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THE VALUE OF CERTAIN COMBINATORICS SUM

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Dedicated to Professor Harry Miller on the occasion of his 70th birthday

Abstract. In this paper we analyze the values and the properties of the function $S(n,l) := \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^l$ $(n,l \in \mathbb{N}\cup\{0\})$, for $n < l$. At first, we obtain two recurrence relations. Namely, we prove that for every $n \in \mathbb{N}\cup\{0\}$ and every $l \in \mathbb{N}$ such that $l > n$, we have

$$S(n+1,l) = \sum_{k=1}^{l-n}\binom{l}{k}S(n,l-k),$$

and also, for every $n \in \mathbb{N}\cup\{0\}$and every $l \in \mathbb{N}$, we have

$$S(n+1,l) = (n+1)S(n,l-1) + (n+1)S(n+1,l-1).$$

Further, we conclude that for every $n \geq 2$ and every $l \geq n$ the following representation formula holds

$$S(n,l) = \sum_{k_1=1}^{l-(n-1)}\binom{l}{k_1}\sum_{k_2=1}^{l-k_1-(n-2)}\binom{l-k_1}{k_2} \cdot \sum_{k_3=1}^{l-k_1-k_2-(n-3)}\binom{l-k_1-k_2}{k_3}\cdots\sum_{k_{n-1}=1}^{l-\sum_{i=1}^{n-2}k_i-1}\binom{l-\sum_{i=1}^{n-2}k_i}{k_{n-1}}.$$

We obtain an explicit formula for the calculation $S(n,l)$, especially for $l = n+1,\ldots,n+5$, and later we give a general result.

1. Preliminaries

Define the function S of two nonnegative integers as

$$S(n,l) := \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^l \ \ (n,l \in \mathbb{N}\cup\{0\}).$$

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Especially, we define that $S(0,0) := 1$. It is a known fact that for $n \in \mathbb{N}$ and $l \in \{0,1,\ldots,n-1\}$, $S(n,l)=0$. Namely, by differentiating the function $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$ with respect to x, we obtain

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}x^{k-1}. \tag{R1}$$

In particular, from (R1) for $x=-1$, we have that for $n \geq 2$

$$\sum_{k=0}^{n}(-1)^k k\binom{n}{k} = 0.$$

By differentiating in relation (R1) we obtain

$$n(n-1)(1+x)^{n-2} = \sum_{k=0}^{n} k(k-1)\binom{n}{k}x^{k-2}. \tag{R2}$$

From (R2) for $x=-1$, for $n \geq 3$ we have $\sum_{k=0}^{n}(-1)^k k(k-1)\binom{n}{k} = 0$. Since $\sum_{k=0}^{n}(-1)^k k\binom{n}{k} = 0$, for $n \geq 3$ we have

$$\sum_{k=0}^{n}(-1)^k k^2\binom{n}{k} = 0.$$

Further, by differentiating of function $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$ $l-$times, where $l<n$, and using mathematical induction one can easy to prove that

$$\sum_{k=0}^{n}(-1)^k k^l\binom{n}{k} = 0,$$

for every $n \in \mathbb{N}$ and $l \in \{0,1,\ldots,n-1\}$.

It is known that for every $n \in \mathbb{N}$, $S(n,n)=n!$ (for example, see [1], page 28). For $n<l$, the sum $S(n,l)$ is interesting because in combinatorics it represents the number of all possible permutations with repetitions of a set of l elements with n classes ([2], page 216), i.e., for $n<l$,

$$S(n,l) = \sum_{\substack{k_1,\ldots,k_n \in \mathbb{N} \\ k_1+\cdots+k_n=l}} \frac{l!}{k_1!\ldots k_n!} = l! \sum_{\substack{k_1,\ldots,k_n \in \mathbb{N} \\ k_1+\cdots+k_n=l}} \frac{1}{k_1!\ldots k_n!}. \tag{R3}$$

Also, there exists a relationship for the sum $S(n,l)$ using Stirling's numbers of the second kind. Namely, for $n,l \in \mathbb{N}$, $n \leq l$,

$$S(n,l) = n! \cdot S_2(l,n),$$

where with $S_2(l,n)$ we denote Stirling's numbers of the second kind ([2], pages 100-101).

