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ON THE LEBESGUE SUMMABILITY OF MULTIPLE TRIGONOMETRIC SERIES

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Dedicated to Professor Harry I. Miller on the occasion of his 70th birthday

Abstract. The Lebesgue summability of a trigonometric series is defined in terms of the symmetric differentiability of the sum of the formally integrated trigonometric series in question. In this paper we extend the theorems of Fatou and Zygmund from single to multiple trigonometric series.

1. Introduction: single trigonometric series

Let $\{c_n : n \in \mathbb{Z}\}$ be a sequence of complex numbers, in symbols: $\{c_n\} \subset \mathbb{C}$. We consider the trigonometric series

$$\sum_{n\in\mathbb{Z}} c_n e^{inx} \tag{1.1}$$

with the symmetric partial sums

$$s_N(x) := \sum_{|n|\le N} c_n e^{inx}, \quad N = 0, 1, 2, \dots .$$

Formal integration of series (1.1) gives

$$c_0 x + \sum_{|n|\ge 1} c_n \frac{e^{inx}}{in} =: L(x), \tag{1.2}$$

provided that the series in (1.2) converges. For example, if

$$\sum_{|n|\ge 1} \left|\frac{c_n}{n}\right|^2 < \infty,$$

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then the series in (1.2) converges almost everywhere, since it is the Fourier series of a function in L^2 and Carleson's celebrated theorem applies. On the other hand, the series in (1.2) need not converge at every point even if

$$c_n \to 0 \quad \text{as} \quad |n| \to \infty,$$

while series (1.1) converges everywhere (see details in [3, p. 321]).

We recall that if the function $L(x)$ in (1.2) exists in some neighborhood of a point $x \in \mathbb{R}$ and if

$$\Delta_h L(x) := \frac{1}{2h}\{L(x+h) - L(x-h)\} \to s \text{ as } h \to 0, \tag{1.3}$$

then series (1.1) is said to be summable to $s \in \mathbb{C}$ at $x \in \mathbb{R}$ by the Lebesgue method of summability, or briefly: it is Lebesgue summable to s. Observe that $\Delta_h L(x)$ is the symmetric difference quotient and its limit, if exists as $h \to 0$, is the symmetric derivative $DL(x) := s$.

The following theorem was proved by Zygmund (see [3, p. 322]).

Theorem 1. *If* $\{c_n\} \subset \mathbb{C}$ *is such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{|n|\le N} |nc_n| = 0, \tag{1.4}$$

then the series in (1.2) *converges for all* x, *and we have uniformly in* x *that*

$$\lim_{h\to 0}\{\Delta_h L(x) - s_N(x)\} = 0, ; \textit{ where } N := \left[\frac{1}{|h|}\right], \tag{1.5}$$

and $[\cdot]$ *means the integer part of a real number.*

In the other words, a necessary and sufficient condition for series (1.1) to have a (finite or infinite) limit s at some point x is that it is Lebesgue summable to the same s at x.

Clearly, condition (1.4) is satisfied if

$$nc_n \to 0 \text{ as } |n| \to \infty;$$

and in this special case Theorem 1 was proved by Fatou [1].

For references in Section 5, we present the representation of the difference between the braces in (1.5) in terms of the coefficient sequence $\{c_n\}$. By (1.2) and (1.3), we have

$$\begin{aligned}\Delta_h L(x) - s_N(x) &= \sum_{|n|\ge 1} c_n e^{inx}\frac{\sin nh}{nh} - \sum_{1\le|n|\le N} c_n e^{inx} \\ &= \sum_{1\le|n|\le N} c_n\, e^{inx}\Big(\frac{\sin nh}{nh} - 1\Big) + \sum_{|n|>N} c_n e^{inx}\frac{\sin nh}{nh}. \end{aligned}\tag{1.6}$$

