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POWER INEQUALITIES FOR THE NUMERICAL RADIUS OF A PRODUCT OF TWO OPERATORS IN HILBERT SPACES

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Abstract. Some power inequalities for the numerical radius of a product of two operators in Hilbert spaces with applications for commutators and self-commutators are given.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [11, p. 1]:

$$W(T)=\{\langle Tx,x\rangle,\ x\in H,\ \|x\|=1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is given by [11, p. 8]:

$$w(T)=\sup\{|\lambda|,\lambda\in W(T)\}=\sup\{|\langle Tx,x\rangle|,\|x\|=1\}. \tag{1.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T:H\to H$. This norm is equivalent to the operator norm. In fact, the following more precise result holds [11, p. 9]:

$$w(T)\le\|T\|\le 2w(T), \tag{1.2}$$

for any $T\in B(H)$

For other results on numerical radii, see [12], Chapter 11.

If A,B are two bounded linear operators on the Hilbert space $(H,\langle\cdot,\cdot\rangle)$, then

$$w(AB)\le 4w(A)w(B). \tag{1.3}$$

In the case that $AB=BA$, then

$$w(AB)\le 2w(A)w(B). \tag{1.4}$$

The following results are also well known [11, p. 38]:

If A is a unitary operator that commutes with another operator B, then

$$w(AB)\le w(B). \tag{1.5}$$

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If A is an isometry and $AB = BA$, then (1.5) also holds true.

We say that A and B *double commute* if $AB = BA$ and $AB^* = B^*A$. If the operators A and B double commute, then [11, p. 38]

$$w(AB) \le w(B) \|A\|. \tag{1.6}$$

As a consequence of the above, we have [11, p. 39]:

Let A be a normal operator commuting with B, then

$$w(AB) \le w(A) w(B). \tag{1.7}$$

For other results and historical comments on the above see [11, p. 39–41]. For recent inequalities involving the numerical radius, see [1] -[9], [13], [14]-[16] and [17].

2. Inequalities for a product of two operators

Theorem 1. *For any $A, B \in B(H)$ and $r \ge 1$, we have the inequality*

$$w^r(B^*A) \le \frac{1}{2} \|(A^*A)^r + (B^*B)^r\|. \tag{2.1}$$

The constant $\frac{1}{2}$ is best possible.

Proof. By the Schwarz inequality in the Hilbert space $(H; \langle .,. \rangle)$ we have

$$\begin{aligned} |\langle B^*Ax, x\rangle| &= |\langle Ax, Bx\rangle| \le \|Ax\| \cdot \|Bx\| \\ &= \langle A^*Ax, x\rangle^{1/2} \cdot \langle B^*Bx, x\rangle^{1/2}, \qquad x \in H. \end{aligned} \tag{2.2}$$

Utilizing the arithmetic mean - geometric mean inequality and then the convexity of the function $f(t) = t^r$, $r \ge 1$, we have successively,

$$\begin{aligned} \langle A^*Ax, x\rangle^{1/2} \cdot \langle B^*Bx, x\rangle^{1/2} &\le \frac{\langle A^*Ax, x\rangle + \langle B^*Bx, x\rangle}{2} \\ &\le \left(\frac{\langle A^*Ax, x\rangle^r + \langle B^*Bx, x\rangle^r}{2}\right)^{\frac{1}{r}} \end{aligned} \tag{2.3}$$

for any $x \in H$.

It is known that if P is a positive operator then for any $r \ge 1$ and $x \in H$ with $\|x\| = 1$ we have the inequality (see for instance [15])

$$\langle Px, x\rangle^r \le \langle P^r x, x\rangle. \tag{2.4}$$

Applying this property to the positive operator A^*A and B^*B, we deduce that

$$\begin{aligned} \left(\frac{\langle A^*Ax, x\rangle^r + \langle B^*Bx, x\rangle^r}{2}\right)^{\frac{1}{r}} &\le \left(\frac{\langle (A^*A)^r x, x\rangle + \langle (B^*B)^r x, x\rangle}{2}\right)^{\frac{1}{r}} \\ &= \left(\frac{\langle [(A^*A)^r + (B^*B)^r] x, x\rangle}{2}\right)^{\frac{1}{r}} \end{aligned} \tag{2.5}$$

for any $x \in H$, $\|x\| = 1$.