In the following section we give our results about the values of the function $S(n,l)$ for $n < l$.

2. Results

Lemma 1. *For every $n \in \mathbb{N} \cup \{0\}$ and every $l \in \mathbb{N}$ such that $l > n$, we have*

$$S(n+1,l) = \sum_{k=1}^{l-n} \binom{l}{k} S(n,l-k) \tag{1}$$

Proof. For a fixed $n \in \mathbb{N} \cup \{0\}$ and $l \in \mathbb{N}$, $l > n$, we have

$$\begin{aligned}
S(n+1,l) &= (-1)^{n+1} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} k^l \\
&= (-1)^{n+1} \sum_{k=1}^{n+1} (-1)^k \left[\binom{n}{k} + \binom{n}{k-1} \right] k^l \\
&= (-1)^{n+1} \sum_{k=1}^{n+1} (-1)^k \binom{n}{k} k^l + (-1)^{n+1} \sum_{k=1}^{n+1} (-1)^k \binom{n}{k-1} k^l \\
&= (-1)^{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^l + (-1)^{n+1} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} (k+1)^l \\
&= -S(n,l) + (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \\
&\qquad \cdot \left[k^l + \binom{l}{1} k^{l-1} + \cdots + \binom{l}{l-1} k + 1 \right] \\
&= -S(n,l) + S(n,l) + \binom{l}{1} S(n,l-1) + \dots \\
&\qquad + \binom{l}{l-1} S(n,1) + S(n,0) \\
&= \sum_{k=1}^{l} \binom{l}{k} S(n,l-k).
\end{aligned}$$

If $k \in \{l-n+1,\dots,l\}$, then $l-k \in \{0,\dots,n-1\}$, so that for $k \in \{l-n+1,\dots,l\}$ we have $S(n,l-k) = 0$. Therefore (1) holds. □

Theorem 2. *For every $n \in \mathbb{N} \cup \{0\}$ and every $l \in \mathbb{N}$,*

$$S(n+1,l) = (n+1)S(n,l-1) + (n+1)S(n+1,l-1). \tag{2}$$

Proof.

$$\begin{aligned}
S(n+1,l) &= (n+1)^l - \binom{n+1}{1} n^l + \binom{n+1}{2}(n-1)^l + \dots \\
&\qquad + (-1)^n \binom{n+1}{n} 1^l \\
&= (n+1)\left[(n+1)^{l-1} - \binom{n+1}{1} n^{l-1} + \dots + (-1)^n \binom{n+1}{n} 1^{l-1}\right] \\
&+ (n+1)\left[\binom{n+1}{1} n^{l-1} - \binom{n+1}{2}(n-1)^{l-1} + \dots + (-1)^{n-1}\binom{n+1}{n} 1^{l-1}\right] \\
&\quad - \binom{n+1}{1} n^l + \binom{n+1}{2}(n-1)^l - \dots + (-1)^n \binom{n+1}{n} 1^l \\
&= (n+1)S(n+1,l-1) + \left[(n+1)(n+1) - n(n+1)\right] n^{l-1} \\
&\quad + \left[(n-1)\binom{n+1}{2} - (n+1)\binom{n+1}{2}\right](n-1)^{l-1} \\
&\quad + \dots + \left[(-1)^n \binom{n+1}{n} + (n+1)(-1)^{n-1}\binom{n+1}{n}\right] 1^{l-1} \\
&= (n+1)S(n+1,l-1) + \binom{n+1}{1} n^{l-1} - 2\binom{n+1}{2}(n-1)^{l-1} \\
&\quad + \dots + (-1)^{n-1} n \binom{n+1}{n} 1^{l-1}.
\end{aligned}$$

For every $k \in \{2, \dots, n\}$ we have

$$\begin{aligned}
k\binom{n+1}{k} &= k\frac{(n+1)!}{k!(n+1-k)!} \\
&= (n+1)\frac{n!}{(k-1)!(n-(k-1))!} = (n+1)\binom{n}{k-1}.
\end{aligned}$$