2. Known results: Double trigonometric series

Let $\{c_{n_1,n_2} : (n_1, n_2) \in \mathbb{Z}^2\}$ be a double sequence of complex numbers, in symbols: $\{c_{n_1,n_2}\} \in \mathbb{C}$. We consider the double trigonometric series

$$\sum_{n_1\in\mathbb{Z}} \sum_{n_2\in\mathbb{Z}} c_{n_1,n_2}\ e^{i(n_1x_1+n_2x_2)} \tag{2.1}$$

with the symmetric rectangular partial sums

$$s_{N_1,N_2}(x_1, x_2) := \sum_{|n_1|\le N_1} \sum_{|n_2|\le N_2} c_{n_1,n_2}\ e^{i(n_1x_1+n_2x_2)},\ \ N_1, N_2 = 0, 1, 2, \dots .$$

Formal integration of series (2.1) with respect to both x_1 and x_2 gives

$$\begin{aligned} c_{0,0}x_1x_2 + x_2 \sum_{|n_1|\ge1} c_{n_1,0}\frac{e^{in_1x_1}}{in_1} + x_1 \sum_{|n_2|\ge1} c_{0,n_2}\frac{e^{in_2x_2}}{in_2} \\ + \sum_{|n_1|\ge1} \sum_{|n_2|\ge1} c_{n_1,n_2}\frac{e^{i(n_1x_1+n_2x_2)}}{i^2n_1n_2} =: L(x_1, x_2), \end{aligned} \tag{2.2}$$

provided that each of the series in (2.2) converges.

Motivated by the definition of Lebesgue summability of series (1.1) in Section 1, the double series (2.1) is said to be Lebesgue summable to $s \in \mathbb{C}$ at a point $(x_1, x_2) \in \mathbb{R}^2$ if $L(\cdot, \cdot)$ exists in some neighborhood of (x_1, x_2) and if

$$\begin{aligned} \Delta_{h_1,h_2}L(x_1, x_2) := \frac{1}{4h_1h_2}\{L(x_1 + h_1, x_2 + h_2) - L(x_1 - h_1, x_2 + h_2) \\ - L(x_1 + h_1, x_2 - h_2) + L(x_1 - h_1, x_2 - h_2)\} \to s \text{ as } h_1, h_2 \to 0. \end{aligned} \tag{2.3}$$

We note that $\Delta_{h_1,h_2}L(x_1, x_2)$ may be considered as a symmetric difference quotient and its limit, if exists as $h_1, h_2 \to 0$ independently of one another, may be called the symmetric derivative $DL(x_1, x_2) := s$.

The following extension of Theorem 1 from single to double trigonometric series was proved in [2].

Theorem 2. *If* $\{c_{n_1,n_2}\} \subset \mathbb{C}$ *is such that*

$$\lim_{N_1\to\infty} \frac{1}{N_1} \sum_{1\le|n_1|\le N_1} \sum_{n_2\in\mathbb{Z}} |n_1c_{n_1,n_2}| = 0, \tag{2.4}$$

$$\lim_{N_2\to\infty} \frac{1}{N_2} \sum_{1\le|n_2|\le N_2} \sum_{n_1\in\mathbb{Z}} |n_2c_{n_1,n_2}| = 0, \tag{2.5}$$

then each of the series in (2.2) *converges for all* (x_1, x_2)*, and we have uniformly in* (x_1, x_2) *that*

$$\lim_{h_1,h_2\to 0}\{\Delta_{h_1,h_2}L(x_1,x_2) - s_{N_1,N_2}(x_1,x_2)\} = 0, \tag{2.6}$$

where

$$N_1 := \Big[\frac{1}{|h_1|}\Big] \quad N_2 := \Big[\frac{1}{|h_2|}\Big].$$

We note that the following two conditions are sufficient for the fulfillment of conditions (2.4) and (2.5), respectively:

$$\lim_{|n_1|\to\infty} \sum_{n_2\in\mathbb{Z}} |n_1 c_{n_1,n_2}| = 0,$$

$$\lim_{|n_2|\to\infty} \sum_{n_1\in\mathbb{Z}} |n_2 c_{n_1,n_2}| = 0.$$

For a reference in Section 5, we present the representation of the difference between the braces in (2.6) in terms of the double coefficient sequence $\{c_{n_1,n_2}\}$. By (2.2) and (2.3), we have

