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AN EXAMPLE OF A GLOBALLY ASYMPTOTICALLY STABLE ANTI-MONOTONIC SYSTEM OF RATIONAL DIFFERENCE EQUATIONS IN THE PLANE

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Dedicated to Professor Harry Miller on the occasion of his 70th birthday

Abstract. We consider the following system of rational difference equations in the plane:

$$\begin{cases} x_{n+1} = \dfrac{\alpha_1}{A_1 + B_1 x_n + C_1 y_n} \\ y_{n+1} = \dfrac{\alpha_2}{A_2 + B_2 x_n + C_2 y_n} \end{cases}, \quad n = 0, 1, 2, \dots$$

where the parameters $\alpha_1, \alpha_2, A_1, A_2, B_1, B_2, C_1, C_2$ are positive numbers and initial conditions x_0 and y_0 are nonnegative numbers. We prove that the unique positive equilibrium of this system is globally asymptotically stable. Also, we determine the rate of convergence of a solution that converges to the equilibrium $E = (\bar{x}, \bar{y})$ of this systems.

1. Introduction and preliminary results

Let $\mathcal{I}$ and $\mathcal{J}$ be intervals of real numbers. Consider a first order system of difference equations of the form

$$\begin{cases} x_{n+1} = f(x_n, y_n) \\ y_{n+1} = g(x_n, y_n) \end{cases}, \quad n = 0, 1, 2, \dots \tag{1}$$

where

$$f : \mathcal{I} \times \mathcal{J} \to \mathcal{I}, \; g : \mathcal{I} \times \mathcal{J} \to \mathcal{J} \text{ and } (x_0, y_0) \in \mathcal{I} \times \mathcal{J}.$$

When the function $f(x, y)$ is increasing in x and decreasing in y and the function $g(x, y)$ is decreasing in x and increasing in y, the system (1) is called *competitive*. One can consider a map $T = (f(x, y), g(x, y))$ associated with the system (1) and define the notions of competitive map accordingly.

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If $\mathbf{v} = (u, v) \in \mathbb{R}^2$, we denote with $\mathcal{Q}_\ell(\mathbf{v})$, $\ell \in \{1, 2, 3, 4\}$, the four quadrants in $\mathbb{R}^2$ relative to $\mathbf{v}$, i.e., $\mathcal{Q}_1(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 \ : x \geq u,\ y \geq v\}$, $\mathcal{Q}_2(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 \ : x \leq u,\ y \geq v\}$, and so on. Define the *South-East* partial order $\preceq_{se}$ on $\mathbb{R}^2$ by $(x, y) \preceq_{se} (s, t)$ if and only if $x \leq s$ and $y \geq t$. Similarly, we define the *North-East* partial order $\preceq_{ne}$ on $\mathbb{R}^2$ by $(x, y) \preceq_{ne} (s, t)$ if and only if $x \leq s$ and $y \leq t$. For $\mathcal{A} \subset \mathbb{R}^2$ and $\mathrm{x} \in \mathbb{R}^2$, define the *distance from* x *to* $\mathcal{A}$ as $\mathrm{dist}(\mathrm{x}, \mathcal{A}) := \inf\{\|\mathrm{x} - \mathrm{y}\| \ : \mathrm{y} \in \mathcal{A}\}$. By $\operatorname{int}\mathcal{A}$ we denote the interior of a set $\mathcal{A}$.

It is easy to show that a map F is competitive if it is non-decreasing with respect to the South-East partial order, that is if the following holds:

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{se} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \Rightarrow F\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{se} F\begin{pmatrix} x^2 \\ y^2 \end{pmatrix}. \tag{2}$$

For standard definitions of attracting fixed point, saddle point, stable manifold, and related notions see [7, 9, 10] and [16].

When the function $f(x, y)$ is increasing in x and increasing in y and the function $g(x, y)$ is increasing in x and increasing in y System (1) is called *cooperative*. *Strongly competitive* systems of difference equations or strongly competitive maps are those for which the functions f and g are coordinate-wise strictly monotone.

System (1) where the functions f and g have monotonic character opposite of the monotonic character in competitive system will be called *anti-competitive*, while system (1) where the functions f and g have monotonic character opposite of the monotonic character in cooperative system will be called *anti-cooperative*. Anti-competitive and anti-cooperative systems will be called *anti-monotone* systems.

