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BASINS OF ATTRACTION OF EQUILIBRIUM POINTS OF MONOTONE DIFFERENCE EQUATIONS

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Dedicated to Professor Harry Miller on the occasion of his 70th birthday

Abstract. We investigate the global character of the difference equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k+1}), \quad n = 0, 1, \dots$$

with several equilibrium points, where f is increasing in all its variables. We show that a considerable number of well known difference equations can be embeded into this equation through the iteration process. We also show that a negative feedback condition can be used to determine a part of the basin of attraction of different equilibrium points, and that the boundaries of the basins of attractions of different locally asymptotically stable equilibrium points are in fact the global stable manifolds of neighboring saddle or non-hyperbolic equilibrium points.

1. Introduction

Let I be some interval of real numbers and let $f \in C^1[I \times I, I]$. Let $\bar{x}_1, \bar{x}_2 \in I$, $0 \le \bar{x}_1 < \bar{x}_2$ be two equilibrium points of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots \tag{1}$$

where f is a continuous and increasing function in both variables. There are several global attractivity results for Eq.(1) which give the sufficient conditions for all solutions to approach a unique equilibrium. These results were used efficiently in monograph [15] to study the global behavior of solutions of second order linear fractional difference equation of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + B x_n + C x_{n-1}}, \quad n = 0, 1, \dots \tag{2}$$

with non-negative initial conditions and parameters.

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We list three such results:

The first theorem, which has also been very useful in applications to mathematical biology, see [13], was really motivated by a problem in [9].

Theorem 1. ([8]. See also [9, 13], and [14], p. 53). *Let $I \subseteq [0, \infty)$ be some interval and assume that $f \in C[I \times I, (0, \infty)]$ satisfies the following conditions:*

(i) *$f(x, y)$ is non-decreasing in each of its arguments;*
(ii) *Eq.(1) has a unique positive equilibrium point $\overline{x} \in I$ and the function $f(x, x)$ satisfies the* **negative feedback condition**

$$(x - \overline{x})(f(x, x) - x) < 0 \quad \textit{for every} \quad x \in I - \{\overline{x}\}. \tag{3}$$

Then every positive solution of Eq.(1) with initial conditions in I converges to $\overline{x}$.

The second result was obtained in [15] and it was extended to the case of higher order difference equations and systems in [19, 26, 32].

Theorem 2. *Let $[a, b]$ be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) *$f(x, y)$ is non-decreasing in each of its arguments;*
(b) *Eq.(1) has a unique equilibrium $\overline{x} \in [a, b]$.*

Then every solution of Eq.(1) converges to $\overline{x}$.

The following result has been obtained recently in [1].

Theorem 3. *Let $I \subseteq R$ and let $f \in C[I \times I, I]$ be a function which increases in both variables. Then for every solution of Eq.(1) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution do exactly one of the following:*

(i) *Eventually they are both monotonically increasing.*
(ii) *Eventually they are both monotonically decreasing.*
(iii) *One of them is monotonically increasing and the other is monotonically decreasing.*

Remark 1. Theorem 1 is actually a special case of Theorem 2. Indeed (3) implies that there exist a and $b, a < \bar{x} < b$ such that $f(a, a) > a, f(b, b) < b$, which in view of monotonicity of f implies that

$$f : [a, b] \times [a, b] \to [a, b],$$

and so all conditions of Theorem 2 are satisfied. Furthermore, Theorem 2 is a special case of Theorem 3 if we additionally assume non-existence of period-two solutions. None of these results provide any information about the basins of attraction of different equilibrium points when there exist several equilibrium points, as in the case of equations (4) and (21).

Example 1. Here we consider the following equation

$$x_{n+1} = \frac{1}{2}(x_n + x_{n-1} + \sin(x_n) + \sin(x_{n-1})), \quad n = 0, 1, \dots \tag{4}$$

on an interval $[0, 2K\pi]$, where K is an integer. This equation has $2K + 1$ equilibrium points $\bar{x}_m = m\pi, m = 0, 1, \dots, 2K$. An immediate checking shows that $\bar{x}_{2m+1}$ are locally asymptotically stable, while $\bar{x}_{2m}$ are saddle equilibrium points with eigenvalues $\lambda_{\pm} = \frac{1\pm\sqrt{5}}{2}$. The expression $f(x, x) - x$, where $f(x, y) = \frac{1}{2}(x + y + \sin(x) + \sin(y))$ becomes

$$f(x, x) - x = \sin x < 0 \quad \text{if} \quad x \in ((2m + 1)\pi, (2m + 2)\pi).$$

This shows that the negative feedback condition is satisfied in the interval $(\bar{x}_{2m+1}, \bar{x}_{2m+2})$ and so in view of Theorem 1 or Theorem 2 the product of this interval by itself is a part of the basin of attraction of $\bar{x}_{2m+1}$. Similarly $(\bar{x}_{2m}, \bar{x}_{2m+1})^2$ is a part of the basin of attraction of $\bar{x}_{2m+1}$. In view of Theorem 3 all solutions of Eq.(4) are eventually monotonic and so are convergent. Notice that Eq.(4) has no minimal period-two solutions. A direct calculation shows that the eigenvectors which correspond to two eigenvalues $\lambda_{\pm}$ at the equilibrium points $(\bar{x}_{2m}, \bar{x}_{2m})$ are $[1, \ \lambda_{\pm}]$.

The problem for Eq.(4) is to determine precisely the basin of attraction of all local attractors $\bar{x}_{2m+1}$. By using software that simulates discrete dynamical systems, such as *Dynamica* 2, see [17], we obtain the basins of attraction depicted in Figure 1.

Now, using Theorem 14 which is the main result of this paper, we obtain the precise description of the basin of attraction $\mathcal{B}_{2m+1}$ of $\bar{x}_{2m+1}$ and $\mathcal{B}_{2m}$ of $\bar{x}_{2m}$ as:

$$\mathcal{B}_{2m+1} = \{(x, y) : \exists y_u, y_l : \ y_l < y < y_u \quad (x, y_l) \in \mathcal{W}_{2m}, (x, y_u) \in \mathcal{W}_{2m+2}\}$$

and

$$\mathcal{B}_{2m} = \mathcal{W}_{2m},$$

where $\mathcal{W}_{2m} = \mathcal{W}_{(\bar{x}_{2m}, \bar{x}_{2m})}$ is the global stable manifold of the equilibrium point $(\bar{x}_{2m}, \bar{x}_{2m})$. In other words, the basin of attraction of $(\bar{x}_{2m+1}, \bar{x}_{2m+1})$ is the set of all points in the plane of initial conditions which are between the stable manifolds of two consecutive saddle equilibrium points $(\bar{x}_{2m}, \bar{x}_{2m})$ and $(\bar{x}_{2m+2}, \bar{x}_{2m+2})$.

These results are summarized in the plot in Figure 1.