$$\begin{aligned}
&\Delta_{h_1,h_2}L(x_1,x_2) - s_{N_1,N_2}(x_1,x_2)\\
&\quad = \Big\{\sum_{|n_1|\ge 1} c_{n_1,0}\; e^{in_1x_1}\frac{\sin n_1h_1}{n_1h_1} - \sum_{1\le|n_1|\le N_1} c_{n_1,0}\; e^{in_1x_1}\Big\}\\
&\quad + \Big\{\sum_{|n_2|\ge 1} c_{0,n_2}\; e^{in_2x_2}\frac{\sin n_2h_2}{n_2h_2} - \sum_{1\le|n_2|\le N_2} c_{0,n_2}\; e^{in_2x_2}\Big\}\\
&\quad + \Big\{\sum_{|n_1|\ge 1}\sum_{|n_2|\ge 1} c_{n_1,n_2}\; e^{i(n_1x_1+n_2x_2)}\frac{\sin n_1h_1}{n_1h_1}\frac{\sin n_2h_2}{n_2h_2}\\
&\qquad - \sum_{1\le|n_1|\le N_1}\sum_{1\le|n_2|\le N_2} c_{n_1,n_2}\; e^{i(n_1x_1+n_2x_2)}.
\end{aligned} \tag{2.7}$$

3. New results: Multiple trigonometric series

Let $d \ge 1$ be an integer and $\{c_{n_1,n_2,\dots,n_d} : (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d\}$ a d-multiple sequence of complex numbers, in symbols: $\{c_{n_1,\dots,n_d}\} \subset \mathbb{C}$. We consider the d-multiple trigonometric series

$$\sum_{n_1\in\mathbb{Z}} \cdots \sum_{n_d\in\mathbb{Z}} c_{n_1,\dots,n_d} \exp\Big(i\sum_{k=1}^{d} n_k x_k\Big) \tag{3.1}$$

with the symmetric rectangular partial sums

$$s_{N_1,\dots,N_d}(x_1,\dots,x_d) := \sum_{|n_1|\le N_1} \cdots \sum_{|n_d|\le N_d} c_{n_1,\dots,n_d} \exp\Big(i\sum_{k=1}^{d} n_k x_k\Big),$$

where $N_1,\dots,N_d = 0,1,2,\dots$.

Integrating formally series (3.1) with respect to each of the variables $x_1,\dots,x_d$ in turn, in case $d=3$ we obtain

$$\begin{aligned}
&c_{0,0,0}x_1x_2x_3 + x_2x_3 \sum_{|n_1|\ge 1} c_{n_1,0,0}\frac{e^{n_1x_1}}{in_1}\\
&+x_1x_3 \sum_{|n_2|\ge 1} c_{0,n_2,0}\frac{e^{in_2x_2}}{in_2} + x_1x_2 \sum_{|n_3|\ge 1} c_{0,0,n_3}\frac{e^{in_3x_3}}{in_3}\\
&+x_3 \sum_{|n_1|\ge 1}\sum_{|n_2|\ge 1} c_{n_1,n_2,0}\,\frac{e^{i(n_1x_1+n_2x_2)}}{i^2n_1n_2}\\
&+x_2 \sum_{|n_1|\ge 1}\sum_{|n_3|\ge 1} c_{n_1,0,n_3}\frac{e^{i(n_1x_1+n_3x_3)}}{i^2n_1n_3}\\
&+x_1 \sum_{|n_2|\ge 1}\sum_{|n_3|\ge 1} c_{0,n_2,n_3}\,\frac{e^{i(n_2x_2+n_3x_3)}}{i^2n_2n_3}\\
&+\sum_{|n_1|\ge 1}\sum_{|n_2|\ge 1}\sum_{|n_3|\ge 1} c_{n_1,n_2,n_3}\frac{e^{i(n_1x_1+n_2x_2+n_3x_3)}}{i^3n_1n_2n_3} =: L(x_1,x_2,x_3),
\end{aligned} \tag{3.2}$$

provided that each series in (3.2) converges. In Theorem 3 below, we will give sufficient conditions for the convergence of these series, which takes place even uniformly in (x_1,x_2,x_3).