Competitive and cooperative systems have been investigated by many authors, see [2, 3, 4, 9, 12, 17] and others. The study of anti-monotone systems started recently in [5]. Recently large number of open problems about the competitive systems of the special form

$$\begin{cases} x_{n+1} = \dfrac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n} \\[2ex] y_{n+1} = \dfrac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{cases}, \quad n = 0, 1, 2, \ldots \tag{3}$$

where $\alpha_1, \beta_1, \gamma_1, A_1, B_1, C_1, \alpha_2, \beta_2, \gamma_2, A_2, B_2, C_2$ are nonnegative constants and $x_0, y_0 \geq 0$ are initial conditions such that $A_1 + B_1 x_0 + C_1 y_0 > 0$ and $A_2 + B_2 x_0 + C_2 y_0 > 0$, have been posed in [1]. The system considered here is labeled as the system $(37, 37)$ in [1]. The rational system of difference equations play an important role in modeling in biology and economics, see [6] and [7].

The following result gives a convergence result for a system in $\mathcal{R}^2$ when there exists an invariant rectangle and the map of the system satisfies certain monotonicity and algebraic conditions. See [8] and [6, 11].

Theorem 1. *Let* $\mathcal{R} = [a,b] \times [c,d]$ *and*

$$f : \mathcal{R} \to [a,b],\ g : \mathcal{R} \to [c,d]$$

be a continuous functions such that:

(a) $f(x,y)$ *is decreasing in both variables and* $g(x,y)$ *is decreasing in both variables for each* $(x,y) \in \mathcal{R}$;

(b) *If* $(m_1, M_1, m_2, M_2) \in \mathcal{R}^2$ *is a solution of*

$$\begin{cases} M_1 = f(m_1, m_2),\ m_1 = f(M_1, M_2) \\ M_2 = g(m_1, m_2),\ m_2 = g(M_1, M_2) \end{cases} \tag{4}$$

then $m_1 = M_1$ *and* $m_2 = M_2$. *Then the system* (1) *has a unique equilibrium* $(\bar{x}, \bar{y})$ *and every solution of* (1) *with* $(x_0, y_0) \in \mathcal{R}$ *converges to the unique equilibrium* $(\bar{x}, \bar{y})$. *In addition, the equilibrium* $(\bar{x}, \bar{y})$ *is globally asymptotically stable.*

Variations on Theorem 1 have appeared and have been used very effectively to completely settle global attractivity issues of many systems of rational equations, see [6, 11, 12].

In this paper we want to give an example of an anti-monotonic system with a unique equilibrium which is globally asymptotically stable.

In Section 2 we consider the following system of difference equations

$$\begin{cases} x_{n+1} = \dfrac{\alpha_1}{A_1 + B_1 x_n + C_1 y_n} \\ y_{n+1} = \dfrac{\alpha_2}{A_2 + B_2 x_n + C_2 y_n} \end{cases}, \quad n = 0, 1, 2, \ldots, \tag{5}$$

where the parameters $\alpha_1, \alpha_2, A_1, A_2, B_1, B_2, C_1, C_2$ are positive numbers and initial conditions x_0 and y_0 are positive initial conditions. This system has exactly one equilibrium point $E(\bar{x}, \bar{y})$ which is locally asymptotically stable for all values of parameters. We use Theorem 1 to show that the equilibrium point $E(\bar{x}, \bar{y})$ is globally asymptotically stable.

Finally, in Section 3 we give the rate of convergence of a solution that converges to the equilibrium $E(\bar{x}, \bar{y})$ of the system of difference equations (5) for all values of parameters. The rate of convergence of solutions that converge to an equilibrium has been obtained for some two-dimensional systems in [13] and [14].

The following results give the rate of convergence of solutions of a system of difference equations

$$\mathbf{x}_{n+1} = [A + B(n)]\mathbf{x}_n, \tag{6}$$

where $\mathbf{x}_n$ is a k-dimensional vector, $A \in \mathbf{C}^{k\times k}$ is a constant matrix, and $B : \mathbf{Z}^+ \to \mathbf{C}^{k\times k}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \text{ when } n \to 0, \tag{7}$$

where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm; $\|\cdot\|$ also denotes the Euclidean norm in $\mathbb{R}^2$ given by

