1 + \frac{Q_i}{\lambda_i^{\theta+1} \int_0^\pi |y(t)|^{\theta+1} dt} \right)^{\frac{\theta}{\theta+1}} \int_0^\pi |y(s)|^{\theta+1} ds \\
&+ \left(h_{\delta,2} + h_{\delta,1} + \sum_{i=1}^m g_{\delta,i} + ||e|| + ||p|| \right) \pi^\theta \left(\int_0^\pi |y(t)|^{\theta+1} dt \right)^{\frac{1}{\theta+1}} \\
&\le k \int_0^\pi y^2(t)dt + (||q_1|| + \epsilon) \\
&\cdot \max_{(t,s)\in[0,\pi]^2} h(t,s)\pi^\theta \int_0^\pi |y(s)|^{\theta+1} ds + (||q_2|| + \epsilon) \int_0^\pi |y(t)|^{\theta+1} dt \\
&+ \sum_{i\in I_1} (||p_i||_\infty + \epsilon) \left(\frac{Q_i}{\sigma} + Q_i \right)^{\frac{\theta}{\theta+1}} \left(\int_0^\pi |y(s)|^{\theta+1} ds \right)^{\frac{\theta}{\theta+1}} \\
&+ \sum_{i\in I_2} \lambda_i^\theta (||p_i||_\infty + \epsilon) \left[1 + \left(1 + \frac{\theta}{\theta+1} \right) \frac{Q_i}{\lambda_i^{\theta+1} \int_0^\pi |y(t)|^{\theta+1} dt} \right] \int_0^\pi |y(s)|^{\theta+1} ds \\
&+ \left(h_{\delta,2} + h_{\delta,1} + \sum_{i=1}^m g_{\delta,i} + ||e|| + ||p|| \right) \pi^\theta \left(\int_0^\pi |y(t)|^{\theta+1} dt \right)^{\frac{1}{\theta+1}} \\
&= k \int_0^\pi y^2(t)dt + (||q_1|| + \epsilon) \\
&\cdot \max_{(t,s)\in[0,\pi]^2} h(t,s)\pi^\theta \int_0^\pi |y(s)|^{\theta+1} ds + (||q_2|| + \epsilon) \int_0^\pi |y(t)|^{\theta+1} dt
\end{aligned}$$

$$+\sum_{i\in I_1}(||p_i||_\infty+\epsilon)\left(\frac{Q_i}{\sigma}+Q_i\right)^{\frac{\theta}{\theta+1}}\left(\int_0^\pi|y(s)|^{\theta+1}ds\right)^{\frac{\theta}{\theta+1}}$$
$$+\sum_{i\in I_2}\lambda_i^\theta(||p_i||_\infty+\epsilon)\int_0^\pi|y(s)|^{\theta+1}ds+\sum_{i\in I_2}\lambda_i^\theta(||p_i||_\infty+\epsilon)\left(1+\frac{\theta}{\theta+1}\right)\frac{Q_i}{\lambda_i^{\theta+1}}$$
$$+\left(h_{\delta,2}+h_{\delta,1}+\sum_{i=1}^m g_{\delta,i}+||e||+||p||\right)\pi^\theta\left(\int_0^\pi|y(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}.$$

If $\theta=1$, then

$$\left(\beta-k-(||q_1||+\epsilon)\max_{(t,s)\in[0,\pi]^2}h(t,s)\pi-(||q_2||+\epsilon)-\sum_{i\in I_2}(||p_i||_\infty+\epsilon)\lambda_i\right)$$
$$\cdot\int_0^1|y(s)|^2ds\le\sum_{i\in I_1}(||p_i||_\infty+\epsilon)\left(\frac{Q_i}{\sigma}+Q_i\right)^{\frac{1}{2}}\left(\int_0^\pi||x(s)|^2ds\right)^{\frac{1}{2}}$$
$$+\sum_{i\in I_2}\lambda_i^\theta(||p_i||_\infty+\epsilon)\left(1+\frac{1}{2}\right)\frac{Q_i}{\lambda_i^2}$$
$$+\left(h_{\delta,2}+h_{\delta,1}+\sum_{i=1}^m g_{\delta,i}+||e||+||p||\right)\pi\left(\int_0^\pi|y(t)|^2dt\right)^{\frac{1}{2}}.$$

It follows from (11) that there is an $M_1>0$ such that $\int_0^\pi|y(t)|^2dt\le M_1$.