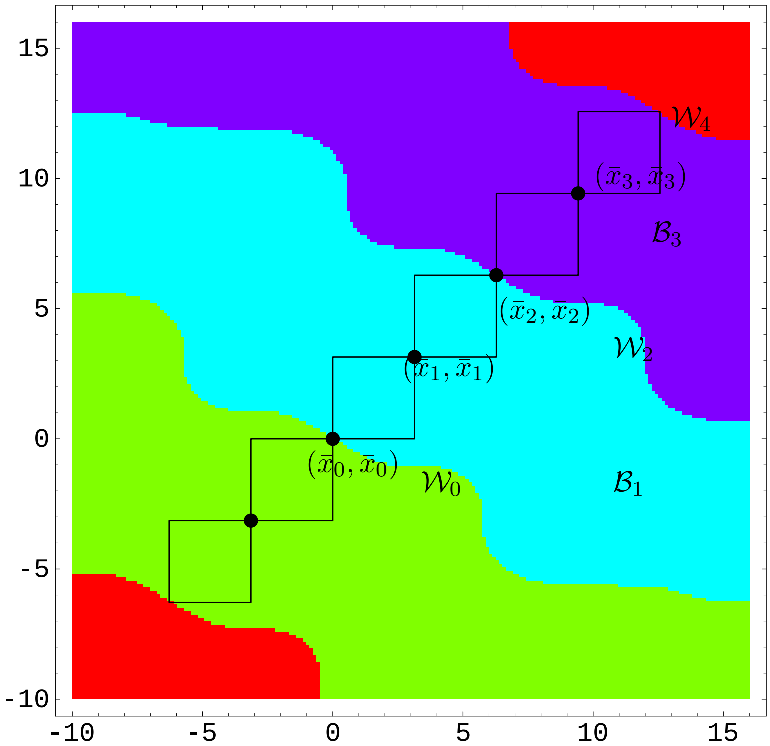

Figure 1: Figure shows the basins of attraction of Eq.(4). Figure was generated by *Dynamica* 3 [17].

2. Preliminaries

We now give some basic notions about fixed points and monotonic maps in the plane.

Consider a map T on a nonempty set $\mathcal{S} \subset \mathbb{R}^2$, and let $\overline{\mathbf{e}} \in \mathcal{S}$. The point $\overline{\mathbf{e}} \in \mathcal{S}$ is called a *fixed point* if $T(\overline{\mathbf{e}}) = \overline{\mathbf{e}}$. An *isolated* fixed point is a fixed point that has a neighborhood with no other fixed points in it. A fixed point $\overline{\mathbf{e}} \in \mathcal{S}$ is an *attractor* if there exists a neighborhood $\mathcal{U}$ of $\overline{\mathbf{e}}$ such that $T^n(\mathbf{x}) \to \overline{\mathbf{e}}$ as $n \to \infty$ for $\mathbf{x} \in \mathcal{U}$; the *basin of attraction* is the set of all $\mathbf{x} \in \mathcal{S}$ such that $T^n(\mathbf{x}) \to \overline{\mathbf{e}}$ as $n \to \infty$. A fixed point $\overline{\mathbf{e}}$ is a global attractor on a set $\mathcal{K}$ if $\overline{\mathbf{e}}$ is an attractor and $\mathcal{K}$ is a subset of the basin of attraction of $\overline{\mathbf{e}}$. If T is differentiable at a fixed point $\overline{\mathbf{e}}$, and if the Jacobian $J_T(\overline{\mathbf{e}})$ has one eigenvalue with modulus less than one and a second eigenvalue with modulus greater than one, $\overline{\mathbf{e}}$ is said to be a *saddle*. See [27] for additional definitions (stable and unstable manifolds, asymptotic stability).

Consider a partial ordering $\preceq$ on $\mathbb{R}^2$. Two points $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are said to be *related* if $\mathbf{v} \preceq \mathbf{w}$ or $\mathbf{w} \preceq \mathbf{v}$. Also, a strict inequality between points may be defined as $\mathbf{v} \prec \mathbf{w}$ if $\mathbf{v} \preceq \mathbf{w}$ and $\mathbf{v} \neq \mathbf{w}$. A stronger inequality may be defined as $\mathbf{v} = (v_1, v_2) \ll \mathbf{w} = (w_1, w_2)$ if $\mathbf{v} \preceq \mathbf{w}$ with $v_1 \neq w_1$ and $v_2 \neq w_2$. For $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^2$, the *order interval* $[\![\mathbf{u}, \mathbf{v}]\!]$ is the set of all $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{u} \preceq \mathbf{x} \preceq \mathbf{v}$.

A map T on a nonempty set $\mathcal{S} \subset \mathbb{R}^2$ is a continuous function $T : \mathcal{S} \to \mathcal{S}$. The map T is *monotone* if $\mathbf{v} \preceq \mathbf{w}$ implies $T(\mathbf{v}) \preceq T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{S}$, and it is *strongly monotone* on $\mathcal{S}$ if $\mathbf{v} \prec \mathbf{w}$ implies that $T(\mathbf{v}) \ll T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{S}$. The map is *strictly monotone* on $\mathcal{S}$ if $\mathbf{v} \prec \mathbf{w}$ implies that $T(\mathbf{v}) \prec T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{S}$. Clearly, being related is invariant under iteration of a strongly monotone map.

Throughout this paper we shall use the *North-East ordering* for which the positive cone is the first quadrant, i.e. this partial ordering is defined by

$$(x_1, y_1) \preceq_{ne} (x_2, y_2) \quad \Leftrightarrow \quad x_1 \leq x_2 \text{ and } y_1 \leq y_2 \tag{5}$$

A map T on a nonempty set $\mathcal{S} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is called cooperative and a map monotone with respect to the South-East ordering

$$(x_1, y_1) \preceq_{se} (x_2, y_2) \quad \Leftrightarrow \quad x_1 \leq x_2 \text{ and } y_1 \geq y_2 \tag{6}$$

is called competitive.

If T is differentiable map on a nonempty set $\mathcal{S}$, a sufficient condition for T to be strongly monotonic with respect to the NE ordering is that the Jacobian matrix at all points $\mathbf{x}$ has the sign configuration

$$\operatorname{sign}(J_T(\mathbf{x})) = \begin{bmatrix} + & + \\ + & + \end{bmatrix}, \tag{7}$$

provided that $\mathcal{S}$ is open and convex.

For $\mathbf{x} \in \mathbb{R}^2$, define $\mathcal{Q}_\ell(\mathbf{x})$ for $\ell = 1, \ldots, 4$ to be the usual four quadrants based at $\mathbf{x}$ and numbered in a counterclockwise direction, for example, $Q_1(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^2 : \ x_1 \leq y_1,\ x_2 \leq y_2\}$. The (open) ball of radius r centered at $\mathbf{x}$ is denoted with $\mathcal{B}(\mathbf{x}, r)$. If $\mathcal{K} \subset \mathbb{R}^2$ and $r > 0$, write $\mathcal{K} + \mathcal{B}(\mathbf{0}, r) := \{\mathbf{x} : \mathbf{x} = \mathbf{k} + \mathbf{y}$ for some $\mathbf{k} \in \mathcal{K}$ and $\mathbf{y} \in \mathcal{B}(\mathbf{0}, r)\}$. If $\mathbf{x} \in [-\infty, \infty]^2$ is such that $\mathbf{x} \preceq \mathbf{y}$ for every $\mathbf{y}$ in a set $\mathcal{Y}$, we write $\mathbf{x} \preceq \mathcal{Y}$. The inequality $\mathcal{Y} \preceq \mathbf{x}$ is defined similarly.

The following definition is from [31].

Definition 1. *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. A competitive map $T : \mathcal{S} \to \mathcal{S}$ is said to satisfy condition $(O+)$ if for every x, y in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $x \preceq_{ne} y$, and T is said to satisfy condition $(O-)$ if for every x, y in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $y \preceq_{ne} x$.*

The following theorem was proved by de Mottoni-Schiaffino [6] for the Poincaré map of a periodic competitive Lotka-Volterra system of differential equations. Smith generalized the proof to competitive and cooperative maps [28, 29].