Motivated by the definition of Lebesgue summability in the cases $d=1$ and 2, we say that the triple series in (3.1) for $d=3$ is Lebesgue summable to $s\in\mathbb{C}$ at a point $(x_1,x_2,x_3)\in\mathbb{R}^3$ if $L(\cdot,\cdot,\cdot)$ under those conditions. exists in some neighborhood of (x_1,x_2,x_3) and if

$$\begin{aligned}
&\Delta_{h_1,h_2,h_3}L(x_1,x_2,x_3)\\
&:= \frac{1}{8h_1h_2h_3}\{L(x_1+h_1,x_2+h_2,x_3+h_3) - L(x_1-h_1,x_2+h_2,x_3+h_3)\\
&\quad - L(x_1+h_1,x_2-h_2,x_3+h_3) - L(x_1+h_1,x_2+h_2,x_3-h_3)\\
&\quad + L(x_1-h_1,x_2-h_2,x_3+h_3) + L(x_1-h_1,x_2+h_2,x_3-h_3)\\
&\quad + L(x_1+h_1,x_2-h_2,x_3-h_3)\\
&\quad - L(x_1-h_1,x_2-h_2,x_3-h_3) \to s \text{ as } h_1,h_2,h_3\to 0.
\end{aligned} \tag{3.3}$$

We note that $\Delta_{h_1,h_2,h_3}(x_1, x_2, x_3)$ may be considered as a symmetric difference quotient and its limit, if it exists as $h_1, h_2, h_3 \to 0$ independently of one another, may be called the symmetric derivative $DL(x_1, x_2, x_3) := s$.

The form of the result of formal differentiation of series (3.1) with respect to each of the variables $x_1, \dots, x_d$ in turn, as well as the form of the corresponding function $\Delta_{h_1,\dots,h_d}(x_1, \dots, x_d)$ in the case of general $d \geq 4$ are obvious.

Our main new result is formulated in Theorem 3. It extends Theorems 1 and 2 for d-dimensional trigonometric series as follows.

Theorem 3. *If* $\{c_{n_1,\dots,n_d} : (n_1, \dots, n_d) \in \mathbb{Z}^d\}$ *is such that*

$$\lim_{N_1\to\infty} \frac{1}{N_1} \sum_{1\leq |n_1|\leq N_1} \sum_{n_2\in\mathbb{Z}} \dots \sum_{n_d\in\mathbb{Z}} |n_1 c_{n_1,n_2,\dots,n_d}| = 0, \tag{3.4}$$

$$\lim_{N_2\to\infty} \frac{1}{N_2} \sum_{1\leq |n_2|\leq N_2} \sum_{n_1\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}} \dots \sum_{n_d\in\mathbb{Z}} |n_2 c_{n_1,n_2,n_3,\dots,n_d}| = 0, \dots, \tag{3.5}$$

$$\lim_{N_d\to\infty} \frac{1}{N_d} \sum_{1\leq |n_d|\leq N_d} \sum_{n_1\in\mathbb{Z}} \dots \sum_{n_{d-1}\in\mathbb{Z}} |n_d c_{n_1,\dots,n_{d-1},n_d}| = 0, \tag{3.6}$$

then each of the series occurring in the definition of $L(x_1, \dots, x_d)$ *(cf.* (3.2)*) converges for all* $(x_1, \dots, x_d)$ *and we have uniformly in* $(x_1, \dots, x_d)$ *that*

$$\lim_{h_1,\dots,h_d\to 0} \{\Delta_{h_1,\dots,h_d}(x_1, \dots, x_d) - s_{N_1,\dots,N_d}(x_1, \dots, x_d)\} = 0, \tag{3.7}$$

where

$$N_k := \left[\frac{1}{|h_k|}\right], \quad k = 1, 2, \dots, d.$$

In other words, under conditions (3.4), (3.5), …, (3.6), a necessary and sufficient condition for the d-multiple series (3.1) to have a (finite or infinite) limit s at some point $(x_1, \dots, x_d) \in \mathbb{R}^d$ is that it is Lebesgue summable to the same limit at $(x_1, \dots, x_d)$.