If $\theta>1$, then

$$\beta\int_0^1|x(s)|^{\theta+1}ds\le|k|\pi^{\frac{\theta-1}{\theta+1}}\left(\int_0^\pi|y(t)|^{\theta+1}dt\right)^{\frac{2}{\theta+1}}$$
$$+(||q_1||+\epsilon)\max_{(t,s)\in[0,\pi]^2}h(t,s)\pi^\theta\int_0^\pi|y(s)|^{\theta+1}ds+(||q_2||+\epsilon)\int_0^\pi|y(t)|^{\theta+1}dt$$
$$+\sum_{i\in I_1}(||p_i||_\infty+\epsilon)\left(\frac{Q_i}{\sigma}+Q_i\right)^{\frac{\theta}{\theta+1}}\left(\int_0^\pi||x(s)|^{\theta+1}ds\right)^{\frac{\theta}{\theta+1}}$$
$$+\sum_{i\in I_2}\lambda_i^\theta(||p_i||_\infty+\epsilon)\int_0^\pi||x(s)|^{\theta+1}ds+\sum_{i\in I_2}\lambda_i^\theta(||p_i||_\infty+\epsilon)\left(1+\frac{\theta}{\theta+1}\right)\frac{Q_i}{\lambda_i^{\theta+1}}$$
$$+\left(h_{\delta,2}+h_{\delta,1}+\sum_{i=1}^m g_{\delta,i}+||e||+||p||\right)\pi^\theta\left(\int_0^\pi|y(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}.$$

Then

$$\left(\beta - (||q_1|| + \epsilon) \max_{(t,s)\in[0,\pi]^2} h(t,s)\pi^\theta - (||q_2|| + \epsilon) - \sum_{i\in I_2}(||p_i||_\infty + \epsilon)\lambda_i^\theta\right)$$

$$\cdot \int_0^1 |y(s)|^{\theta+1} ds \leq \sum_{i\in I_1}(||p_i||_\infty + \epsilon)\left(\frac{Q_i}{\sigma} + Q_i\right)^{\frac{\theta}{\theta+1}} \left(\int_0^\pi ||x(s)|^{\theta+1} ds\right)^{\frac{\theta}{\theta+1}}$$

$$+ |k|\pi^{\frac{\theta-1}{\theta+1}} \left(\int_0^\pi |y(t)|^{\theta+1} dt\right)^{\frac{2}{\theta+1}} + \sum_{i\in I_2} \lambda_i^\theta (||p_i||_\infty + \epsilon)\left(1 + \frac{\theta}{\theta+1}\right)\frac{Q_i}{\lambda_i^{\theta+1}}$$

$$+ \left(h_{\delta,2} + h_{\delta,1} + \sum_{i=1}^m g_{\delta,i} + ||e|| + ||p||\right)\pi^\theta \left(\int_0^\pi |y(t)|^{\theta+1} dt\right)^{\frac{1}{\theta+1}}.$$