Theorem 4. *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. If T is a competitive map for which $(O+)$ holds then for all $x \in \mathcal{S}$, $\{T^n(x)\}$ is eventually componentwise monotone. If the orbit of x has compact closure, then it converges to a fixed point of T. If instead $(O-)$ holds, then for all $x \in \mathcal{S}$, $\{T^{2n}\}$ is eventually componentwise monotone. If the orbit of x has compact closure in $\mathcal{S}$, then its omega limit set is either a period-two orbit or a fixed point.*

It is well known that a stable period-two orbit and a stable fixed point may coexist, see Dancer and Hess [5].

The following result is from [31], with the domain of the map specialized to be the cartesian product of intervals of real numbers. It gives a sufficient condition for conditions $(O+)$ and $(O-)$.

Theorem 5. *Let $R \subset \mathbb{R}^2$ be the cartesian product of two intervals in $\mathbb{R}$. Let $T : R \to R$ be a C^1 competitive map. If T is injective and $\det J_T(x) > 0$ for all $x \in R$ then T satisfies $(O+)$. If T is injective and $\det J_T(x) < 0$ for all $x \in R$ then T satisfies $(O-)$.*

Theorems 4 and 5 are quite applicable as we have shown in [4], in the case of competitive systems in the plane consisting of linear fractional equations.

The next results, from [22], are useful for determining basins of attraction of fixed points of cooperative maps. These results generalize the corresponding result for hyperbolic case from [20]. Related results have been obtained by H. L. Smith in [28, 29, 30].

Theorem 6. *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap \operatorname{int}(\mathcal{Q}_1(\overline{\mathrm{x}}) \cup \mathcal{Q}_3(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

a. *The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.*
b. *The Jacobian $J_T(\overline{\mathrm{x}})$ of T at $\overline{\mathrm{x}}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.*

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

We shall see in Theorem 9 that the situation where the endpoints of $\mathcal{C}$ are boundary points of $\mathcal{R}$ is of interest. The following result gives a sufficient condition for this case.

Theorem 7. *For the curve $\mathcal{C}$ of Theorem 6 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

i. *The map T has no fixed points nor periodic points of minimal period-two in Δ.*
ii. *The map T has no fixed points in Δ, $\det J_T(\overline{\mathrm{x}}) > 0$, and $T(\mathrm{x}) = \overline{\mathrm{x}}$ has no solutions $\mathrm{x} \in \Delta$.*
iii. *The map T has no points of minimal period-two in Δ, $\det J_T(\overline{\mathrm{x}}) < 0$, and $T(\mathrm{x}) = \overline{\mathrm{x}}$ has no solutions $\mathrm{x} \in \Delta$.*

In many cases one can expect the curve $\mathcal{C}$ to be smooth.

Theorem 8. *Under the hypotheses of Theorem 6, suppose there exists a neighborhood $\mathcal{U}$ of $\overline{\mathrm{x}}$ in $\mathbb{R}^2$ such that T is of class C^k on $\mathcal{U} \cup \Delta$ for some $k \geq 1$, and that the Jacobian of T at each $\mathrm{x} \in \Delta$ is invertible. Then the curve $\mathcal{C}$ in the conclusion of Theorem 6 is of class C^k.*

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 6 reduces just to $|\lambda| < 1$. This follows from a change of variables [31] that allows the Perron-Frobenius Theorem to be applied to give that at any point, the Jacobian of a strongly competitive map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis. The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 9. (A) *Assume the hypotheses of Theorem 6, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 6. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_- := \{\mathrm{x} \in \mathcal{R} \setminus \mathcal{C} : \exists \mathrm{y} \in \mathcal{C} \text{ with } \mathrm{x} \preceq_{se} \mathrm{y}\}$$

and

$$\mathcal{W}_+ := \{\mathrm{x} \in \mathcal{R} \setminus \mathcal{C} : \exists \mathrm{y} \in \mathcal{C} \text{ with } \mathrm{y} \preceq_{se} \mathrm{x}\}, \tag{8}$$

such that the following statements are true.

(i) *$\mathcal{W}_-$ is invariant, and $\mathrm{dist}(T^n(\mathrm{x}), \mathcal{Q}_2(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $\mathrm{x} \in \mathcal{W}_-$.*
(ii) *$\mathcal{W}_+$ is invariant, and $\mathrm{dist}(T^n(\mathrm{x}), \mathcal{Q}_4(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $\mathrm{x} \in \mathcal{W}_+$.*

(B) *If, in addition to the hypotheses of part* (A), $\overline{\mathrm{x}}$ *is an interior point of* $\mathcal{R}$ *and* T *is* C^2 *and strongly competitive in a neighborhood of* $\overline{\mathrm{x}}$*, then* T *has no periodic points in the boundary of* $\mathcal{Q}_1(\overline{\mathrm{x}}) \cup \mathcal{Q}_3(\overline{\mathrm{x}})$ *except for* $\overline{\mathrm{x}}$*, and the following statements are true.*

(iii) *For every* $\mathrm{x} \in \mathcal{W}_-$ *there exists* $n_0 \in \mathbb{N}$ *such that* $T^n(\mathrm{x}) \in \operatorname{int} \mathcal{Q}_2(\overline{\mathrm{x}})$ *for* $n \geq n_0$.

(iv) *For every* $\mathrm{x} \in \mathcal{W}_+$ *there exists* $n_0 \in \mathbb{N}$ *such that* $T^n(\mathrm{x}) \in \operatorname{int} \mathcal{Q}_4(\overline{\mathrm{x}})$ *for* $n \geq n_0$.

Basins of attraction of *period-two solutions* or period-two orbits of certain systems or maps can be effectively treated with Theorems 6 and 9. See [18, 19, 24] for the hyperbolic case, for the non-hyperbolic case see [22].

If T is a map on a set $\mathcal{R}$ and if $\overline{\mathrm{x}}$ is a fixed point of T, the *stable set* $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set $\{x \in \mathcal{R} : T^n(\mathrm{x}) \to \overline{\mathrm{x}}\}$ and *unstable set* $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set

$$\Big\{ \mathrm{x} \in \mathcal{R} : \text{there exists } \{\mathrm{x}_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(\mathrm{x}_n) = \mathrm{x}_{n+1}, \\ \mathrm{x}_0 = \mathrm{x}, \text{ and } \lim_{n \to -\infty} \mathrm{x}_n = \overline{\mathrm{x}} \Big\}$$

When T is non-invertible, the set $\mathcal{W}^s(\overline{\mathrm{x}})$ may not be connected and made up of infinitely many curves, or $\mathcal{W}^u(\overline{\mathrm{x}})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $\mathcal{W}^s(\overline{\mathrm{x}})$ and $\mathcal{W}^u(\overline{\mathrm{x}})$ are the stable and unstable manifolds of $\overline{x}$.

Theorem 10. *In addition to the hypotheses of part* (B) *of Theorem* 9*, suppose that* $\mu > 1$ *and that the eigenspace* E^μ *associated with* μ *is not a coordinate axis. If the curve* $\mathcal{C}$ *of Theorem* 6 *has endpoints in* $\partial\mathcal{R}$*, then* $\mathcal{C}$ *is the stable set* $\mathcal{W}^s(\overline{\mathrm{x}})$ *of* $\overline{\mathrm{x}}$*, and the unstable set* $\mathcal{W}^u(\overline{\mathrm{x}})$ *of* $\overline{x}$ *is a curve in* $\mathcal{R}$ *that is tangential to* E^μ *at* $\overline{\mathrm{x}}$ *and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of* $\mathcal{W}^u(\overline{\mathrm{x}})$ *in* $\mathcal{R}$ *are fixed points of* T.

