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ON THE DISCRETE NONLINEAR HAMMERSTEIN SYSTEMS WITH NON-SYMMETRIC KERNELS

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Dedicated to Professor Harry I. Miller on the occasion of his 70th birthday

ABSTRACT. We study the nonlinear Hammerstein system

$$x(t)=\sum_{s=1}^{\infty}k(s,t)f(s,x(s))+g(t)\quad (t\in\mathbb{N})$$

with non-symmetric kernel $k(s,t)$.

1. INTRODUCTION

The discrete nonlinear Hammerstein system of equations, which we study in this article, occurs in certain stochastic problems. We establish some results about unique solvability of the nonlinear discrete Hammerstein system with non-symmetric kernel $k(s,t)\,(s,t\in\mathbb{N})$

$$x(t)=\sum_{s=1}^{\infty}k(s,t)f(s,x(s))+g(t)\quad (t\in\mathbb{N}). \tag{1}$$

Here $k:\mathbb{N}\times\mathbb{N}\to\mathbb{R}$ defines a linear bounded operator

$$Kx(t)=\sum_{s=1}^{\infty}k(s,t)x(s), \tag{K}$$

and $f:\mathbb{N}\times\mathbb{R}\to\mathbb{R}$ is a real function which generates a nonlinear operator superposition F; x belongs to l_p. The problem of solvability of the system (1) is equivalent to the problem of solvability of the operator equation

$$x=KFx+g\quad (x,g\in l_p) \tag{2}$$

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The function g is a given real function of a natural argument; the function x is an unknown real sequence.

The first results about unique solvability of the system (1) have been obtained in [8], see also [6]. In the studies of unique solvability of the Hammerstein equations, generally speaking, there exist two kinds of assumptions. The first one is about linear part K of this equation, and the second are assumptions on the nonlinearity F of the equation (2). As for the operator K the standard assumption on the matrix-kernel is symmetry $k(s,t) = k(t,s)$, and some of the results in that constellation were given in [5]. The case of non-symmetric kernels in the system (1) has not been studied so far. Also, the case of Hammerstein integral equations were studied, for example, in [11-12], see also [3] and [4]. In this article applying Minty's fix-point theorem for monotone operators [13], see also [7], we get some new facts about the solvability of the system (1).

2. Linear and non-linear part of the equation

Since the formulation of our results will be in $l_p (1 \leq p \leq +\infty)$ spaces, we need to recall some facts from the operator theory in classical spaces of sequences, and make several easy assumptions.

The linear operator K, which acts from $l_{p'}$ into l_p, $(p' = p(p-1)^{-1}; \infty' = 1)$ defined by the non-symmetric matrix-kernel $k : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is a bounded operator in l_2 $(l_{p'})$ space, and at the same time is a compact operator in those spaces if the following holds [9-10]

$$\sum_{s=1}^{\infty} \sum_{t=1}^{\infty} \mid k(s,t) \mid^{max\{2,p'\}} < \infty. \tag{3}$$

Below we denote, as usual, the scalar product in l_2 by

$$\langle x, y \rangle = \sum_{s=1}^{\infty} x(s)y(s). \tag{4}$$

Suppose that operator (K) acts not only in l_2, but also from $l_{p'}$ into l_p, where $2 \leq p \leq \infty$. Let

$$A = \frac{1}{2}(K + K^*) \tag{5}$$

denote the *self-adjoint part* of K, where K^* is the adjoint operator defined by $K^*x(t) = \sum_{s=1}^{\infty} k(t,s)x(s)$. For our further use we should introduce

$$B = \frac{1}{2}(K - K^*). \tag{6}$$

One can see that both operators A and B act from the space $l_{p'}$ into the space l_p. Let $A = UL$ be a polar decomposition of the operator A into a superposition of a unitary operator U, acting in l_2 and the positive operator defined by L, $C = L^{\frac{1}{2}}$. We assume here that A is a positive operator. It is known that a linear bounded operator A is called positive defined in l_2 if $\langle Ax, x\rangle \geq 0$ holds for all $x \in l_2$. Moreover, the operator A can be represented as $A = CC^*$ (U=I, L=A) where $C = A^{\frac{1}{2}}$ is the square root of A acting from l_2 into l_p. The adjoint operator C^* acts from $l_{p'}$ into l_2, because in this situation, we have $l_{p'} \subset l_2 \subset l_p$ $(2 \leq p \leq \infty)$, i.e. $l_{p'}, l_2$ and l_p make a *regular triple-spaces*, see [11].