It follows from (12) that there is an $M_2 > 0$ such that $\int_0^\pi |y(t)|^{\theta+1} dt \leq M_2$.
Hence above discussions imply that there is $M > 0$ such that

$$\int_0^\pi |y(s)|^{\theta+1} ds \leq M = \max\{M_1, M_2\}.$$

Step 2. Prove that there is a constant $B > 0$ so that $||x|| \leq B$.
Since

$$\int_0^\pi [y'(t)]^2 dt = \lambda k \int_0^\pi y^2(t) dt - \lambda \int_0^\pi p(t)y(t) dt$$

$$+ \lambda \int_0^\pi G\left(t, \int_0^1 h(t,s)y(s)ds, y(t), y(t-\tau_1(t)), \ldots, y(t-\tau_m(t))\right) y(t) dt$$

$$\leq \lambda k \int_0^\pi y^2(t) dt - \lambda \int_0^\pi p(t)y(t) dt - \lambda\beta \int_0^\pi |y(t)|^{\theta+1} dt$$

$$+ \lambda \int_0^\pi g\left(t, \int_0^1 h(t,s)y(s)ds, y(t), y(t-\tau_1(t)), \ldots, y(t-\tau_m(t))\right) y(t) dt$$

$$\leq k \int_0^\pi y^2(t) dt + \int_0^\pi |p(t)||y(t)| dt$$

$$\int_0^\pi \left| g\left(t, \int_0^1 h(t,s)y(s)ds, y(t), y(t-\tau_1(t)), \ldots, y(t-\tau_m(t))\right)\right| |y(t)| dt$$

$$\leq k \int_0^\pi y^2(t) dt + \int_0^\pi \left| h_1\left(t, \int_0^\pi h(t,s)y(s)ds\right)\right| |y(t)| dt$$

$$+ \int_0^\pi |h_2(t,y(t))||y(t)| dt + \sum_{i=1}^m \int_0^\pi |g_i(t, y(t-\tau_i(t))|y(t)| dt + \int_0^\pi |e(t)||y(t)| dt.$$

Similar to the proof of that of Step 1, we get

$$\begin{aligned}\int_0^\pi [y'(t)]^2 dt \le k\int_0^\pi y^2(t)dt + (||q_1|| + \epsilon)\max_{(t,s)\in[0,\pi]^2} h(t,s)\pi^\theta \int_0^\pi |y(s)|^{\theta+1} ds \\ + (||q_2|| + \epsilon)\int_0^\pi |y(t)|^{\theta+1} dt \\ + \sum_{i\in I_1} (||p_i||_\infty + \epsilon)\left(\frac{Q_i}{\sigma} + Q_i\right)^{\frac{\theta}{\theta+1}} \left(\int_0^\pi |y(s)|^{\theta+1} ds\right)^{\frac{\theta}{\theta+1}} \\ + \sum_{i\in I_2} \lambda_i^\theta (||p_i||_\infty + \epsilon)\int_0^\pi |y(s)|^{\theta+1} ds \\ + \sum_{i\in I_2} \lambda_i^\theta (||p_i||_\infty + \epsilon)\left(1 + \frac{\theta}{\theta+1}\right)\frac{Q_i}{\lambda_i^{\theta+1}} + ||e||\pi^\theta \left(\int_0^\pi |y(t)|^{\theta+1} dt\right)^{\frac{1}{\theta+1}}.\end{aligned}$$

It follows from $\int_0^\pi |y(t)|^{\theta+1} dt \le M$ that there is $M_3 > 0$ such that $\int_0^\pi [y'(t)]^2 dt \le M_3$. For each $t \in [0, \pi]$, we get