Remark 2. The results for cooperative systems are analogous where in the above mentioned results competitive should be replaced by cooperative, $\mathcal{Q}_2(\overline{x})$ and $\mathcal{Q}_4(\overline{x})$ should be replaced with $\mathcal{Q}_1(\overline{x})$ and $\mathcal{Q}_3(\overline{x})$ respectively, and increasing should be replaced by decreasing. See [31] for a substitution that transform competitive map into cooperative map.

The following result gives the necessary and sufficient condition for the local stability of

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{9}$$

when f is non-decreasing in all its arguments, see [12].

Theorem 11. *Let*

$$z_{n+1} = \sum_{i=0}^{k} p_i z_{n-i}, \quad n = 0, 1, \dots \tag{10}$$

be the standard linearization about the equilibrium $\overline{x}$ *of Eq.* (9) *where* $p_i = \partial f / \partial x_{n-i}(\bar{x}, \dots, \bar{x}) \geq 0$, $i = 0, \dots, k$. *Then the equilibrium* $\overline{x}$ *of Eq.* (9) *is one of the following:*

(a) *locally asymptotically stable if* $\sum_{i=0}^{k} p_i < 1$,
(b) *non-hyperbolic and locally stable if* $\sum_{i=0}^{k} p_i = 1$,
(c) *unstable if* $\sum_{i=0}^{k} p_i > 1$.

Remark 3. We say that f is *strongly increasing* in both arguments if it is increasing, differentiable and have both partial derivatives positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of Eq. (1) follows from the fact that if f is strongly increasing, then a map associated to Eq. (1) is a cooperative map on $\mathcal{I} \times \mathcal{I}$ while the second iterate of a map associated to Eq. (1) is a strictly cooperative map on $\mathcal{I} \times \mathcal{I}$.

Set $x_{n-1} = u_n$ and $x_n = v_n$ in Eq. (1) to obtain the equivalent system

$$\begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = f(v_n, u_n) \end{array}, \quad n = 0, 1, \dots .$$

Let $F(u, v) = (v, f(v, u))$. Then F maps $I \times I$ into itself and is a cooperative map. The second iterate $T := F^2$ is given by

$$T(u, v) = (f(v, u),\ f(f(v, u), v))$$

and it is clearly strictly cooperative on $\mathcal{I} \times \mathcal{I}$.

Remark 4. The characteristic equation of Eq. (1) at an equilibrium point $(\bar{x}, \bar{x})$:

$$\lambda^2 - D_1 f(\bar{x}, \bar{x})\lambda - D_2 f(\bar{x}, \bar{x}) = 0, \tag{11}$$

has two real roots λ, μ which satisfy $\lambda < 0 < \mu$, and $|\lambda| < \mu$, whenever f is strictly increasing in both variables. Here $D_i f, i = 1, 2$ denotes the partial derivative with respect to the i-th variable. Thus the applicability of Theorems 6 and 9 depends on the nonexistence of minimal period-two solution.

The theory of monotone maps has been extensively developed at the level of ordered Banach spaces and applied to many types of equations such as ordinary, partial and discrete, see [5, 6, 10, 11, 31, 32, 33, 34]. In particular, [10] has an extensive updated bibliography of different aspects of the theory of monotone maps. The theory of monotone discrete maps is more specialized and so one should expect stronger results in this case. An excellent review of basic results is given in [11, 26, 31]. In particular, two-dimensional discrete maps are studied in great details and very precise results which describe the global dynamics and the basins of attractions of equilibrium points as well as global stable manifolds are given in [6, 18, 20, 22, 28, 29, 30, 31, 32].

In this paper we consider Eq. (1) which has several equilibrium points and provide the sufficient conditions for all solutions to converge to an equilibrium point. More precisely, we will give sufficient conditions for the precise description of the basins of attraction of different equilibrium points. An application of our results gives precise description of the basis of attraction of equations (4) and (21).

3. Main results

The following result is needed to state our main result:

Theorem 12. *Consider Eq.*(1) *subject to the following conditions:*

(C1) $f \in C[I^2, I]$ *is increasing in both arguments and* I *is an interval*
(C2) *Eq.*(1) *has no prime period-two solutions*

Then every bounded solution of Eq.(1) *converges to an equilibrium.*

Proof. We will check that the map F associated to Eq.(1) satisfies condition $(O-)$. Indeed, assume
$$F(x) \preceq_{se} F(y)$$
for any $x = (u_1, v_1), y = (u_2, v_2)$, that is
$$v_1 \leq v_2, \quad f(v_1, u_1) \geq f(v_2, u_2).$$
In view of monotone character of f this implies $u_2 \leq u_1$ which implies $y \preceq_{se} x$. Observe that we do not require continuity of f for this proof. Thus the conditions of Theorem 4 are satisfied and so the conclusion follows from this theorem.

We will provide the second proof of this result under the additional assumption that f is strictly increasing, which we need for application of Theorems 6 and 9. We check that the conditions of Theorem 5 are satisfied for the second iterate $T = F^2$ of the map associated with Eq.(1).

If $D_1 g(u, v)$ and $D_2 g(u, v)$ denote the partial derivatives of a function $g(u, v)$ with respect to u and v, the Jacobian matrix of T is
$$J_T(u,v) = \begin{pmatrix} D_2 f(v,u) & D_1 f(v,u) \\ D_1 f(f(v,u),v)\, D_2 f(v,u) & \begin{array}{c} D_1 f(f(v,u),v)\, D_1 f(v,u) \\ +D_2 f(f(v,u),v) \end{array} \end{pmatrix}. \tag{12}$$

The determinant of (12) is given by
$$\det J_T(u,v) = D_2 f(v,u) D_2 f(f(v,u),v) > 0.$$

To check injectivity of T we set

$$T((u_1, v_1)) = T((u_2, v_2))$$

which implies

$$f(v_1, u_1) = f(v_2, u_2), \quad f(f(v_1, u_1), v_1) = f(f(v_2, u_2), v_2)$$

and so $f(f(v_1, u_1), v_1) = f(f(v_1, u_1), v_2)$. By using the monotonicity of f we conclude that $v_1 = v_2$ which in view of $f(v_1, u_1) = f(v_2, u_2)$ gives $u_1 = u_2$. Theorems 4 and 5 impliy that the subsequences $\{x_{2k}\}_{k=0}^{\infty}$ and $\{x_{2k-1}\}_{k=0}^{\infty}$ of every solution of Eq.(1) are eventually monotonic. Furthermore, every bounded solution converges to a period-two solution and in view of $(C2)$ to an equilibrium. □

Corollary 1. *Consider Eq.*(1) *subject to* $(C1)$ *and the negative feedback condition* (3)*. Then every bounded solution of Eq.*(1) *converges to an equilibrium.*

Proof. We show that the negative feedback condition (3) implies $(C2)$, that is, the non-existence of a prime period-two solution. If $f(x, x) > x$ or $f(x, x) < x$ for all $x < \bar{x}$ or $x > \bar{x}$, then $v > u$ implies that $u = f(v, u) > f(u, u)$ for $u < \bar{x}$ and $v = f(u, v) < f(v, v)$ for $v > \bar{x}$, which shows that Eq.(1) can not have period-two solution. □

Remark 5. Theorem 12 was established independently in [1] by direct proof. We are providing the proof here for two reasons. First, we want to show that the result follows from more general Theorem 4. Second, we want to show that the conditions (a) and (b) of Theorem 6 are satisfied for Eq.(1) when f is strictly increasing and the corresponding map is C^2 in some neighborhood of the equilibrium point.