Now, on making use of the inequalities (2.2), (2.3) and (2.5), we get the inequality

$$|\langle (B^*A)^r x, x\rangle|^r \le \frac{1}{2}\langle [(A^*A)^r + (B^*B)^r] x, x\rangle \tag{2.6}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.6) and since the operator $[(A^*A)^r + (B^*B)^r]$ is self-adjoint, we deduce the desired inequality (2.1).

For $r = 1$ and $B = A$, we get in both sides of (2.1) the same quantity $\|A\|^2$ which shows that the constant $\frac{1}{2}$ is best possible in general in the inequality (2.1). □

Corollary 1. *For any $A \in B(H)$ and $r \ge 1$ we have the inequalities*

$$w^r(A) \le \frac{1}{2}\|(A^*A)^r + I\| \tag{2.7}$$

and

$$w^r\left(A^2\right) \le \frac{1}{2}\|(A^*A)^r + (AA^*)^r\|, \tag{2.8}$$

respectively.

A different approach is considered in the following result:

Theorem 2. *For any $A, B \in B(H)$ and any $\alpha \in (0,1)$ and $r \ge 1$, we have the inequality*

$$w^{2r}(B^*A) \le \left\|\alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha)(B^*B)^{\frac{r}{1-\alpha}}\right\|. \tag{2.9}$$

Proof. By Schwarz's inequality, we have

$$\begin{aligned} |\langle (B^*A) x, x\rangle|^2 &\le \langle (A^*A) x, x\rangle \cdot \langle (B^*B) x, x\rangle \\ &= \left\langle \left[(A^*A)^{\frac{1}{\alpha}}\right]^{\alpha} x, x\right\rangle \cdot \left\langle \left[(B^*B)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} x, x\right\rangle, \end{aligned} \tag{2.10}$$

for any $x \in H$.

It is well known that (see for instance [15]) if P is a positive operator and $q \in (0,1]$ then for any $u \in H$, $\|u\| = 1$, we have

$$\langle P^q u, u\rangle \le \langle Pu, u\rangle^q. \tag{2.11}$$

Applying this property to the positive operators $(A^*A)^{\frac{1}{\alpha}}$ and $(B^*B)^{\frac{1}{1-\alpha}}$ $(\alpha \in (0,1))$, we have

$$\begin{aligned} &\left\langle \left[(A^*A)^{\frac{1}{\alpha}}\right]^{\alpha} x, x\right\rangle \cdot \left\langle \left[(B^*B)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} x, x\right\rangle \\ &\qquad \le \left\langle (A^*A)^{\frac{1}{\alpha}} x, x\right\rangle^{\alpha} \cdot \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x\right\rangle^{1-\alpha}, \end{aligned} \tag{2.12}$$

for any $x \in H$, $\|x\| = 1$.

Now, utilizing the weighted arithmetic mean - geometric mean inequality, i.e., $a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha) b$, $\alpha \in (0,1)$, $a, b \geq 0$, we get

$$\left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle^{\alpha} \cdot \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle^{1-\alpha} \leq \alpha \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle \tag{2.13}$$

for any $x \in H$, $\|x\| = 1$.

Moreover, by the elementary inequality following from the convexity of the function $f(t) = t^r$, $r \geq 1$, namely

$$\alpha a + (1-\alpha) b \leq (\alpha a^r + (1-\alpha) b^r)^{\frac{1}{r}}, \qquad \alpha \in (0,1), \ a, b \geq 0,$$

we deduce that

$$\begin{aligned} &\alpha \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle \\ &\leq \left[\alpha \left\langle (A^*A)^{\frac{1}{\alpha}} x, x \right\rangle^r + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}} x, x \right\rangle^r \right]^{\frac{1}{r}} \\ &\leq \left[\alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{r}{1-\alpha}} x, x \right\rangle \right]^{\frac{1}{r}}, \end{aligned} \tag{2.14}$$

for any $x \in H$, $\|x\| = 1$, where, for the last inequality we used the inequality (2.4) for the positive operators $(A^*A)^{\frac{1}{\alpha}}$ and $(B^*B)^{\frac{1}{1-\alpha}}$.

Now, on making use of the inequalities (2.10), (2.12), (2.13) and (2.14), we get

$$|\langle (B^*A) x, x \rangle|^{2r} \leq \left\langle \left[\alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha) (B^*B)^{\frac{r}{1-\alpha}} \right] x, x \right\rangle \tag{2.15}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.15) produces the desired inequality (2.9). □

Remark 1. The particular case $\alpha = \frac{1}{2}$ produces the inequality

$$w^{2r}(B^*A) \leq \frac{1}{2} \left\| (A^*A)^{2r} + (B^*B)^{2r} \right\|, \tag{2.16}$$

for $r \geq 1$. Notice that $\frac{1}{2}$ is best possible in (2.16) since for $r = 1$ and $B = A$ we get in both sides of (2.16) the same quantity $\|A\|^4$.