We note that the following conditions are sufficient for the fulfillment of conditions (3.4), (3.5), …, (3.6), respectively:

$$\lim_{|n_1|\to\infty} \sum_{n_2\in\mathbb{Z}} \dots \sum_{n_d\in\mathbb{Z}} |n_1 c_{n_1,n_2,\dots,n_d}| = 0,$$

$$\lim_{|n_2|\to\infty} \sum_{n_1\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}} \dots \sum_{n_d\in\mathbb{Z}} |n_2 c_{n_1,n_2,n_3,\dots,n_d}| = 0, \dots,$$

$$\sum_{|n_d|\to\infty} \sum_{n_1\in\mathbb{Z}} \dots \sum_{n_{d-1}\in\mathbb{Z}} |n_d c_{n_1,\dots,n_{d-1},n_d}| = 0.$$

4. Auxiliary results

Lemma 1. (see [2, Lemma 1]). *For any sequence $\{c_n\} \subset \mathbb{C}$, the following two conditions are equivalent:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{|n|\le N} |nc_n| = 0$$

and

$$\lim_{N\to\infty} N \sum_{|n|\ge N} \left|\frac{c_n}{n}\right| = 0.$$

The next Lemma 2 is folklore.

Lemma 2. *The following estimate holds for all $0 < t \le 1$:*

$$0 < 1 - \frac{\sin t}{t} \le \frac{t^2}{3!}.$$

5. Proof of Theorem 3

In the case $d = 1$, Theorem 3 was proved by Zygmund [3, p. 322], and his proof was somewhat simplified by us in [2, p. 559]. In the case $d = 2$, Theorem 3 was proved also in [2, pp. 559-561], and our proof was based on a natural decomposition of the lattice points of the first quadrant of the real plane into four disjoint regions (see the representation of the difference in question in [2, p. 560, formula (4.9)]). In the case $d = 3$, Theorem 3 could be proved in an analogous way; that is, by decomposing the lattice points of the first octant of the 3-dimensional real space into eight disjoint regions, and then by estimating the corresponding differences over various appropriate unions of these regions. The problem with this method of proof is that these estimations get more and more complicated.

Therefore, we will prove Theorem 3 by induction with respect to the dimension d. In order to avoid cumbersome notations, we present the induction step from $(d-1)$ to d in the case $d = 3$. By conditions (3.4), (3.5), (3.6) in the case $d = 3$, it is easy to check that each series in (3.1) converges uniformly in (x_1, x_2, x_3).

Now, we proceed to prove (3.7) in the case $d = 3$, while relying on the validity of (3.7) in the cases $d = 1$ and $d = 2$. By (3.2) and (3.3), we may write that

$$\Delta_{h_1,h_2,h_3} L(x_1,x_2,x_3) - s_{N_1,N_2,N_3}(x_1,x_2,x_3)$$
$$= \Big\{ \sum_{|n_1|\ge 1} c_{n_1,0,0,}\ e^{in_1x_1} \frac{\sin n_1h_1}{n_1h_1} - \sum_{1\le|n_1|\le N_1} c_{n_1,0,0}\ e^{in_1x_1} \Big\}$$

$$+\Big\{\sum_{|n_2|\ge 1} c_{0,n_2,0}\ e^{in_2x_2}\frac{\sin n_2h_2}{n_2h_2} - \sum_{1\le|n_2|\le N_2} c_{0,n_2,0}\ e^{in_2x_2}\Big\}$$

$$+\Big\{\sum_{|n_3|\ge 1} c_{0,0,n_3}\ e^{in_3x_3}\frac{\sin n_3h_3}{n_3h_3} - \sum_{1\le|n_3|\le N_3} c_{0,0,n_3}\ e^{in_3x_3}\Big\}$$