Theorem 13 and Corollary 1 are results that hold for second order difference equations and can not be extended to higher order difference equations because Theorems 4 and 5 which are fundamental in the proof can not be extended to higher order difference equations. Next, we will prove a slightly less general result for Eq.(1) which can be extended to higher order difference equations.

Theorem 13. *Consider Eq.*(1) *subject to* $(C1)$ *and the following conditions:*

(C3) *There exist two equilibrium points* $0 \leq x_1 < x_2$ *of Eq.*(1)

(C4) *Either the negative feedback condition* (NFC) *with respect to* $\overline{x}_1$ *holds*

$$(x - \overline{x}_1)(f(x, x) - x) < 0, \ \forall x \in (\overline{x}_1, \overline{x}_2)$$

or

(C5) *the negative feedback condition (NFC) with respect to* $\overline{x}_2$ *holds*

$$(x - \overline{x}_2)(f(x, x) - x) < 0, \ \forall x \in (\overline{x}_1, \overline{x}_2).$$

Then every bounded solution of Eq.(1) *converges to an equilibrium. The box* $(\overline{x}_1, \overline{x}_2)^2$ *is a part of the basin of attraction of* $\overline{x}_1$ *if* $(C4)$ *is satisfied or* $\overline{x}_2$ *if* $(C5)$ *is satisfied.*

Proof. Assume that $x_{-1}, x_0 \in [0, \overline{x}_1]$ and that at least one of them is smaller than $\overline{x}_1$. By using the monotonicity of f we obtain

$$x_1 = f(x_0, x_{-1}) < f(\overline{x}_1, \overline{x}_1) = \overline{x}_1.$$

By using the inductive argument we can show that $x_n \in [0, \overline{x}_1), \forall n \geq -1$, which proves that $[0, \overline{x}_1]$ is an attracting interval. In addition f satisfies

$$0 < f(u, v) \leq f(\overline{x}_1, \overline{x}_1) = \overline{x}_1, \ \ \forall u, v \in [0, \overline{x}_1].$$

Thus $[0, \overline{x}_1)$ is an invariant interval and $f : [0, \overline{x}_1]^2 \to [0, \overline{x}_1]$. Consequently, all conditions of Theorem 2 are satisfied and so we conclude that

$$\lim_{n\to\infty} x_n = \overline{x}_1 \tag{13}$$

for every solution $\{x_n\}$ of Eq.(1).

A similar argument can be applied in the case where $x_{-1}, x_0 \geq \overline{x}_2$ and at least one inequality is strict. In this case, if $U > \overline{x}_2$ is a number such that

$$x_n \leq U, \quad \forall n \geq -1$$

such that

$$f(u, v) \leq f(U, U) \leq U$$

which shows that $[\overline{x}_2, U]$ is an invariant (and attracting) interval. Thus an application of Theorem 2 leads to

$$\lim_{n\to\infty} x_n = \overline{x}_2. \tag{14}$$

Next, assume that $x_{-1}, x_0 \in [\overline{x}_1, \overline{x}_2]$. For the sake of definiteness assume that

$$\overline{x}_1 \leq x_{-1} \leq x_0 \leq \overline{x}_2$$

where at least one inequality is strict.

By using (C_2), we have

$$\overline{x}_1 = f(\overline{x}_1, \overline{x}_1) \leq x_1 = f(x_0, x_{-1}) \leq f(\overline{x}_2, \overline{x}_2) = \overline{x}_2.$$

By using induction we can show that $x_n \in [\overline{x}_1, \overline{x}_2], n \geq -1$ and so $[\overline{x}_1, \overline{x}_2]$ is an attracting interval. Let

$$\overline{x}_1 < U = \max\{x_{-1}, x_0\} < \overline{x}_2.$$

If (C_4) holds then

$$\overline{x}_1 < f(\overline{x}_1, \overline{x}_1) < x_1 = f(x_0, x_{-1}) \leq f(U, U) < U.$$

That is $\overline{x}_1 < x_1 < U$. Furthermore, if $u, v \in [\overline{x}_1, U]$ then (C_4) implies

$$\overline{x}_1 = f(\overline{x}_1, \overline{x}_1) \leq f(u, v) \leq f(U, U) < U.$$

Consequently, by applying Theorem 2 in an invariant interval $[\bar{x}_1, U]$ we get that the solution satisfies (13).

Assume now that $\overline{x}_2 = x_{-1} > x_0 > \overline{x}_1$. Let $L = \min\{x_{-1}, x_0\}$. If (C_5) holds, then we obtain

$$x_1 = f(x_0, x_{-1}) > f(L, L) > L$$

and by (C_2) we have

$$L < f(L, L) \leq f(u, v) \leq f(\overline{x}_2, \overline{x}_2) = \overline{x}_2,$$

for all $u, v \in [L, \overline{x}_2]$. Thus $[L, \overline{x}_2]$ is an invariant interval for f.

By applying Theorem 2 we obtain that the solutions satisfy (14).

The case $\overline{x}_2 = x_{-1} > x_0 = \overline{x}_1$ or $\overline{x}_2 = x_0 > x_{-1} = \overline{x}_1$ can be reduced to one of the previous cases. Namely, it is easy to show that in this case $x_1, x_2 \in (\overline{x}_1, \overline{x}_2)$ and so we can define L and U starting with x_1 and x_2.

Now we want to consider the remaining cases for the initial conditions such as $x_{-1} \leq \bar{x}_2 < x_0$ and prove that all bounded solutions converge to an equilibrium. To complete this task we will use the basic results of monotone maps, see [31], [19].

Based on our previous claims we know that the basin of attraction $B(E_1)$ of E_1 contains $Q_3(E_1)$ and also $Q_1(E_1) \cap Q_3(E_2)$ if (C_4) is satisfied while the basin of attraction $B(E_2)$ contains $Q_1(E_2)$, and in the case when (C_5) is satisfied, $Q_1(E_1) \cap Q_3(E_2)$.

Take any point $P(x_{-1}, x_0)$ outside of the set $A = Q_1(E_2) \cup Q_4(E_4) \cup Q_1(E_1) \cap Q_3(E_2)$. Then, one can find two points $P_l \in Q_3(E_1)$ and $P_u \in Q_1(E_2)$ such that

$$P_l \preceq P \preceq P_u$$

with respect to the North-East ordering.

By using the monotonicity of a map F associated with Eq.(1) we have

$$F(P_l) \preceq F(P) \preceq F(P_u)$$

and

$$F^n(P_l) \preceq F^n(P) \preceq F^n(P_u).$$

As we have seen

$$\lim_{n \to \infty} F^n(P_l) = E_1 \text{ and } \lim_{n \to \infty} F^n(P_u) = E_2,$$

which by continuity (see [20]) implies that

$$\lim_{n \to \infty} F^n(P) \in Q_1(E_1) \cap Q_3(E_2)$$

and so

$$\lim_{n \to \infty} F^n(P) = E_1$$

if (C_4) holds, and

$$\lim_{n\to\infty} F^n(P) = E_2$$

if (C_5) holds.

This means that the solution $\{x_n\}$ satisfies (13) when (C_4) holds or (14) when (C_5) holds. $\square$

Theorem 13 can be combined with Theorems 6 and 9 to give the precise description of basins of attraction of equilibrium points of Eq.(1). Here is our main result:

Theorem 14. *Consider Eq.*(1) *where f is increasing function in its arguments and assume that there is no minimal period-two solution. Assume that $E_1(x_1, y_1), E_2(x_2, y_2)$, and $E_3(x_3, y_3)$ are three consecutive equilibrium points in North-East ordering that satisfy*

$$(x_1, y_1) \preceq_{ne} (x_2, y_2) \preceq_{ne} (x_3, y_3)$$

and that E_1, E_3 are saddle points with the neighborhoods where f is strictly increasing and E_2 is a local attractor.