Corollary 2. *For any $A \in B(H)$ and $\alpha \in (0,1)$, $r \geq 1$, we have the inequalities*

$$w^{2r}(A) \leq \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha) I \right\| \tag{2.17}$$

and

$$w^{2r}\left(A^2\right) \leq \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha) (AA^*)^{\frac{r}{1-\alpha}} \right\|, \tag{2.18}$$

respectively.

Moreover, we have

$$\|A\|^{4r} \leq \left\| \alpha \left(A^{*}A\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(A^{*}A\right)^{\frac{r}{1-\alpha}} \right\|. \tag{2.19}$$

3. Inequalities for the sum of two products

The following result may be stated:

Theorem 3. *For any $A, B, C, D \in B(H)$ and $r, s \geq 1$ we have*

$$\left\| \frac{B^{*}A + D^{*}C}{2} \right\|^{2} \leq \left\| \frac{(A^{*}A)^{r} + (C^{*}C)^{r}}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(B^{*}B)^{s} + (D^{*}D)^{s}}{2} \right\|^{\frac{1}{s}}. \tag{3.1}$$

Proof. By the Schwarz inequality in the Hilbert space $(H; \langle ., . \rangle)$ we have

$$\begin{aligned} & |\langle (B^{*}A + D^{*}C)\, x, y \rangle|^{2} \\ & = |\langle B^{*}Ax, y \rangle + \langle D^{*}Cx, y \rangle|^{2} \\ & \leq [|\langle B^{*}Ax, y \rangle| + |\langle D^{*}Cx, y \rangle|]^{2} \\ & \leq \left[\langle A^{*}Ax, x \rangle^{\frac{1}{2}} \cdot \langle B^{*}By, y \rangle^{\frac{1}{2}} + \langle C^{*}Cx, x \rangle^{\frac{1}{2}} \cdot \langle D^{*}Dy, y \rangle^{\frac{1}{2}} \right]^{2}, \end{aligned} \tag{3.2}$$

for any $x, y \in H$.

Now, on utilizing the elementary inequality

$$(ab + cd)^{2} \leq \left(a^{2} + c^{2}\right)\left(b^{2} + d^{2}\right), \qquad a, b, c, d \in \mathbb{R},$$

we then conclude that

$$\begin{aligned} & \left[\langle A^{*}Ax, x \rangle^{\frac{1}{2}} \cdot \langle B^{*}By, y \rangle^{\frac{1}{2}} + \langle C^{*}Cx, x \rangle^{\frac{1}{2}} \cdot \langle D^{*}Dy, y \rangle^{\frac{1}{2}} \right]^{2} \\ & \qquad \leq (\langle A^{*}Ax, x \rangle + \langle C^{*}Cx, x \rangle) \cdot (\langle B^{*}By, y \rangle + \langle D^{*}Dy, y \rangle), \end{aligned} \tag{3.3}$$

for any $x, y \in H$.

Now, on making use of a similar argument to the one in the proof of Theorem 1, we have for $r, s \geq 1$ that

$$\begin{aligned} & (\langle A^{*}Ax, x \rangle + \langle C^{*}Cx, x \rangle) \cdot (\langle B^{*}By, y \rangle + \langle D^{*}Dy, y \rangle) \\ & \quad \leq 4 \left\langle \left[\frac{(A^{*}A)^{r} + (C^{*}C)^{r}}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(B^{*}B)^{s} + (D^{*}D)^{s}}{2} \right] y, y \right\rangle^{\frac{1}{s}} \end{aligned} \tag{3.4}$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Consequently, by (3.2) – (3.4) we have

$$\begin{aligned} & \left| \left\langle \left[\frac{B^{*}A + D^{*}C}{2} \right] x, y \right\rangle \right|^{2} \\ & \quad \leq \left\langle \left[\frac{(A^{*}A)^{r} + (C^{*}C)^{r}}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(B^{*}B)^{s} + (D^{*}D)^{s}}{2} \right] y, y \right\rangle^{\frac{1}{s}} \end{aligned} \tag{3.5}$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Taking the supremum over $x, y \in H$, $\|x\| = \|y\| = 1$ we deduce the desired inequality (3.1). □