$$+\Big\{\sum_{|n_1|\ge 1}\sum_{|n_2|\ge 1} c_{n_1,n_2,0}\ e^{i(n_1x_1+n_2x_2)}\frac{\sin n_1h_1}{n_1h_1}\frac{\sin n_2h_2}{n_2h_2}$$

$$-\sum_{1\le|n_1|\le N_1}\sum_{1\le|n_2|\le N_2} c_{n_1,n_2,0}\ e^{i(n_1x_1+n_2x_2)}\Big\}$$

$$+\Big\{\sum_{|n_1|\ge 1}\sum_{|n_3|\ge 1} c_{n_1,0,n_3}\ e^{i(n_1x_1+n_3x_3)}\frac{\sin n_1h_1}{n_1h_1}\ \frac{\sin n_3h_3}{n_3h_3}$$

$$-\sum_{1\le|n_1|\le N_1}\sum_{1\le|n_3|\le N_3} c_{n_1,0,n_3}\ e^{i(n_1x_1+n_3x_3)}\Big\}$$

$$+\Big\{\sum_{|n_2|\ge 1}\sum_{|n_3|\ge 1} c_{0,n_2,n_3}\ e^{i(n_2x_2+n_3x_3)}\frac{\sin n_2h_2}{n_2h_2}\frac{\sin n_3h_3}{n_3h_3}$$

$$-\sum_{1\le|n_2|\le N_2}\sum_{1\le|n_3|\le N_3} c_{0,n_2,n_3}\ e^{i(n_2x_2+n_3x_3)}\Big\}$$

$$+\Big\{\sum_{|n_1|\ge 1}\sum_{|n_2|\ge 1}\sum_{|n_3|\ge 1} c_{n_1,n_2,n_3}\ e^{i(n_1x_1+n_2x_2+n_3x_3)}\ \frac{\sin n_1h_1}{n_1h_1}\ \frac{\sin n_2h_2}{n_2h_2}\ \frac{\sin n_3h_3}{n_3h_3}$$

$$-\sum_{1\le|n_1|\le N_1}\sum_{1\le|n_2|\le N_2}\sum_{1\le|n_3|\le N_3} c_{n_1,n_2,n_3}\ e^{i(n_1x_1+n_2x_2+n_3x_3)}\Big\}$$

$$=: \Big\{\Delta^{(1)}_{h_1}(x_1) - s^{(1)}_{N_1}(x_1)\Big\} + \Big\{\Delta^{(2)}_{h_2}(x_2) - s^{(2)}_{N_2}(x_2)\Big\} + \Big\{\Delta^{(3)}_{h_3}(x_3) - s^{(3)}_{N_3}(x_3)\Big\}$$

$$+\Big\{\Delta^{(4)}_{h_1,h_2}(x_1,x_2) - s^{(4)}_{N_1,N_2}(x_1,x_2)\} + \{\Delta^{(5)}_{h_1,h_3}(x_1,x_3) - s^{(5)}_{N_1,N_3}(x_1,x_3)\Big\}$$

$$+\Big\{\Delta^{(6)}_{h_2,h_3}(x_2,x_3) - s^{(6)}_{N_2,N_3}(x_2,x_3)\Big\}$$

$$+\Big\{\Delta^{(7)}_{h_1,h_2,h_3}(x_1,x_2,x_3) - s^{(7)}_{N_1,N_2,N_3}(x_1,x_2,x_3)\Big\},\quad \text{say.} \tag{5.1}$$

Due to (3.4) - (3.6) for $d = 3$, we may apply Theorem 1 to get that the first three differences in braces on the right-hand side of (5.1) converge to 0 as $h_1, h_2, h_3 \to 0$, respectively:

$$\lim_{h_k\to 0}\Big\{\Delta^{(k)}_{h_k}(x_k) - s^{(k)}_{N_k}(x_k)\Big\} = 0, \quad k = 1,2,3 \tag{5.2}$$

(cf. (1.6)). Similarly, for the next three differences in braces on the right-hand side of (5.1) way may apply Theorem 2 to conclude that these differences also converge to 0 as $h_1, h_2, h_3 \to 0$, respectively:

$$\lim_{h_k, h_\ell \to 0} \Big\{ \Delta^{(k+\ell+1)}_{h_k, h_\ell}(x_k, x_\ell) - s^{(k+\ell+1)}_{N_k, N_\ell}(x_k, x_\ell) \Big\} = 0, \tag{5.3}$$

where $k = 1, 2$ and $\ell = k+1, k+2 \le 3$ (cf. (2.7)).