$$\|\mathbf{x}\| = \|\,(x, y)\,\| = \sqrt{x^2 + y^2}. \tag{8}$$

Theorem 2. ([15]) *Assume that condition* (7) *holds. If* $\mathbf{x}_n$ *is a solution of system* (6)*, then either* $\mathbf{x}_n = \mathbf{0}$ *for all large* n *or*

$$\varrho = \lim_{n\to\infty} \sqrt[n]{\|\mathbf{x}_n\|} \tag{9}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 3. ([15]) *Assume that condition* (7) *holds. If* $\mathbf{x}_n$ *is a solution of the system* (6)*, then either* $\mathbf{x}_n = \mathbf{0}$ *for all large* n *or*

$$\varrho = \lim_{n\to\infty} \frac{\|\mathbf{x}_{n+1}\|}{\|\mathbf{x}_n\|} \tag{10}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

2. Dynamics of the system (5)

In this section we consider system of difference equations (5).

Theorem 4. *System* (5) *has the unique positive equilibrium* $E = (\bar{x}, \bar{y})$ *which is globally asymptotically stable.*

Proof. The equilibrium point of the system (5) satisfies the following system of equations

$$\begin{aligned} \bar{x} &= \frac{\alpha_1}{A_1 + B_1\bar{x} + C_1\bar{y}} \\ \bar{y} &= \frac{\alpha_2}{A_2 + B_2\bar{x} + C_2\bar{y}} \end{aligned} \tag{11}$$

System (11) implies

$$B_1\bar{x}^2 + \bar{x}(A_1 + C_1\bar{y}) - \alpha_1 = 0 \tag{12}$$

$$C_2\bar{y}^2 + \bar{y}(A_2 + B_2\bar{x}) - \alpha_2 = 0 \tag{13}$$

Equations (12) and (13) imply, respectively

$$\bar{x} = \frac{-A_1 - C_1\bar{y} + \sqrt{(A_1 + C_1\bar{y})^2 + 4\alpha_1}}{2B_1},$$

and

$$\bar{y} = \frac{-A_2 - B_2\bar{x} + \sqrt{(A_2 + B_2\bar{x})^2 + 4\alpha_2}}{2C_2}.$$

Clearly, the unique positive equilibrium is $E = (\bar{x}, \bar{y})$. Also, system (11) implies

$$\bar{x} = \frac{\alpha_2 - A_2\bar{y} - C_2\bar{y}^2}{B_2\bar{y}}, \tag{14}$$

$$\bar{y} = \frac{\alpha_1 - A_1\bar{x} - B_1\bar{x}^2}{C_1\bar{x}}. \tag{15}$$

The map T associated to the system (5) is

$$T(x, y) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha_1}{A_1 + B_1x + C_1y} \\ \dfrac{\alpha_2}{A_2 + B_2x + C_2y} \end{pmatrix} \tag{16}$$

The Jacobian matrix of T is

$$J_T = \begin{pmatrix} -\dfrac{\alpha_1 B_1}{(A_1 + B_1x + C_1y)^2} & -\dfrac{\alpha_1 C_1}{(A_1 + B_1x + C_1y)^2} \\ -\dfrac{\alpha_2 B_2}{(A_2 + B_2x + C_2y)^2} & -\dfrac{\alpha_2 C_2}{(A_2 + B_2x + C_2y)^2} \end{pmatrix}. \tag{17}$$

By using the equations (14) and (15), value of the Jacobian matrix of T at the equilibrium point $E = (\bar{x}, \bar{y})$ is

$$J_T(\bar{x}, \bar{y}) = \begin{pmatrix} -\dfrac{B_1\bar{x}^2}{\alpha_1} & -\dfrac{C_1\bar{x}^2}{\alpha_1} \\ -\dfrac{B_2\bar{y}^2}{\alpha_2} & -\dfrac{C_2\bar{y}^2}{\alpha_2} \end{pmatrix}. \tag{18}$$

The determinant of (18) is given by

$$\det J_T(\bar{x}, \bar{y}) = \frac{\bar{x}^2\bar{y}^2}{\alpha_1\alpha_2}(B_1C_2 - B_2C_1).$$

The trace of (18) is

$$Tr J_T(\bar{x}, \bar{y}) = -\frac{\alpha_2 B_1\bar{x}^2 + \alpha_1 C_2\bar{y}^2}{\alpha_1\alpha_2}.$$

The characteristic equation has the form

$$\lambda^2+\lambda\frac{\alpha_2 B_1\bar{x}^2+\alpha_1 C_2\bar{y}^2}{\alpha_1\alpha_2}+\frac{\bar{x}^2\bar{y}^2}{\alpha_1\alpha_2}(B_1C_2-B_2C_1)=0.$$

Instead of proving local stability by standard test, which is a fairly complicated task, we will prove global asymptotic stability which will implies the local stability as well. We will use Theorem 1.