Remark 2. If $s = r$, then the inequality (3.1) is equivalent with

$$\left\| \frac{B^*A + D^*C}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\| \cdot \left\| \frac{(B^*B)^r + (D^*D)^r}{2} \right\|. \quad (3.6)$$

Corollary 3. *For any $A, C \in B(H)$ we have*

$$\left\| \frac{A + C}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|, \quad (3.7)$$

where $r \geq 1$. Also, we have

$$\left\| \frac{A^2 + C^2}{2} \right\|^{2} \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(AA^*)^s + (CC^*)^s}{2} \right\|^{\frac{1}{s}} \quad (3.8)$$

for all $r, s \geq 1$, and in particular

$$\left\| \frac{A^2 + C^2}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\| \cdot \left\| \frac{(AA^*)^r + (CC^*)^r}{2} \right\| \quad (3.9)$$

for $r \geq 1$.

The inequality (3.7) follows from (3.1) for $B = D = I$, while the inequality (3.8) is obtained from the same inequality (3.1) for $B = A^*$ and $D = C^*$.

Another particular result of interest is the following one:

Corollary 4. *For any $A, B \in B(H)$ we have*

$$\left\| \frac{B^*A + A^*B}{2} \right\|^{2} \leq \left\| \frac{(A^*A)^r + (B^*B)^r}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(A^*A)^s + (B^*B)^s}{2} \right\|^{\frac{1}{s}} \quad (3.10)$$

for $r, s \geq 1$ and, in particular,

$$\left\| \frac{B^*A + A^*B}{2} \right\|^{r} \leq \left\| \frac{(A^*A)^r + (B^*B)^r}{2} \right\| \quad (3.11)$$

for any $r \geq 1$.

The inequality (3.9) follows from (3.1) for $D = A$ and $C = B$.

Another particular case that might be of interest is the following one.

Corollary 5. *For any $A, D \in B(H)$ we have*

$$\left\| \frac{A + D}{2} \right\|^{2} \leq \left\| \frac{(A^*A)^r + I}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(DD^*)^s + I}{2} \right\|^{\frac{1}{s}}, \quad (3.12)$$

where $r, s \geq 1$. *In particular*

$$\|A\|^2 \leq \left\| \frac{(A^*A)^r + I}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(AA^*)^s + I}{2} \right\|^{\frac{1}{s}}. \tag{3.13}$$

Moreover, for any $r \geq 1$ *we have*

$$\|A\|^{2r} \leq \left\| \frac{(A^*A)^r + I}{2} \right\| \cdot \left\| \frac{(AA^*)^r + I}{2} \right\|.$$

The proof is obvious by the inequality (3.1) on choosing $B = I$, $C = I$ and writing the inequality for D^* instead of D.

Remark 3. If $T \in B(H)$ and $T = A + iC$, i.e., A and C are its Cartesian decomposition, then we get from (3.7) that

$$\|T\|^{2r} \leq 2^{2r-1} \left\| A^{2r} + C^{2r} \right\|,$$

for any $r \geq 1$.

Also, since $A = \operatorname{Re}(T) = \frac{T+T^*}{2}$ and $C = \operatorname{Im}(T) = \frac{T-T^*}{2i}$, then from (3.7) we get the following inequalities as well

$$\|\operatorname{Re}(T)\|^{2r} \leq \left\| \frac{(T^*T)^r + (TT^*)^r}{2} \right\|$$

and

$$\|\operatorname{Im}(T)\|^{2r} \leq \left\| \frac{(T^*T)^r + (TT^*)^r}{2} \right\|$$

for any $r \geq 1$.

In terms of the *Euclidean radius* of two operators $w_e(\cdot, \cdot)$, where, as in [1],

$$w_e(T, U) := \sup_{\|x\|=1} \left(|\langle Tx, x\rangle|^2 + |\langle Ux, x\rangle|^2 \right)^{\frac{1}{2}},$$

we have the following result as well.