Therefore, we have further

$$\begin{aligned}
S(n+1,l) &= (n+1)S(n+1,l-1) \\
&\quad + (n+1)\left[n^{l-1} - \binom{n}{1}(n-1)^{l-1} + \dots + (-1)^{n-1}\binom{n}{n-1} 1^{l-1}\right] \\
&= (n+1)S(n+1,l-1) + (n+1)S(n,l-1).
\end{aligned}$$

□

By (1), we obtain for $l \geq 2$,

$$S(2,l) = \sum_{k=1}^{l-1} \binom{l}{k} S(1,l-k) = \sum_{k=1}^{l-1} \binom{l}{k} = 2^l - 2, \tag{3}$$

because $S(1,l)=1$ for every $l\in\mathbb{N}$. Also, for $l\geq 3$,

$$S(3,l)=\sum_{k=1}^{l-2}\binom{l}{k}S(2,l-k)=\sum_{k=1}^{l-2}\binom{l}{k}\sum_{i=1}^{l-k-1}\binom{l-k}{i}. \tag{4}$$

Now, for $l\geq 4$ we have

$$S(4,l)=\sum_{k=1}^{l-3}\binom{l}{k}S(3,l-k)=\sum_{k=1}^{l-3}\binom{l}{k}\sum_{i=1}^{l-k-2}\binom{l-k}{i}\sum_{j=1}^{l-k-i-1}\binom{l-k-i}{j}. \tag{5}$$

If we continue this procedure $(n-1)$-times, then we obtain the following result.

Theorem 3. *For every $n\geq 2$ and every $l\geq n$ it holds*

$$S(n,l)=\sum_{k_1=1}^{l-(n-1)}\binom{l}{k_1}\sum_{k_2=1}^{l-k_1-(n-2)}\binom{l-k_1}{k_2}\cdot\sum_{k_3=1}^{l-k_1-k_2-(n-3)}\binom{l-k_1-k_2}{k_3}\cdots\sum_{k_{n-1}=1}^{l-\sum_{i=1}^{n-2}k_i-1}\binom{l-\sum_{i=1}^{n-2}k_i}{k_{n-1}}. \tag{6}$$

Example 4. a) By (4) we have

$$\begin{aligned}S(3,5)&=\sum_{k=1}^{3}\binom{5}{k}\sum_{i=1}^{4-k}\binom{5-k}{i}\\&=\binom{5}{1}\left[\binom{4}{1}+\binom{4}{2}+\binom{4}{3}\right]+\binom{5}{2}\left[\binom{3}{1}+\binom{3}{2}\right]+\binom{5}{3}\binom{2}{1}\\&=5\cdot 14+10\cdot 6+10\cdot 2=150.\end{aligned}$$

b) By (5) we have

$$\begin{aligned}S(4,6)&=\sum_{k=1}^{3}\binom{6}{k}\sum_{i=1}^{4-k}\binom{6-k}{i}\sum_{j=1}^{5-k-i}\binom{6-k-i}{j}\\&=\binom{6}{1}\binom{5}{1}\sum_{j=1}^{3}\binom{4}{j}+\binom{6}{1}\binom{5}{2}\sum_{j=1}^{2}\binom{3}{j}+\binom{6}{1}\binom{5}{3}\binom{2}{1}\\&\quad+\binom{6}{2}\binom{4}{1}\sum_{j=1}^{2}\binom{3}{j}+\binom{6}{2}\binom{4}{2}\binom{2}{1}+\binom{6}{3}\binom{3}{1}\binom{2}{1}\\&=30(2^4-2)+60(2^3-2)+120+60(2^3-2)+180+120=1560.\end{aligned}$$

Proposition 5. *For every* $n \in \mathbb{N} \cup \{0\}$

$$S(n, n+1) = \frac{n}{2}(n+1)! \tag{7}$$

Proof. 1) Obviously, the assertion of the proposition holds for $n = 0$ and $n = 1$. Assume that (7) holds for a fixed $n \in \mathbb{N}$. Then, by (2) we have

$$\begin{aligned} S(n+1, n+2) &= (n+1)S(n, n+1) + (n+1)S(n+1, n+1) \\ &= (n+1)\frac{n}{2}(n+1)! + (n+1)(n+1)! = \frac{n+1}{2}(n+2)! \end{aligned}$$