First, system (5) implies

$$x_{n+1}\le\frac{\alpha_1}{A_1}\quad\text{and}\quad y_{n+1}\le\frac{\alpha_2}{A_2}.$$

This shows that

$$\left[0,\frac{\alpha_1}{A_1}\right]\times\left[0,\frac{\alpha_2}{A_2}\right],$$

is an invariant and attracting box.

The second condition of Theorem 1 has the following form

$$M_1=\frac{\alpha_1}{A_1+B_1m_1+C_1m_2},\quad M_2=\frac{\alpha_2}{A_2+B_2m_1+C_2m_2}$$

$$m_1=\frac{\alpha_1}{A_1+B_1M_1+C_1M_2},\quad m_2=\frac{\alpha_2}{A_2+B_2M_1+C_2M_2}.$$

These equations imply:

$$A_1M_1+B_1M_1m_1+C_1M_1m_2=\alpha_1,\quad A_2M_2+B_2m_1M_2+C_2M_2m_2=\alpha_2$$

$$A_1m_1+B_1M_1m_1+C_1m_1M_2=\alpha_1,\quad A_2m_2+B_2M_1m_2+C_2M_2m_2=\alpha_2.$$

By using algebraic manipulations, we obtain

$$A_1(M_1-m_1)+C_1(M_1m_2-m_1M_2)=0, \tag{19}$$

$$A_2(M_2-m_2)+B_2(m_1M_2-M_1m_2)=0. \tag{20}$$

This implies

$$-\frac{A_1}{C_1}(M_1-m_1)=\frac{A_2}{B_2}(M_2-m_2)\quad(M_1\ge m_1,\quad M_2\ge m_2).$$

Assuming that $M_2>m_2$ this implies $m_1>M_1$, which is a contradiction. Since $M_2=m_2$ and $M_1=m_1$. The conclusion of this theorem follows from Theorem 1 and the fact that Theorem 1 does not give only global attractivity but global stability as well. □

Remark 1. We can prove that each of the following systems of rational difference equations in the plane

$$\begin{cases} x_{n+1} = \dfrac{\alpha_1}{A_1 + x_n} \\ y_{n+1} = \dfrac{\alpha_2}{A_2 + x_n} \end{cases}, \quad n = 0, 1, 2, \ldots$$

$$\begin{cases} x_{n+1} = \dfrac{\alpha_1}{A_1 + y_n} \\ y_{n+1} = \dfrac{\alpha_2}{A_2 + y_n} \end{cases}, \quad n = 0, 1, 2, \ldots$$

$$\begin{cases} x_{n+1} = \dfrac{\alpha_1}{A_1 + y_n} \\ y_{n+1} = \dfrac{\alpha_2}{A_2 + x_n} \end{cases}, \quad n = 0, 1, 2, \ldots$$

possesses a unique equilibrium which is globally asymptotically stable. However, the proofs are quite different than the proof of Theorem 4 and are based on the explicit formula for the solution of the Riccati's equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + B x_n}, \quad n = 0, 1, \ldots,$$

see [6, 7].

3. Rate of convergence

Our goal in this Section is to determine the rate of convergence of every solution of the system (5) in the regions where the parameters $\alpha_1, \alpha_2, A_1, A_2, B_1, B_2, C_1, C_2$ are positive numbers and initial conditions x_0 and y_0 are arbitrary, nonnegative numbers.