$$\begin{aligned}\frac{1}{2}[y(t)]^2 &= \int_0^t y(s)y'(s)ds = \left|\int_0^t y(s)y'(s)ds\right| \le \int_0^\pi |y(s)||y'(s)|ds \\ &\le \left(\int_0^\pi |y(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_0^\pi |y'(t)|^2 dt\right)^{\frac{1}{2}} \le \left(\int_0^\pi |y(t)|^2 dt\right)^{\frac{1}{2}} M_3^{\frac{1}{2}} \\ &\le \begin{cases} M_1^{\frac{1}{2}} M_3^{\frac{1}{2}}, & \theta = 1, \\ \left[\pi^{\frac{\theta-1}{\theta+1}} \left(\int_0^\pi |y(t)|^{\theta+1} dt\right)^{\frac{2}{\theta+1}}\right]^{\frac{1}{2}} M_3^{\frac{1}{2}}, & \theta > 1 \end{cases} \\ &\le \begin{cases} M_1^{\frac{1}{2}} M_3^{\frac{1}{2}}, & \theta = 1, \\ \left[\pi^{\frac{\theta-1}{\theta+1}} M_2^{\frac{2}{\theta+1}}\right]^{\frac{1}{2}} M_3^{\frac{1}{2}}, & \theta > 1. \end{cases}\end{aligned}$$

It follows that there is a constant $B > 0$ such that $\sup_{t\in[0,\pi]} |y(t)| \le B$. It follows that

$$||y|| \le \max\left\{B, \max_{t\in[-\tau,0]} |\phi(t)|, \max_{t\in[\pi,\pi+\delta]} |\psi(t)|\right\} \text{ for all } x \in \Omega_1.$$

Then Ω_1 is bounded.

Let $\Omega \supseteq \Omega_1$ be a bounded open subset of X centered at zero. It is easy to see that $Lx \ne \lambda Nx$ for $\lambda \in (0, 1]$ and $x \in D(L) \bigcap \partial\Omega$. It follows from Lemma 2.1 that $Lx = Nx$ has at least one solution x in Ω. Then x is a solution of BVP(1). □

Theorem 2.2. *Suppose that* (H_4) *holds. Then BVP(1) has at least one solution if* $|k| < 1$.

Proof. Similar to that of the proof of Theorem 2.1, we get (10). It follows that

$$\int_0^\pi [y'(t)]^2 dt = \lambda \left(k \int_0^\pi [y(t)]^2 dt \right.$$
$$+ \int_0^\pi G\left(t, \int_0^1 h(t,s)y(s)ds, y(t), y(t-\tau_1(t)), \dots, y(t-\tau_m(t))\right) y(t)dt$$
$$\left. - \int_0^\pi p(t)y(t)dt \right).$$

Let

$$y_1(t) = \begin{cases} y(t), & t \in [0,\pi], \\ -y(t), & t \in -\pi, 0]. \end{cases}$$

Suppose

$$y_1(t) = \sum_{n=1}^{\infty} a_n \sin nt,$$

then

$$y_1'(t) = \sum_{n=1}^{\infty} n a_n \cos nt.$$

It is easy to see that $\int_{-\pi}^{\pi} y_1(t)dt = 0$. It follows from the Parseval equality, $\int_{-\pi}^{\pi}[y(t)]^2 dt = \sum_{n=1}^{\infty} |a_n|^2$ and $\int_{-\pi}^{\pi}[y'(t)]^2 dt = \sum_{n=1}^{\infty} n^2 |a_n|^2$, that $\int_0^\pi [y(t)]^2 dt \le \int_0^\pi [y'(t)]^2 dt$, and the equality holds if and only if $y(t) = c \sin t$, see [17]. Hence

$$\int_0^\pi [y'(t)]^2 dt \le |k| \int_0^\pi [y(t)]^2 dt + \int_0^\pi \left| h_1\left(t, \int_0^1 h(t,s)y(s)ds\right)\right| |y(t)|dt$$
$$+ \int_0^\pi |h_2(t,y(t))||y(t)dt + \sum_{i=1}^m \int_0^\pi |g_i(t, y(t-\tau_i(t))||y(t)dt + \int_0^\pi |e(t)||y(t)|dt$$
$$+ ||p|| \int_0^\pi |y(t)|dt \le |k| \int_0^\pi [y'(t)]^2 dt$$
$$+ \int_0^\pi \left| h_1\left(t, \int_0^1 h(t,s)y(s)ds\right)\right| |y(t)|dt + \int_0^\pi |h_2(t,y(t))||y(t)dt$$
$$+ \sum_{i=1}^m \int_0^\pi |g_i(t, y(t-\tau_i(t))||y(t)dt + (||e|| + ||p||) \int_0^\pi |y(t)|dt.$$