2) By (1) we have

$$\begin{aligned} S(n, n+1) &= (n+1)S(n-1, n) + \binom{n+1}{2} S(n-1, n-1) \\ &= (n+1)S(n-1, n) + \binom{n+1}{2}(n-1)! \end{aligned}$$

If we divide the last relation with $(n+1)!$, then we obtain

$$\frac{S(n, n+1)}{(n+1)!} - \frac{S(n-1, n)}{n!} = \frac{1}{2}.$$

Put $f(n) := \frac{S(n,n+1)}{(n+1)!}$. Then we have the equation $f(n) - f(n-1) = \frac{1}{2}$. If $f(n) = an$ for some real constant a, then $an - a(n-1) = \frac{1}{2}$ implies $a = \frac{1}{2}$. Hence, $f(n) = \frac{S(n,n+1)}{(n+1)!} = \frac{n}{2}$, i.e. (7) holds. □

Now we want to find the formulas for calculation of $S(n, n+2)$ and $S(n, n+3)$. By (1) and (7), for $n \in \mathbb{N}$ we have

$$\begin{aligned} S(n, n+2) &= (n+2)S(n-1, n+1) + \binom{n+2}{2} S(n-1, n) \\ &\quad + \binom{n+2}{3} S(n-1, n-1) \\ &= (n+2)S(n-1, n+1) + \binom{n+2}{2}\frac{n-1}{2} n! + \binom{n+2}{3}(n-1)! \end{aligned}$$

If we divide the last relation with $(n+2)!$ and by putting $f(n) := \frac{S(n,n+2)}{(n+2)!}$, then we have

$$f(n) - f(n-1) = \frac{n-1}{4} + \frac{1}{6} = \frac{3n-1}{12}.$$

If we assume that $f(n) = an^2 + bn$ for some real constants a and b, then the equation

$$an^2 + bn - a(n-1)^2 - b(n-1) = \frac{3n-1}{12}$$

has a unique solution : $a = \frac{3}{24}$, $b = \frac{1}{24}$. Hence,

$$S(n, n+2) = \frac{3n^2 + n}{24}(n+2)! \tag{8}$$

Obviously, relation (8) holds for $n = 0$, too. By (1), (7) and (8) for $n \in \mathbb{N}$ we have

$$\begin{aligned} S(n, n+3) &= (n+3)S(n-1, n+2) + \binom{n+3}{2}S(n-1, n+1) \\ &\quad + \binom{n+3}{3}S(n-1, n) + \binom{n+3}{4}S(n-1, n-1) \\ &= (n+3)S(n-1, n+2) + \binom{n+3}{2}\frac{3(n-1)^2 + n - 1}{24}(n+1)! \\ &\quad + \binom{n+3}{3}\frac{n-1}{2}n! + \binom{n+3}{4}(n-1)! \end{aligned}$$

If we divide the last relation with $(n+3)!$ and by putting $f(n) := \frac{S(n,n+3)}{(n+3)!}$, then we obtain

$$f(n) - f(n-1) = \frac{3n^2 - 5n + 2}{48} + \frac{n-1}{12} + \frac{1}{24} = \frac{3n^2 - n}{48}.$$

If we assume that $f(n) = an^3 + bn^2 + cn$ for some real constants a, b and c, then the equation

$$an^3 + bn^2 + cn - a(n-1)^3 - b(n-1)^2 - c(n-1) = \frac{3n^2 - n}{48}$$

has a unique solution : $a = \frac{1}{48}$, $b = \frac{1}{48}$, $c = 0$. Hence,

$$S(n, n+3) = \frac{n^3 + n^2}{48}(n+3)! \tag{9}$$

Relation (9) holds also for $n = 0$. In the same manner one can prove that for every $n \in \mathbb{N} \cup \{0\}$,

$$S(n, n+4) = \frac{15n^4 + 30n^3 + 5n^2 - 2n}{5760}(n+4)! \tag{10}$$

and

$$S(n, n+5) = \frac{9n^5 + 30n^4 + 15n^3 - 6n^2}{34560}(n+5)! \tag{11}$$

Note that each of the formulas (8), (9), (10) and (11) can be proved by mathematical induction, too.