Theorem 4. *For any* $A, B, C, D \in B(H)$ *and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *we have the inequality*

$$w_e^2(B^*A, D^*C) \leq \|(A^*A)^p + (C^*C)^p\|^{1/p} \cdot \|(B^*B)^q + (D^*D)^q\|^{1/q}. \tag{3.14}$$

Proof. For any $x \in H$, $\|x\| = 1$ we have the inequalities

$$\begin{aligned}
&|\langle B^*Ax, x\rangle|^2 + |\langle D^*Cx, x\rangle|^2 \\
&\leq \langle A^*Ax, x\rangle \cdot \langle B^*Bx, x\rangle + \langle C^*Cx, x\rangle \cdot \langle D^*Dx, x\rangle \\
&\leq \left(\langle A^*Ax, x\rangle^p + \langle C^*Cx, x\rangle^p\right)^{1/p} \cdot \left(\langle B^*Bx, x\rangle^q + \langle D^*Dx, x\rangle^q\right)^{1/q} \\
&\leq \left(\langle (A^*A)^p x, x\rangle + \langle (C^*C)^p x, x\rangle\right)^{1/p} \cdot \left(\langle (B^*B)^q x, x\rangle + \langle (D^*D)^q x, x\rangle\right)^{1/q} \\
&\leq \langle [(A^*A)^p + (C^*C)^p] x, x\rangle^{1/p} \cdot \langle [(B^*B)^q + (D^*D)^q] x, x\rangle^{1/q}.
\end{aligned}$$

Taking the supremum over $x \in H$, $\|x\| = 1$ and noticing that the operators $(A^*A)^p + (C^*C)^p$ and $(B^*B)^q + (D^*D)^q$ are self-adjoint, we deduce the desired inequality (3.14). □

The following particular case is of interest.

Corollary 6. *For any $A, C \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$w_e^2(A, C) \le 2^{1/q} \|(A^*A)^p + (C^*C)^p\|^{1/p}.$$

The proof follows from (3.14) for $B = D = I$.

Corollary 7. *For any $A, D \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$w_e^2(A, D) \le \|(A^*A)^p + I\|^{1/p} \cdot \|(D^*D)^q + I\|^{1/q}.$$

4. Norm inequalities for the commutator

The commutator of two bounded linear operators T and U is the operator $TU - UT$. For the usual norm $\|\cdot\|$ and for any two operators T and U, by using the triangle inequality and the submultiplicity of the norm, we can state the following inequality

$$\|TU - UT\| \le 2\|U\|\|T\|. \tag{4.1}$$

In [10], the following result has been obtained as well

$$\|TU - UT\| \le 2\min\{\|T\|, \|U\|\}\min\{\|T - U\|, \|T + U\|\}. \tag{4.2}$$

By utilizing Theorem 3 we can state the following result for the numerical radius of the commutator.

Proposition 1. *For any $T, U \in B(H)$ and $r, s \ge 1$ we have*

$$\|TU - UT\|^2 \le 2^{2-\frac{1}{r}-\frac{1}{s}} \|(T^*T)^r + (U^*U)^r\|^{\frac{1}{r}} \cdot \|(TT^*)^s + (UU^*)^s\|^{\frac{1}{s}}. \tag{4.3}$$

Proof. Follows by Theorem 3 on choosing $B = T^*$, $A = U$, $D = -U^*$ and $C = T$. □

Remark 4. In particular, for $r = s$ we get from (4.3) that

$$\|TU - UT\|^{2r} \le 2^{2r-2} \|(T^*T)^r + (U^*U)^r\| \cdot \|(TT^*)^r + (UU^*)^r\| \tag{4.4}$$

and for $r = 1$ we get

$$\|TU - UT\|^2 \le \|T^*T + U^*U\| \cdot \|TT^* + UU^*\|. \tag{4.5}$$

For a bounded linear operator $T \in B(H)$, the self-commutator is the operator $T^*T - TT^*$. Observe that the operator $V := -i(T^*T - TT^*)$ is self-adjoint and $w(V) = \|V\|$, i.e.,

$$w(T^*T - TT^*) = \|T^*T - TT^*\|.$$

Now, utilizing (4.3) for $U = T^*$ we can state the following corollary.

Corollary 8. *For any $T \in B(H)$ we have the inequality*

$$\|T^*T - TT^*\|^2 \le 2^{2-\frac{1}{r}-\frac{1}{s}} \|(T^*T)^r + (TT^*)^r\|^{\frac{1}{r}} \cdot \|(T^*T)^s + (TT^*)^s\|^{\frac{1}{s}} . \quad (4.6)$$

In particular, we have

$$\|T^*T - TT^*\|^r \le 2^{r-1} \|(T^*T)^r + (TT^*)^r\| , \quad (4.7)$$

for any $r \ge 1$.

Moreover, for $r = 1$ we have

$$\|T^*T - TT^*\| \le \|T^*T + TT^*\| . \quad (4.8)$$

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