Thus, it remains to prove that the last difference in braces on the right-hand side of (5.1) also converges to 0 as $h_1, h_2, h_3 \to 0$:

$$\lim_{h_1, h_2, h_3 \to 0} \Big\{ \Delta^{(7)}_{h_1, h_2, h_3}(x_1, x_2, x_3) - s^{(7)}_{N_1, N_2, N_3} \Big\} = 0. \tag{5.4}$$

To this effect, we define the double sequence $\{C_{n_1 n_2}\} \subset \mathbb{C}$ as follows:

$$C_{n_1, n_2} := \sum_{|n_3| \ge 1} c_{n_1, n_2, n_3} e^{i n_3 x_3} \frac{\sin n_3 h_3}{n_3 h_3}, \quad (n_1, n_2) \in \mathbb{Z}^2.$$

Clearly, we may write that

$$\begin{aligned}
&\Delta^{(7)}_{h_1, h_2, h_3}(x_1, x_2, x_3) - s^{(7)}_{N_1, N_2, N_3}(x_1, x_2, x_3) \\
&= \Big\{ \sum_{|n_1| \ge 1} \sum_{n_2 | \ge 1} C_{n_1, n_2} \; e^{i(n_1 x_1 + n_2 x_2)} \frac{\sin n_1 h_1}{n_1 h_1} \; \frac{\sin n_2 h_2}{n_2 h_2} \\
&\quad - \sum_{1 \le |n_1| \le N_1} \sum_{1 \le |n_2| \le N_2} C_{n_1, n_2} e^{i(n_1 x_1 + n_2 x_2)} \Big\} \\
&\quad + \Big\{ \sum_{1 \le |n_1| \le N_1} \sum_{1 \le |n_2| \le N_2} C_{n_1, n_2} e^{i(n_1 x_1 + n_2 x_2)} \\
&\quad - \sum_{1 \le |n_1| \le N_1} \sum_{1 \le |n_2| \le N_2} \sum_{1 \le |n_3| \le N_3} c_{n_1, n_2, n_3} \; e^{i(n_1 x_1 + n_2 x_2 + n_3 x_3} \Big\} \\
&=: T^{(1)}_{N_1, N_2}(x_1, x_2, x_3) + T^{(2)}_{N_1, N_2, N_3}(x_1, x_2, x_3), \quad \text{say.}
\end{aligned} \tag{5.5}$$

Since

$$\frac{1}{N_1} \sum_{1 \le |n_1| \le N_1} \sum_{n_2 \in \mathbb{Z}} |n_1 C_{n_1, n_2}| \le \frac{1}{N_1} \sum_{1 \le |n_1| \le N_1} \sum_{n_2 \in \mathbb{Z}} \sum_{|n_3| \ge 1} |n_1 c_{n_1, n_2, n_3}|,$$

it follows from (3.4) that condition (2.4) is satisfied with C_{n_1, n_2} in place of c_{n_1, n_2}. In the same way, it follows from (3.5) that condition (2.5) is also satisfied with C_{n_1, n_2} is place of c_{n_1, n_2}. Thus, we may apply Theorem 2 for the double sequence $\{C_{n_1, n_2}\}$ and to obtain that

$$\lim_{h_1, h_2 \to 0} T^{(1)}_{N_1, N_2}(x_1, x_2, x_3) = 0, \quad N_1 := \Big[\frac{1}{|h_2|}\Big], N_2 := \Big[\frac{1}{|h_2|}\Big]. \tag{5.6}$$