Remark 6. Using the same procedure one can obtain a formula for $S(n, n+6)$, $S(n, n+7)$, and so forth. However, it is hard to find a general formula.

Theorem 7. *For every $k \in \mathbb{N}$ there exists a polynomial $f_k(n) = a_1 n^k + \cdots + a_k n$ with rational coefficients and with property that for every $n \in \mathbb{N} \cup \{0\}$*

$$S(n, n+k) = f_k(n)(n+k)! \tag{12}$$

Also, the following holds

$$\sum_{i=1}^{k} a_i = \frac{1}{(k+1)!}. \tag{13}$$

Proof. The assertion of the theorem is proved for $k = 1, \ldots, 5$. Note, that (12) and (13) hold also for $k = 0$ with $f_0(n) = 1$. Assume that the assertion of theorem holds for $S(n, n+i)$, $i \in \{1, \ldots, k\}$. By (1), for every $n \in \mathbb{N}$ we have

$$\begin{aligned} S(n, n+k+1) &= \sum_{i=1}^{k+2} \binom{n+k+1}{i} S(n-1, n+k+1-i) \\ &= (n+k+1) S(n-1, n+k) \\ &\quad + \sum_{i=2}^{k+2} \binom{n+k+1}{i} S(n-1, n+k+1-i). \end{aligned}$$

If $i \in \{2, \ldots, k+2\}$,then $n+k+1-i \in \{n-1, \ldots, n+k-1\}$, and, therefore, $0 \le (n+k+1-i) - (n-1) \le k$. By assumption, for every $i \in \{2, \ldots, k+2\}$ there exists a polynomial of degree $(n+k+1-i)-(n-1) = k+2-i$, such that $S(n-1, n+k+1-i) = f_{k+2-i}(n-1) \cdot (n+k+1-i)!$ Now we have further

$$S(n, n+k+1) - (n+k+1) S(n-1, n+k) = \sum_{i=2}^{k+2} \frac{(n+k+1)!}{i!} f_{k+2-i}(n-1).$$

If we divide the last relation with $(n+k+1)!$ and by putting $f_{k+1}(n) := \frac{S(n,n+k+1)}{(n+k+1)!}$, then we obtain

$$f_{k+1}(n) - f_{k+1}(n-1) = \sum_{i=2}^{k+2} \frac{1}{i!} f_{k+2-i}(n-1). \tag{*}$$

The sum on the right side of (*) is a polynomial of degree k in variable n. Therefore, there exists a unique polynomial $f_{k+1}(n)$ of degree $(k+1)$ in variable n, with rational coefficients, such that the relation (*) is satisfied. Hence, (12) is satisfied for every $k \in \mathbb{N}$. From (12) and $S(1, l) = 1$, for every $l \in \mathbb{N}$, we conclude that for every $k \in \mathbb{N}$, $f_k(1) = \frac{1}{(k+1)!}$, i.e. (13) holds. $\square$

Example 8. a) By (9) we have

$$\sum_{k=0}^{100}(-1)^k\binom{100}{k}k^{103} = \sum_{k=0}^{100}(-1)^k\binom{100}{k}(100-k)^{103} = S(100,103)$$
$$= \frac{100^3+100^2}{48}\cdot 103!$$

b) By (11) we have

$$\sum_{k=0}^{77}(-1)^k\binom{77}{k}k^{82} = -\sum_{k=0}^{77}(-1)^k\binom{77}{k}(77-k)^{82} = -S(77,82)$$
$$= -\frac{9\cdot 77^5+30\cdot 77^4+15\cdot 77^3-6\cdot 77^2}{34560}\cdot 82!$$

Remark 9. From (R3) and (12) we conclude that for every $k \in \mathbb{N}$ the polynomial $f_k(n)$ equals

$$f_k(n) = \sum_{\substack{k_1,\ldots,k_n\in\mathbb{N}\\ k_1+\cdots+k_n=n+k}} \frac{1}{k_1!\ldots k_n!}.$$

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