Then the basin of attraction $\mathcal{B}(E_2)$ of E_2 is the region between the global stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$. More precisely

$$\mathcal{B}(E_2) = \Big\{(x, y) : \exists y_u, y_l : \ y_l < y < y_u, \\ (x, y_l) \in \mathcal{W}^s(E_1), (x, y_u) \in \mathcal{W}^s(E_3)\Big\}.$$

The basins of attraction $\mathcal{B}(E_1) = \mathcal{W}^s(E_1)$ and $\mathcal{B}(E_3) = \mathcal{W}^s(E_3)$ are exactly the global stable manifolds of E_1 and E_3.

Proof. In view of Theorem 3 all solutions are eventually monotonic and so all bounded solutions are convergent to the equilibrium points. Set

$$\mathcal{B} = \Big\{(x, y) : \exists y_u, y_l : \ y_l < y < y_u, \quad (x, y_l) \in \mathcal{W}^s(E_1), (x, y_u) \in \mathcal{W}^s(E_3)\Big\}.$$

Straightforward calculation shows that the eigenspaces E^λ associated with the stable eigenvalues $\lambda, |\lambda| < 1$ is $[1, \lambda]$ and so is not a coordinate axis. The existence and the properties of the global stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$ are guaranteed by Theorems 6 and 9. Take $(x_{-1}, x_0) \in \mathcal{B}$. Then there exist the points $(x^l_{-1}, x^l_0) \in \mathcal{W}^s(E_1)$ and $(x^u_{-1}, x^u_0) \in \mathcal{W}^s(E_3)$ such that

$$(x^l_{-1}, x^l_0) \preceq (x_{-1}, x_0) \preceq (x^u_{-1}, x^u_0),$$

which implies

$$T^n((x^l_{-1}, x^l_0)) \preceq T^n((x_{-1}, x_0)) \preceq T^n((x^u_{-1}, x^u_0)).$$

By taking $n \to \infty$ we obtain

$$E_1 \preceq T^n((x_{-1}, x_0)) \preceq E_3.$$

In view of the uniqueness of the global stable manifold and Theorem 9 we obtain that

$$\lim_{n\to\infty} T^n((x_{-1}, x_0)) = E_2,$$

which shows that $\mathcal{B} \subset \mathcal{B}(E_2)$.

The proof that $\mathcal{B}(E_1) = \mathcal{W}^s(E_1)$ and $\mathcal{B}(E_3) = \mathcal{W}^s(E_3)$ follows from Theorems 9 and 10. $\square$

Remark 6. The requirement that E_1 and E_3 are saddle points can be replaced by the requirement that at least one of them is non-hyperbolic point that satisfies conditions (a) and (b) of Theorem 6. See Example 6. The assumptions of Theorem 14 are of local character for all three equilibrium points, and no strict increasing character is assumed for E_2. Even with assumptions we can characterize the basin of attraction of E_2. See Example 7.

Remark 7. Theorem 13 can be extended to the case when Eq.(1) has a finite number of equilibrium points. Also Theorem 13 can be extended to the case of the $(k+1)$-st order difference equation (9).

Theorem 15. *Consider Eq.*(9) *subject to the following conditions:*

(C1) $f \in C[[0,\infty)^{k+1}, [0,\infty)]$
(C2) $f(u_0, u_1, \ldots, x_k)$ *is increasing in all arguments*
(C3) *There exist* m *equilibrium points* $0 \le x_1 < x_2 < \ldots < x_m$ *of Eq.*(9)
(C4) *The negative feedback condition with respect to* $\overline{x}_i$ *for some* $i \in \{1, \ldots, m\}$ *holds*

$$(x - \overline{x}_i)(f(x, x, \ldots, x) - x) < 0, \ \forall x \in (\overline{x}_i, \overline{x}_{i+1}).$$

Then every bounded solution of Eq.(1) *converges to an equilibrium. The box* $(\overline{x}_i, \overline{x}_{i+1})^{k+1}$ *is a part of the basin of attraction of* $\overline{x}_i$.

4. Examples

Here we present some examples of applications of our theorems.

Example 2. Equation

$$x_{n+1} = \frac{px_n + x_{n-1}}{r + px_n + x_{n-1}}, \quad n = 0, 1, \ldots \tag{15}$$

where $p, r > 0$ and the initial conditions x_{-1}, x_0 are non-negative was considered in [2]. Eq.(15) has zero equilibrium always and when $p+1 > r$ there exists an additional positive equilibrium $\overline{x} = (p+1-r)/(p+1)$. Clearly the function

$$f(u,v) = \frac{pu + v}{r + pu + v}, \quad u, v \ge 0$$

is strictly increasing function in both arguments and

$$f(u,v)=\frac{pu+v}{r+pu+v}\le\frac{\max\{p,1\}}{\min\{p,r,1\}}=U,\quad u,v\ge 0.$$

It is easy to show that the zero equilibrium is a global attractor for $p+1<r$. Assume that $p+1>r$. Then for $0<x<\overline{x}$ we have

$$f(x,x)-x=x\left(\frac{p+1-r-(p+1)x}{r+(p+1)x}\right)=(p+1)x\left(\frac{\overline{x}-x}{r+(p+1)x}\right)>0,$$

which shows that the negative feedback condition with respect to the positive equilibrium is satisfied. Theorem 13 implies (13) for every solution with at least one positive initial condition.

Example 3. We give an example which is relevant to mathematical biology, see [25] for a special case. Consider

$$x_{n+1}=f_1(x_n)+f_2(x_{n-1}),\quad n=0,1,\ldots,\tag{16}$$

where f_1, f_2 are continuous, increasing functions on $[0,\infty)$. Assume that there exists unique $\overline{x}>0$ such that

$$f_1(0)+f_2(0)=0\quad\text{and}\quad f_1(\overline{x})+f_2(\overline{x})=\overline{x}.$$

Assume that

$$f_1(x)+f_2(x)>x$$

for any $0<x<\overline{x}$. Then every positive solution of Eq. (16) will converge to $\overline{x}$. This result is an immediate application of Theorem 1. The special case of this result when

$$f_i(u)=\frac{\beta_i u}{1+b_i u},\quad \beta_i, b_i>0\quad i=1,2,$$

was given in [25]. Equation

$$x_{n+1}=\frac{\beta_1 x_n}{1+b_1 x_n}+\frac{\beta_2 x_{n-1}}{1+b_2 x_{n-1}}\tag{17}$$

has zero equilibrium always and when $\beta_1+\beta_2>1$ it has also an additional positive equilibrium point $\overline{x}$. Indeed, the positive equilibrium satisfies

$$1=\frac{\beta_1}{1+b_1\overline{x}}+\frac{\beta_2}{1+b_2\overline{x}}=h(\overline{x}).$$

Clearly $h(u)$ is a decreasing function for $u>0$ and

$$h(0)=\beta_1+\beta_2>1,\quad \lim_{u\to\infty}h(u)=0,$$

which shows that there exists unique $\overline{x} > 0$ such that $h(\overline{x}) = 1$. Let us check the negative feedback condition with respect to zero equilibrium for function $f(x, y)$. We have

$$f(x,x) - x = \frac{\beta_1 x}{1 + b_1 x} + \frac{\beta_2 x}{1 + b_2 x} - x = x(\frac{\beta_1}{1 + b_1 x} + \frac{\beta_2}{1 + b_2 x} - 1)$$
$$= x(h(x) - 1) > x(h(\overline{x}) - 1) = 0 \quad \text{if } x < \overline{x}.$$

Theorem 13 implies

$$\lim_{n\to\infty} x_n = \overline{x}.$$

for every solution of Eq.(17).