Definition. *We say the operator K is $\mathbb{P}$-positive if it satisfies the angle-bounded inequality*

$$|\langle Kx, y\rangle - \langle x, Ky\rangle| \leq \beta\sqrt{\langle Kx, x\rangle}\sqrt{\langle Ky, y\rangle} \quad (x, y \in l_2) \tag{7}$$

where $\beta \in \mathbf{R}^+$, and operator (5) *is a positive operator.*

Let us examine now the operators $M = C^{-1}K(C^*)^{-1}$ and $N = K(C^*)^{-1}$, and note that, under our assumptions, both M and N act in the space l_2.

Lemma 1. *If the operator K is $\mathbb{P}$-positive then the operators M and N are bounded.*

Proof. The operator $M = C^{-1}K(C^*)^{-1}$ is bounded in l_2 if and only if $C^{-1}B(C^*)^{-1}$ is bounded in l_2. Moreover, the same statement is valid for the operators $N = K(C^*)^{-1}$ and the $B(C^*)^{-1}$. Firstly, since for any $h \in l_2$, we have

$$\begin{aligned}
&\langle Mh, h\rangle + \langle h, Mh\rangle - 2\langle h, h\rangle \\
&= \langle K(C^*)^{-1}h, (C^*)^{-1}h\rangle + \langle K^*(C^*)^{-1}h, (C^*)^{-1}h\rangle - 2\langle C^*(C^*)^{-1}h, C^*(C^*)^{-1}h\rangle \\
&= \langle K(C^*)^{-1}h, (C^*)^{-1}h\rangle + \langle K^*(C^*)^{-1}h, (C^*)^{-1}h\rangle - 2\langle A(C^*)^{-1}h, (C^*)^{-1}h\rangle = 0
\end{aligned}$$

one can conclude that holds

$$\langle Mh, h\rangle = \|h\|^2 \quad (h \in l_2). \tag{8}$$

The relation (8), in particular, means that both operators M and M^* have a trivial null-space.

On the other hand, for arbitrary $h_1, h_2 \in l_2$, is

$$\begin{aligned}
&|\langle C^{-1}B(C^*)^{-1}h_1, h_2\rangle| = |\langle B(C^*)^{-1}h_1, (C^*)^{-1}h_2\rangle| = |\langle B\phi, \theta\rangle| \\
&= |\frac{1}{2}\langle (K - K^*)\phi, \theta\rangle| = |\frac{1}{2}\langle K\phi, \theta\rangle - \frac{1}{2}\langle K^*\phi, \theta\rangle| = |\frac{1}{2}\langle K\phi, \theta\rangle - \frac{1}{2}\langle \phi, K\theta\rangle| \\
&\leq \frac{1}{2}\beta\sqrt{\langle K\phi, \phi\rangle}\sqrt{\langle K\theta, \theta\rangle} \\
&= \frac{1}{2}\beta\sqrt{\langle K(C^*)^{-1}h_1, (C^*)^{-1}h_1\rangle}\sqrt{\langle K(C^*)^{-1}h_2, (C^*)^{-1}h_2\rangle}
\end{aligned}$$

$$= \frac{1}{2}\beta\sqrt{\langle C^{-1}K(C^*)^{-1}h_1, h_1\rangle}\sqrt{\langle C^{-1}K(C^*)^{-1}h_2, h_2\rangle}$$
$$= \frac{1}{2}\beta\sqrt{\langle Mh_1, h_1\rangle}\sqrt{\langle Mh_2, h_2\rangle} = \frac{1}{2}\beta\|h_1\|\|h_2\|$$