Then

$$
\begin{aligned}
&(1-|k|)\int_0^\pi [y'(t)]^2dt\\
&\quad\le \int_0^\pi \left|h_1\left(t,\int_0^1 h(t,s)y(s)ds\right)\right||y(t)|dt+\int_0^\pi |h_2(t,y(t))||y(t)|dt\\
&\qquad +\sum_{i=1}^m\int_0^\pi |g_i(t,y(t-\tau_i(t))||y(t)|dt+(||e||+||p||)\int_0^\pi |y(t)|dt.
\end{aligned}
$$

Now, using the notations defined in the proof of Theorem 2.1, the methods used in the proof of Theorem 2.1, we get

$$
\begin{aligned}
&(1-|k|)\int_0^\pi [y'(t)]^2dt\\
&\le (||q_1||+\epsilon)\max_{(t,s)\in[0,\pi]^2} h(t,s)\pi^\theta\int_0^\pi |y(s)|^{\theta+1}ds+(||q_2||+\epsilon)\int_0^\pi |y(t)|^{\theta+1}dt\\
&\quad +\sum_{i\in I_1}(||p_i||_\infty+\epsilon)\left(\frac{Q_i}{\sigma}+Q_i\right)^{\frac{\theta}{\theta+1}}\left(\int_0^\pi |y(s)|^{\theta+1}ds\right)^{\frac{\theta}{\theta+1}}\\
&\quad +\sum_{i\in I_2}\lambda_i^\theta(||p_i||_\infty+\epsilon)\int_0^\pi |y(s)|^{\theta+1}ds+\sum_{i\in I_2}\lambda_i^\theta(||p_i||_\infty+\epsilon)\left(1+\frac{\theta}{\theta+1}\right)\frac{Q_i}{\lambda_i^{\theta+1}}\\
&\quad +\left(h_{\delta,2}+h_{\delta,1}+\sum_{i=1}^m g_{\delta,i}+||e||+||p||bigg)\pi^\theta\left(\int_0^\pi |y(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}\right.\\
&\le (||q_1||+\epsilon)\max_{(t,s)\in[0,\pi]^2} h(t,s)\pi^\theta\pi^{\frac{1-\theta}{2}}\left(\int_0^\pi |y(s)|^2ds\right)^{\frac{\theta+1}{2}}\\
&\quad +(||q_2||+\epsilon)\pi^{\frac{1-\theta}{2}}\left(\int_0^\pi |y(s)|^2ds\right)^{\frac{\theta+1}{2}}\\
&\quad +\sum_{i\in I_1}(||p_i||_\infty+\epsilon)\left(\frac{Q_i}{\sigma}+Q_i\right)^{\frac{\theta}{\theta+1}}\pi^{\frac{\theta(1-\theta)}{2(\theta+1)}}\left(\int_0^\pi |y(s)|^2ds\right)^{\frac{\theta}{2}}\\
&\quad +\sum_{i\in I_2}\lambda_i^\theta(||p_i||_\infty+\epsilon)\pi^{\frac{1-\theta}{2}}\left(\int_0^\pi |y(s)|^2ds\right)^{\frac{\theta+1}{2}}\\
&\quad +\sum_{i\in I_2}\lambda_i^\theta(||p_i||_\infty+\epsilon)\left(1+\frac{\theta}{\theta+1}\right)\frac{Q_i}{\lambda_i^{\theta+1}}\\
&\quad +\left(h_{\delta,2}+h_{\delta,1}+\sum_{i=1}^m g_{\delta,i}+||e||+||p||\right)\pi^\theta\pi^{\frac{1-\theta}{2(\theta+1)}}\left(\int_0^\pi |y(s)|^2ds\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\le (||q_1|| + \epsilon) \max_{(t,s)\in[0,\pi]^2} h(t,s)\pi^\theta \pi^{\frac{1-\theta}{2}} \left(\int_0^\pi |y'(s)|^2 ds\right)^{\frac{\theta+1}{2}} \\
&\quad + (||q_2|| + \epsilon)\pi^{\frac{1-\theta}{2}} \left(\int_0^\pi |y'(s)|^2 ds\right)^{\frac{\theta+1}{2}} \\
&+ \sum_{i\in I_1} (||p_i||_\infty + \epsilon) \left(\frac{Q_i}{\sigma} + Q_i\right)^{\frac{\theta}{\theta+1}} \pi^{\frac{\theta(1-\theta)}{2(\theta+1)}} \left(\int_0^\pi |y'(s)|^2 ds\right)^{\frac{\theta}{2}} \\
&\quad + \sum_{i\in I_2} \lambda_i^\theta (||p_i||_\infty + \epsilon)\pi^{\frac{1-\theta}{2}} \left(\int_0^\pi |y'(s)|^2 ds\right)^{\frac{\theta+1}{2}} \\
&\quad + \sum_{i\in I_2} \lambda_i^\theta (||p_i||_\infty + \epsilon) \left(1 + \frac{\theta}{\theta+1}\right) \frac{Q_i}{\lambda_i^{\theta+1}} \\
&+ \left(h_{\delta,2} + h_{\delta,1} + \sum_{i=1}^m g_{\delta,i} + ||e|| + ||p||\right)\pi^\theta \pi^{\frac{1-\theta}{2(\theta+1)}} \left(\int_0^\pi |y'(s)|^2 ds\right)^{\frac{1}{2}}.
\end{aligned}
$$