Theorem 5. *The error vector*

$$\mathbf{e}_n = \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix} = \begin{pmatrix} x_n - \overline{x} \\ y_n - \overline{y} \end{pmatrix}$$

of every solution $\mathbf{x}_n \neq \mathbf{0}$ *of* (5) *satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} \sqrt[n]{\|\mathbf{e}_n\|} = |\lambda_i(J_T(E))| \quad \text{for some } i = 1, 2, \tag{21}$$

and

$$\lim_{n\to\infty} \frac{\|\mathbf{e}_{n+1}\|}{\|\mathbf{e}_n\|} = |\lambda_i(J_T(E))| \quad \text{for some } i = 1, 2, \tag{22}$$

where $|\lambda_i(J_T(E))|$ *is equal to the modulus of one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium* $J_T(E)$*.*

Proof. First we will find a system satisfied by the error terms. The error terms are given as (using (5) and (11))

$$\begin{aligned}
x_{n+1}-\overline{x} &= \frac{\alpha_1}{A_1+B_1x_n+C_1y_n}-\frac{\alpha_1}{A_1+B_1\bar{x}+C_1\overline{y}}\\
&=\alpha_1\frac{B_1(\bar{x}-x_n)+C_1(\bar{y}-y_n)}{(A_1+B_1x_n+C_1y_n)(A_1+B_1\bar{x}+C_1\bar{y})}\\
&=\frac{-\alpha_1B_1}{(A_1+B_1x_n+C_1y_n)(A_1+B_1\bar{x}+C_1\bar{y})}(x_n-\bar{x})\\
&\quad+\frac{-\alpha_1C_1}{(A_1+B_1x_n+C_1y_n)(A_1+B_1\bar{x}+C_1\bar{y})}(y_n-\bar{y})\\
&=\frac{-B_1\bar{x}}{A_1+B_1x_n+C_1y_n}(x_n-\bar{x})+\frac{-C_1\bar{x}}{A_1+B_1x_n+C_1y_n}(y_n-\bar{y}),
\end{aligned}$$

and

$$\begin{aligned}
y_{n+1}-\bar{y} &= \frac{\alpha_2}{A_2+B_2x_n+C_2y_n}-\frac{\alpha_2}{A_2+B_2\bar{x}+C_2\bar{y}}\\
&=\alpha_2\frac{B_2(\bar{x}-x_n)+C_2(\bar{y}-y_n)}{(A_2+B_2x_n+C_2y_n)(A_2+B_2\bar{x}+C_2\bar{y})}\\
&=\frac{-\alpha_2B_2}{(A_2+B_2x_n+C_2y_n)(A_2+B_2\bar{x}+C_2\bar{y})}(x_n-\bar{x})\\
&\quad+\frac{-\alpha_2C_2}{(A_2+B_2x_n+C_2y_n)(A_2+B_2\bar{x}+C_2\bar{y})}(y_n-\bar{y})\\
&=\frac{-B_2\bar{y}}{A_2+B_2x_n+C_2y_n}(x_n-\bar{x})+\frac{-C_2\bar{y}}{A_2+B_2x_n+C_2y_n}(y_n-\bar{y}).
\end{aligned}$$

That is

$$\left.\begin{aligned}
x_{n+1}-\bar{x} &= \frac{-B_1\bar{x}}{A_1+B_1x_n+C_1y_n}\left(x_n-\bar{x}\right)+\frac{-C_1\bar{x}}{A_1+B_1x_n+C_1y_n}\left(y_n-\bar{y}\right),\\
y_{n+1}-\bar{y} &= \frac{-B_2\bar{y}}{A_2+B_2x_n+C_2y_n}\left(y_n-\bar{y}\right)+\frac{-C_2\bar{y}}{A_2+B_2x_n+C_2y_n}\left(x_n-\bar{x}\right).
\end{aligned}\right\} \tag{23}$$

Set

$$e_n^1=x_n-\overline{x}\quad\text{and}\quad e_n^2=y_n-\overline{y}.$$

Then system (23) can be represented as

$$\begin{aligned}
e_{n+1}^1&=a_ne_n^1+b_ne_n^2\\
e_{n+1}^2&=d_ne_n^1+c_ne_n^2
\end{aligned}$$

where

$$a_n = \frac{-B_1\bar{x}}{A_1 + B_1 x_n + C_1 y_n}, \quad b_n = \frac{-C_1\bar{x}}{A_1 + B_1 x_n + C_1 y_n},$$
$$c_n = \frac{-B_2\bar{y}}{A_2 + B_2 x_n + C_2 y_n}, \quad d_n = \frac{-C_2\bar{y}}{A_2 + B_2 x_n + C_2 y_n}.$$