Example 4. We give another example which is relevant to mathematical biology, see [7, 16, 15, 32]. Consider

$$x_{n+1} = \frac{px_n + x_{n-1}}{qx_n + x_{n-1}}, \quad n = 0, 1, \ldots \tag{18}$$

where p, q are positive and the initial conditions x_{-1}, x_0 are nonnegative, and such that $x_{-1} + x_0 > 0$. It has been proved in [16] that the unique equilibrium of Eq.(18) is locally asymptotically stable if $q < pq + 1 + 3p$ and it was conjectured that it is globally asymptotically stable as well. Recently, the global asymptotic stability was established in the region $q < p$ in [18] and [26]. Here we show how we can establish global attractivity of the equilibrium by *embedding* Eq.(18) into a higher order difference equation which is increasing in all its arguments. The embedding is performed by one or more iterations of this equation. This method was used extensively in a recent monograph by Camouzis and Ladas [3] to prove the boundedness of solutions of difference equations and to find the explicit bounds for the solutions, as well as in [12] to find the global attractivity and global stability results for general nonlinear difference equations.

By iterating Eq.(18) once we get

$$x_{n+2} = \frac{qx_n^2 + x_n x_{n-1} + p^2 x_n + px_{n-1}}{qx_n^2 + x_n x_{n-1} + pqx_n + qx_{n-1}} = g(x_n, x_{n-1}), n = 0, \ldots \tag{19}$$

where

$$g(u, v) = \frac{qu^2 + uv + p^2 u + pv}{qu^2 + uv + pqu + qv}$$

is decreasing in both arguments. By iterating equation (19) once more we get an equation of order 6 where the right hand side is increasing in all its arguments:

$$x_{n+2} = g(g(x_{n-2}, x_{n-3}), g(x_{n-3}, x_{n-4})) = F(x_{n-2}, x_{n-3}, x_{n-4}), \tag{20}$$

for $n = 0, 1, \ldots$. A straightforward checking shows that all three equations (18)-(20) have a unique positive equilibrium point and that $F(u, v, z)$

satisfies negative feedback condition (NFC). Thus, the unique positive equilibrium is a global attractor of all solutions of (20), and so of all solutions of (18).

Example 5. Here we consider the following equation

$$x_{n+1} = \frac{1}{2}(x_n + x_{n-1} + \sin(x_{n-1})), \quad n = 0, 1, \ldots \tag{21}$$

on an interval $[0, 2K\pi]$, where K is an integer. This equation has $2K + 1$ equilibrium points $\bar{x}_m = m\pi, m = 0, 1, \ldots, K$. An immediate checking shows that $\bar{x}_{2m+1}$ are locally asymptotically stable, while $\bar{x}_{2m}$ are saddle equilibrium points with eigenvalues $\lambda_{\pm} = \frac{1\pm\sqrt{17}}{4}$. The expression $f(x, x)-x$, where $f(x, y) = \frac{1}{2}(x + y + \sin(y))$ becomes

$$f(x, x) - x = \frac{1}{2}\sin x < 0 \quad \text{if} \quad x \in ((2m + 1)\pi, (2m + 2)\pi).$$

This shows that the negative feedback condition is satisfied in the interval $(\bar{x}_{2m+1}, \bar{x}_{2m+2})$ and so the product of this interval by itself is a part of the basin of attraction of $\bar{x}_{2m+1}$. Similarly $(\bar{x}_{2m}, \bar{x}_{2m+1})^2$ is a part of the basin of attraction of $\bar{x}_{2m+1}$. In particular, Theorem 3 implies that all solutions of Eq.(21) converge to an equilibrium.

In fact, we can give precise descriptions of the basins of attraction of all equilibrium points. Theorem 14 can be used to show that

$$\mathcal{B}_{2m+1} = \{(x, y) : \exists y_u, y_l : \ y_l < y < y_u \quad (x, y_l) \in \mathcal{W}_{2m}, (x, y_u) \in \mathcal{W}_{2m+2}\}$$

and

$$\mathcal{B}_{2m} = \mathcal{W}_{2m},$$

where $\mathcal{W}_{2m} = \mathcal{W}_{(\bar{x}_{2m}, \bar{x}_{2m})}$ is the global stable manifold of the equilibrium point $(\bar{x}_{2m}, \bar{x}_{2m})$. In other words, the basin of attraction of $(\bar{x}_{2m+1}, \bar{x}_{2m+1})$ is the set of all points in the plane of initial conditions which are between the stable manifolds of two consecutive saddle equilibrium points $(\bar{x}_{2m}, \bar{x}_{2m})$ and $(\bar{x}_{2m+2}, \bar{x}_{2m+2})$. Indeed, we only need to show that Eq.(21) does not possess a minimal period-two solution $\Phi, \Psi, \Phi, \Psi, \ldots$. Such a solution would satisfy the system of equations

$$\sin(\Phi) = \Phi - \Psi, \quad \sin(\Psi) = \Psi - \Phi$$

and so

$$\Psi = \Phi - \sin(\Phi) = G(\Phi), \quad \Phi = \Psi - \sin(\Psi) = G(\Psi),$$

which implies that both Φ and Ψ are period-two points of an increasing function $G(x) = x - \sin(x)$, which is impossible.

These results are summarized in the plot in Figure 2.

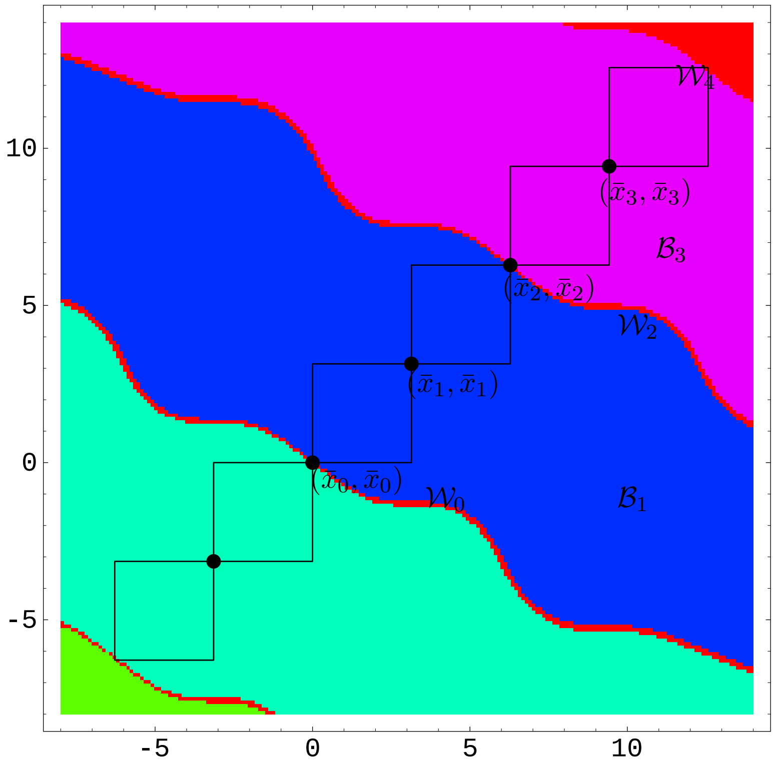

Figure 2: Figure shows the basins of attraction of Eq.(21). Figure was generated by *Dynamica* 3 [17].