It follows from $|k| < 1$ that there is $M > 0$ such that $\int_0^\pi [y'(t)]^2 dt \le M$. One sees that

$$
|y(t)| = \left|\int_0^t y'(s)ds\right| \le \int_0^\pi |y'(t)|dt \le \pi^{\frac{1}{2}} \left(\int_0^\pi [y'(t)]^2 dt\right)^{\frac{1}{2}} \le \pi^{\frac{1}{2}} M^{\frac{1}{2}}.
$$

Hence there exists a constant $B > 0$ such that $\max_{t\in[0,\pi]} |y(t)| \le B$. Then

$$
||y|| = \max_{t\in[-\tau,\pi+\delta]} |y(t)| \le \max\left\{B, \max_{t\in[-\tau,0]} |\phi(t)|, \max_{t\in[\pi,\pi+\delta]} |\psi(t)|\right\}
$$

for all $x \in \Omega_1$. Then Ω_1 is bounded. The remainder of the proof is similar to that of the proof of Theorem 2.1 and is omitted. $\square$

3. Examples

In this section, we present examples of equations, which can not be solved by known theorems in [4-8,10], to illustrate the main result in Section 2.

Example 3.1. Consider the following problem for delay differential equation

$$
\begin{cases}
x''(t) + lx'(t) + kx(t) = \sum_{i=1}^{2n} a_i[x(t)]^i + a_0[x(t)]^{2n+1} \\
\qquad\qquad + \sum_{i=1}^m p_i(t) \left[x\left(\frac{2i-1}{2i}t\right)\right]^{2n+1} + p(t), \quad t \in (0,\pi), \\
x(0) = x(\pi) = 0, \\
x(t) = \phi(t), \quad t \in [-1/2, 0], \quad \phi(0) = 0,
\end{cases} \tag{13}
$$

where $n \ge 0$ an integer, $l, k \in R$, $a_i \in R$ for $i = 1, \ldots, 2n$, $a_0 < 0$, $t - \tau_i(t) = \frac{2i-1}{2i}t$, and p_i and p are continuous functions. Corresponding to BVP(1), we get