Next, we consider the following representation (cf. (1.6)):

$$\begin{aligned} &T^{(2)}_{N_1,N_2,N_3}(x_1,x_2,x_3)\\ &= \sum_{1\le|n_1|\le N_1}\ \sum_{1\le|n_2|\le N_2}\ \sum_{1\le|n_3|\le N_3} c_{n_1,n_2,n_3} e^{i(n_1x_1+n_2x_2+n_2x_3)}\Big(\frac{\sin n_3h_3}{n_3h_3}-1\Big)\\ &\quad + \sum_{1\le|n_1|\le N_1}\ \sum_{1\le|n_2|\le N_2}\ \sum_{|n_3|>N_3} c_{n_1,n_2,n_3} e^{i(n_1x_1+n_2x_2+n_3x_3)}\frac{\sin n_3h_3}{n_3h_3}\\ &\qquad =: T^{(21)}_{N_1,N_2,N_3}(x_1,x_2,x_3)+T^{(22)}_{N_1,N_2,N_3}(x_1,x_2,x_3), \quad \text{say.} \qquad (5.7)\end{aligned}$$

Applying Lemma 2 (with t instead of t^2 on the right-hand side, since $0<t^2\le t$ for $0<t\le 1$) gives

$$\begin{aligned} &|T^{(21)}_{N_1,N_2,N_3}(x_1,x_2,x_3)|\\ &\qquad \le \frac{h_3}{6}\sum_{1\le|n_1|\le N_1}\ \sum_{1\le|n_2|\le N_2}\ \sum_{1\le|n_3|\le N_3}|n_3c_{n_1,n_2,n_3}|\\ &\le \frac{1}{6N_3}\sum_{1\le|n_3|\le N_3}\ \sum_{n_1\in\mathbb{Z}}\ \sum_{n_2\in\mathbb{Z}}|n_3c_{n_1,n_2,n_3}| \to 0 \ \text{ as } N_3:=\Big[\frac{1}{|h_3|}\Big]\to\infty, \quad (5.8)\end{aligned}$$

due to (3.6) (in the case $d=3$).

On the other hand, by Lemma 1 applied for the sequence $\{C_{n_3}\}$ defined by

$$C_{n_3}:=\sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}|c_{n_1,n_2,n_3}|, \quad n_3\in\mathbb{Z},$$

we obtain the following limit relation:

$$\begin{aligned} &|T^{(22)}_{N_1,N_2,N_3}(x_1,x_2,x_3)|\\ &\qquad \le \frac{1}{|h_3|}\sum_{1\le|n_1|\le N_1}\ \sum_{1\le|n_2|\le N_2}\ \sum_{|n_3|\ge N_3}\Big|\frac{c_{n_1,n_2,n_3}}{n_3}\Big|\\ &\qquad \le (N_3+1)\sum_{|n_3|\ge N_3}\ \sum_{n_1\in\mathbb{Z}}\ \sum_{n_2\in\mathbb{Z}}\Big|\frac{c_{n_1,n_2,n_3}}{n_3}\Big|\\ &\qquad = (N_3+1)\sum_{|n_3|\ge N_3}\Big|\frac{C_{n_3}}{n_3}\Big| \to 0 \ \text{ as } N_3:=\Big[\frac{1}{|h_3|}\Big]\to\infty, \quad (5.9)\end{aligned}$$

again due to (3.6) (in the case $d=3$).

Combining (5.5) - (5.9) results in (5.4) as we claimed. Finally, putting (5.1)-(5.4) together yields (3.7) (in the case $d=3$) to be proved.

References

[1] P. Fatou, *Séries trigonométriques et séries de Taylor*, Acta Math., 30 (1906), 335–400.
[2] M. Bagota and F. Móricz, *On the Lebesgue summability of double trigonometric series*, J. Math. Anal. Appl., 348 (2008), 555–561.
[3] A. Zygmund, *Trigonometric Series*, Vol. I., Cambridge Univ. Press, 1959.

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