Example 6. Here we consider the following equation

$$x_{n+1} = \frac{1}{2}(x_n + x_{n-1} + (\sin(x_{n-1}))^2), \quad n = 0, 1, \ldots \tag{22}$$

on an interval $[0, 2K\pi]$, where K is an integer. This equation has $2K + 1$ equilibrium points $\bar{x}_m = m\pi, m = 0, 1, \ldots, K$. An immediate checking shows that $\bar{x}_m$ are non-hyperbolic equilibrium points with eigenvalues equal to 1 and $-1/2$. The expression $f(x, x) - x$, where $f(x, y) = \frac{1}{2}(x + y + (\sin(y))^2)$ becomes

$$f(x, x) - x = \frac{1}{2}(\sin x)^2 > 0 \quad \text{for every} \quad x \neq \bar{x}_m.$$

This shows that the negative feedback condition is satisfied in the interval $(\bar{x}_{m-1}, \bar{x}_m)$ and so the product of this interval by itself is a part of the basin of attraction of $\bar{x}_m$. In particular, Theorem 3 implies that all solutions of Eq.(22) converge to an equilibrium. In fact, we can give precise descriptions of the basins of attraction of all equilibrium points. Theorem 14 can be used to show that

$$\mathcal{B}_m = \{(x,y) : \exists y_u, y_l : \; y_l < y < y_u \quad (x,y_l) \in \mathcal{W}_m, (x,y_u) \in \mathcal{W}_{m+1}\}$$

where $\mathcal{W}_m = \mathcal{W}_{(\bar{x}_m,\bar{x}_m)}$ is the global stable manifold of the equilibrium point $(\bar{x}_m,\bar{x}_m)$. In other words, the basin of attraction of $(\bar{x}_m,\bar{x}_m)$ is the set of all points in the plane of initial conditions which are between the stable manifolds of two consecutive non-hyperbolic equilibrium points $(\bar{x}_m,\bar{x}_m)$ and $(\bar{x}_{m+1},\bar{x}_{m+1})$.

Indeed, we only need to show that Eq.(22) does not possess a minimal period-two solution $\Phi, \Psi, \Phi, \Psi, \ldots$.

A minimal period-two solution would satisfy the system of equations

$$(\sin(\Phi))^2 = \Phi - \Psi, \quad (\sin(\Psi))^2 = \Psi - \Phi$$

and so

$$\Psi = \Phi - (\sin(\Phi))^2 = H(\Phi), \quad \Phi = \Psi - (\sin(\Psi))^2 = H(\Psi),$$

which implies that both Φ and Ψ are period-two points of an increasing function $H(x) = x - (\sin(x))^2$, which is impossible. A direct calculation shows that the eigenvector which corresponds to the smaller eigenvalue $\lambda = -1/2$ is $[2,\ -1]$.

Example 7. Here we consider the following equation

$$x_{n+1} = x_n^3 + x_{n-1}^3, \quad n = 0, 1, \ldots \tag{23}$$

on an interval $(-\infty, \infty)$. This equation has 3 equilibrium points $-1/\sqrt{2}, 0, 1/\sqrt{2}$, where $\pm 1/\sqrt{2}$ are saddle points and 0 is a local attractor, with both roots of characteristic equation equal to 0. Thus, the function $f(u,v) = u^3 + v^3$ is strictly increasing for all values $u \neq 0, v \neq 0$. All conditions of Theorem 14 are satisfied with respect to two saddle equilibrium points (period-two solution clearly does not exist), which guarantee the existence of two stable manifolds $\mathcal{W}^s((1/\sqrt{2}, 1/\sqrt{2}))$ and $\mathcal{W}^s((-1/\sqrt{2}, -1/\sqrt{2}))$, which are continuous, decreasing curves that extend indefinitely. In view of Theorem 9. the region between $\mathcal{W}^s((1/\sqrt{2}, 1/\sqrt{2}))$ and $\mathcal{W}^s((-1/\sqrt{2}, -1/\sqrt{2}))$ is invariant, as well as the regions of points which are north-east of $\mathcal{W}^s((1/\sqrt{2}, 1/\sqrt{2}))$ and south-west of $\mathcal{W}^s((-1/\sqrt{2}, -1/\sqrt{2}))$ and the basin of attraction of $(0,0)$ is precisely the region between $\mathcal{W}^s((1/\sqrt{2}, 1/\sqrt{2}))$ and $\mathcal{W}^s((-1/\sqrt{2}, -1/\sqrt{2}))$. All solutions which starts north-east of $\mathcal{W}^s((1/\sqrt{2}, 1/\sqrt{2}))$ goes to ∞ while all solutions which starts south-west of $\mathcal{W}^s((1/\sqrt{2}, 1/\sqrt{2}))$ goes to $-\infty$. In view of Theorem 3 all solutions are eventually monotonic. See Figure 3.

For example, $x_{-1} > 1, x_0 \geq 0$ implies $x_1 = x_0^3 + x_{-1}^3 \geq x_{-1}^3 = x_{-1}^2 x_{-1} > x_{-1}$ and $x_2 = x_1^3 + x_0^3 > x_{-1}^3 + x_0^3 = x_1$. Similarly we can prove that $\{x_n\}_{n \geq 1}$ is strictly increasing and so is asymptotic to ∞. Similar reasoning holds for $x_{-1} \geq 0, x_0 > 1$. The initial point $(1,1)$ generates the solution $\{1, 1, 2, 9, 737, \ldots\}$.

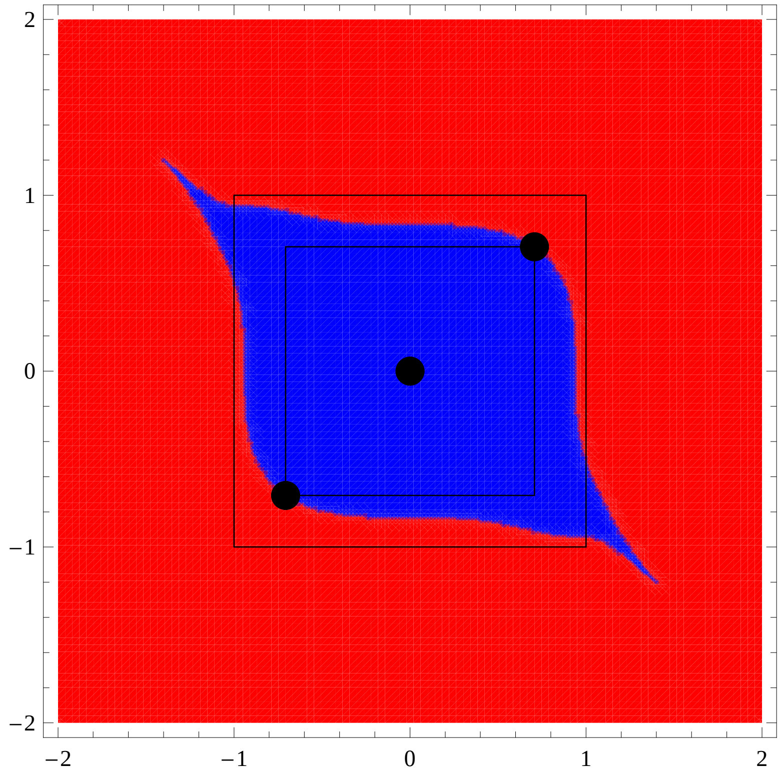

Figure 3: Figure shows the basins of attraction of Eq.(23). Figure was generated by *Dynamica* 3 [17].

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