$$\begin{aligned}
-G(t,x_0,y_0,x_1,\dots,x_m) &= a_0y_0^{2n+1} + \sum_{i=1}^{2n} a_iy_0^i + \sum_{i=1}^{m} p_i(t)x_i^{2k+1} + p(t),\\
-g(t,x_0,y_0,x_1,\dots,x_m) &= a_0y_0^{2n+1},\\
-h(t,x_0,y_0,x_1,\dots,x_m) &= \sum_{i=1}^{2n} a_iy_0^i + \sum_{i=1}^{m} p_i(t)x_i^{2k+1} + p(t),\\
h_1(t,x_0) &= 0,\\
h_2(t,y_0) &= \sum_{i=1}^{2n} a_iy_0^i,\\
g_i(t,x_i) &= p_i(t)x_i^{2n+1},\\
\lambda_i &= \max_{t\in[0,\pi]}\left|\frac{1}{1-\tau_i'(\mu_i(t))}\right| = \frac{2i}{2i-1}.
\end{aligned}$$

We see that $q_2(t)\equiv 0$. It follows from Theorem 2.1 that, for each p, BVP(15) has at least one solution if $n=0$ and

$$k + \sum_{i=1}^{m}\frac{2i}{2i-1}||p_i|| < -a_0,$$

and $n>0$ and

$$\sum_{i=1}^{m}\left(\frac{2i}{2i-1}\right)^{2n+1}||p_i|| < -a_0.$$

Example 3.2. Consider the following problem for the delay differential equation

$$\begin{cases} x''(t)+lx'(t)+kx(t) = p_0(t)[x(t)]^\theta \\ \qquad\qquad + \sum_{i=1}^m p_i(t)\left[x\left(\frac{2i-1}{2i}t\right)\right]^\theta + p(t),\ \ t\in(0,\pi), \\ x(0)=x(\pi)=0, \\ x(t)=\phi(t),\ \ t\in[-1/2,0],\ \ \phi(0)=0, \end{cases} \tag{14}$$

where $l,k\in R$, $t-\tau_i(t) = \frac{2i-1}{2i}t$, and p_i are continuous functions as in problem (1). It follows from Theorem 2.2 that problem (16) has at least one solution if $\theta\in[0,1)$ and $|k|<1$.

Remark 3.1. In Example 3.1, G may grow superlinearly and linearly. In Example 3.2, G grows sublinearly.

Remark 3.2. For BVP(2), we have $k=1$. BVP(2) has no solution. In fact, if (2) has a solution x, then

$$x''(t)+lx'(t)+x(t)=\sin t,\quad t\in(0,\pi),\ \ x(0)=x(\pi)=0.$$

If $l=0$, we get

$$\int_0^\pi x''(t)\sin t\,dt+\int_0^\pi x(t)\sin t\,dt=\int_0^\pi [\sin t]^2 dt.$$

It follows that $\int_0^\pi [\sin t]^2 dt=0$, a contradiction.

If $l\neq 0$, we get

$$\int_0^\pi x''(t)x(t)dt+l\int_0^\pi x'(t)x(t)dt+\int_0^\pi [x(t)]^2 dt=\int_0^\pi x(t)\sin t\,dt,$$

$$\int_0^\pi x''(t)\cos t\,dt+l\int_0^\pi x'(t)\cos t\,dt+\int_0^\pi x(t)\cos t\,dt=\int_0^\pi \sin t\cos t\,dt.$$

It follows that

$$-\int_0^\pi [x'(t)]^2 dt+\int_0^\pi [x(t)]^2 dt$$
$$=\int_0^\pi x(t)\sin t\,dt,\quad l\int_0^\pi x(t)\sin t\,dt=\int_0^\pi \sin t\cos t\,dt.$$

Since $x(t)\not\equiv c\sin t$ we have $\int_0^\pi [x'(t)]^2 dt>\int_0^\pi [x(t)]^2 dt$, we get $\int_0^\pi x(t)\sin tdt$ <0. $l\neq 0$ implies $\int_0^\pi \sin t\cos tdt\neq 0$, a contradiction. One sees that conditions (H_4) and $|k|<1$ in Theorem 2.2 are un-improvable sufficient conditions.