Taking the limits of a_n, b_n, c_n and d_n, we obtain

$$\lim_{n\to\infty} a_n = \frac{-B_1\bar{x}}{A_1 + B_1\bar{x} + C_1\bar{y}}, \quad \lim_{n\to\infty} b_n = \frac{-C_1\bar{x}}{A_1 + B_1\bar{x} + C_1\bar{y}},$$
$$\lim_{n\to\infty} c_n = \frac{-B_2\bar{y}}{A_2 + B_2\bar{x} + C_2\bar{y}}, \quad \lim_{n\to\infty} d_n = \frac{-C_2\bar{y}}{A_2 + B_2\bar{x} + C_2\bar{y}},$$

that is

$$a_n = \frac{-B_1\bar{x}}{A_1 + B_1\bar{x} + C_1\bar{y}} + \alpha_n, \quad b_n = \frac{-C_1\bar{x}}{A_1 + B_1\bar{x} + C_1\bar{y}} + \beta_n,$$
$$c_n = \frac{-B_2\bar{y}}{A_2 + B_2\bar{x} + C_2\bar{y}} + \gamma_n, \quad d_n = \frac{-C_2\bar{y}}{A_2 + B_2\bar{x} + C_2\bar{y}} + \delta_n.$$

where

$$\alpha_n \to 0,\ \beta_n \to 0,\ \gamma_n \to 0 \text{ and } \delta_n \to 0 \text{ when } n \to \infty.$$

Now we have system of the form (5):

$$\mathbf{e}_{n+1} = (A + B(n))\,\mathbf{e}_n,$$

where

$$A = \begin{pmatrix} \dfrac{-B_1\bar{x}}{A_1 + B_1\bar{x} + C_1\bar{y}} & \dfrac{-C_1\bar{x}}{A_1 + B_1\bar{x} + C_1\bar{y}} \\ \dfrac{-B_2\bar{y}}{A_2 + B_2\bar{x} + C_2\bar{y}} & \dfrac{-C_2\bar{y}}{A_2 + B_2\bar{x} + C_2\bar{y}} \end{pmatrix}, \ B(n) = \begin{pmatrix} \alpha_n & \beta_n \\ \delta_n & \gamma_n \end{pmatrix}$$

and

$$\|B(n)\| \to 0 \quad \text{when} \quad n \to \infty.$$

Thus, the limiting system of error terms can be written as:

$$\begin{pmatrix} e^1_{n+1} \\ e^2_{n+1} \end{pmatrix} = \begin{pmatrix} \dfrac{-B_1\bar{x}}{A_1 + B_1\bar{x} + C_1\bar{y}} & \dfrac{-C_1\bar{x}}{A_1 + B_1\bar{x} + C_1\bar{y}} \\ \dfrac{-B_2\bar{y}}{A_2 + B_2\bar{x} + C_2\bar{y}} & \dfrac{-C_2\bar{y}}{A_2 + B_2\bar{x} + C_2\bar{y}} \end{pmatrix} \begin{pmatrix} e^1_n \\ e^2_n \end{pmatrix}.$$

The system is exactly linearized system of (5) evaluated at the equilibrium $E = (\bar{x}, \bar{y})$. Then Theorems 2 and 3 imply the result. □

When $E = (\overline{x}, \overline{y})$, we obtain the following result.

Corollary 1. *Assume that* $\alpha_1, \alpha_2, A_1, A_2, B_1, B_2, C_1, C_2$ *are positive numbers. Then the equilibrium point* $E(\bar{x}, \bar{y})$ *is globally asymptotically stable. The error vector*

$$\mathbf{e}_n = \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

of every solution $\mathbf{x}_n$ *of* (5) *satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} \sqrt[n]{\|\mathbf{e}_n\|} = \lim_{n\to\infty} \sqrt[2n]{x_n^2 + y_n^2} = |\lambda_i\left(J_T\left(E\right)\right)| \quad \text{for some} \quad i = 1, 2,$$

and

$$\lim_{n\to\infty} \frac{\|\mathbf{e}_{n+1}\|}{\|\mathbf{e}_n\|} = \lim_{n\to\infty} \sqrt{\frac{x_{n+1}^2 + y_{n+1}^2}{x_n^2 + y_n^2}} = |\lambda_i\left(J_T\left(E\right)\right)| \quad \text{for some} \quad i = 1, 2,$$

where $|\lambda_i\left(J_T\left(E\right)\right)|$ *is equal to the modulus of one the eigenvalues of the Jacobian matrix evaluated at the equilibrium* E.

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