References

[1] R. P. Agarwal, *Focal Boundary Value Problems for Differential and Difference Equations*, Kluwer, Dordrecht, 1998.
[2] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, 1999.
[3] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.
[4] C. P. Gupta, *Solvability of a boundary value problem with the nonlinearity satisfying a sign condition*, J. Math. Anal. Appl., 129 (1988), 231–242.
[5] Z. Han, *An extension of Duo's theorem and its applications*, Northeast. Math. J., 7 (1991), 480–485.
[6] C. Kuo, *Solvability of a nonlinear two-point boundary value problem at resonance*, J. Differ. Equations, 140 (1997), 1–9.
[7] C. W. Ha and C. Kuo, *Solvability of a nonlinear two-point boundary value problem at resonance (II)*, Topol. Methods Nonlinear Anal., 11 (1998), 159–168.
[8] E. N. Dancer and C. P. Gupta, *A Lyapunov-type result with application to a Dirichlet-type two-point boundary value problem at resonance*, Nonlinear Anal., 22 (1994), 305–318.
[9] R. Ma, *Multiplicity results for a third order boundary value problem at resonance*, Nonlinear Anal., Theory Methods Appl,, 32 (1998), 493–499.
[10] C. Kuo, *Solvability of a nonlinear two-point boundary value problem at resonance (II)*, Nonlinear Anal., Theory Methods Appl., 54 (2003), 565–573.

[11] P. K. Nagle and K. L. Pothoven, *On a third order nonlineqar boundary value problem at resonance*, J. Math. Anal. Appl., 195 (1995), 149–159.

[12] C. P. Gupta, *On a third order boundary value problem at resonance*, Differ. Integral Equ., 2 (1989), 1–12.

[13] C. J. Chyan and J. Henderson, *Positive solutions of* $2m$*th-order boundary value problems*, Appl. Math. Lett., 15 (2002), 767–774.

[14] Y. Liu and W. Ge, *Double positive solutions of fourth-order nonlinear boundary value problems*, Appl. Anal., 82(4)(2003), 369–380.

[15] R. Ma and H. Y. Wang, *On the existence of positive solutions of fourth-order ordinary differential equations*, Appl. Anal. 59 (1995), 225–231.

[16] C. P. Gupta, *Existence and uniqueness theorem for a bending of an elastic beam equation*, Appl. Anal., 26 (1988), 289–340.

[17] E. F. Beckenbach and R. Bellman, *Inequalities*, Erg. Math. N. F. 30, Springer, 1961, p.p. 177–179.

[18] D. S. Cohen, *Multiple stable solutions nonlinear boundary value problems arising in chemical reactor theory*, SIAM J. Appl. Math., 20 (1971), 1–13.

[19] E. N. Dancer, *On the structure of solutions of an equation in catalysis theory when a parameter is large*, J. Differ. Equations, 37 (1980), 404–437.

[20] Z. Han, *Solvability of nonlinear ordinary differential equation when its associated linear equation has no nontrivial or sign-changing solution*, Taiwanese J. Math. 8 (2004), 503–513.

[21] A. Fonda and J. Mawhin, *Quadratic forms, weighted eigenfunctions and boundary value problems for nonlinear second order ordinary differential equations*, Proc. R. Soc. Edinb., 112A (1989), 145–153.

[22] I. Iannacci and M. N. Nakashama, *Nonlinear two-point boundary value problems at resonance without Landesman-Lazer conditions*, Proc. Amer. Math. Soc., 106 (1989), 943–952.

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