therefore the operator $M = C^{-1}K(C^*)^{-1}$ is bounded. As for boundedness of the operator $B(C^*)^{-1}$ we have

$$|\langle B(C^*)^{-1}h, g\rangle| = \left|\frac{1}{2}\langle (K - K^*)(C^*)^{-1}h, g\rangle\right|$$
$$= \left|\frac{1}{2}\langle K(C^*)^{-1}h, g\rangle - \frac{1}{2}\langle K^*(C^*)^{-1}h, g\rangle\right| = \left|\frac{1}{2}\langle K(C^*)^{-1}h, g\rangle\right.$$
$$\left. - \frac{1}{2}\langle (C^*)^{-1}h, Kg\rangle\right| \leq \frac{1}{2}\beta\sqrt{\langle K(C^*)^{-1}h, (C^*)^{-1}h\rangle}\sqrt{\langle Kg, g\rangle}$$
$$= \frac{1}{2}\beta\sqrt{\langle C^{-1}K(C^*)^{-1}h, h\rangle}\sqrt{\langle Kg, g\rangle} = \frac{1}{2}\beta\sqrt{\langle Mh, h\rangle}\sqrt{\langle Kg, g\rangle}$$
$$\leq \frac{1}{2}\widetilde{\beta}\|h\|\|g\|,$$

i.e., the operator N is bounded. □

The operator N has also a trivial null-space, since $N = CM$ (relation (10)) and the operator C each have a trivial null-space as well. Now we can keep the same notation M for the continuous extension (closure) in l_2 of the operator $C^{-1}K(C^*)^{-1}$. The closure in l_2 of the operator $K(C^*)^{-1}$ we denote by N. The operators M and N, in fact, are defined on the closure in l_2 of the range of $C = A^{\frac{1}{2}}$, but it is clear that this closure, in our situation coincides with l_2. In this situation K has two essential decompositions

$$K = CMC^*, \qquad K = NC^*. \tag{9}$$

On the other hand, we see that operators M, N, and K are related by

$$N = CM, \qquad N^* = M^*C^*. \tag{10}$$

In what follows the positive numbers μ for which we have the inequality

$$\|Kh\|^2 \leq \mu\langle Kh, h\rangle, (h \in l_2) \tag{μ}$$

plays an important role. Let us note, that the smallest such a μ is $\mu_K = \|N\|^2$. Indeed,

$$\langle Kh, Kh\rangle = \langle NC^*h, NC^*h\rangle = \|NC^*h\|^2 \leq \|N\|^2\langle C^*h, C^*h\rangle$$
$$= \|N\|^2\langle CC^*h, h\rangle = \|N\|^2\langle Ah, h\rangle = \frac{1}{2}\|N\|^2\langle (K + K^*)h, h\rangle = \|N\|^2\langle Kh, h\rangle$$

The nonlinear part of the equation (2) is the operator superposition

$$Fx(s) = f(s, x(s)) \quad (s \in \mathbb{N}) \tag{11}$$

which acts from l_p into $l_{p'}$, generated by a function $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$. We suppose in addition that $f(s, 0) = 0$ holds, though this condition could be easy omitted. Indeed, in (11) we can replace by the operator $\widetilde{F}x = F(x + \widetilde{x}) - F\widetilde{x}$, which is also a superposition operator, generated by the function $\widetilde{f}(s, u) = f(s, u + \widetilde{x}(s)) - f(s, \widetilde{x}(s))$, where $\widetilde{x} \in l_p$ is arbitrary and $\widetilde{f}$ satisfies condition $\widetilde{f}(s, 0) = 0$.

Due to [5] (see also [2]), the operator F, in the case $1 \le p < \infty$, acts from l_p into $l_{p'}$ if and only if there exist $a(s) \in l_{p'}$, and constants $b \ge 0, \delta > 0, n_0 \in \mathbb{N}$ for which

$$|f(s, u)| \le a(s) + b\,|u|^{p-1} \quad (|u| < \delta, s \ge n_0). \tag{12}$$

In the case $p = \infty$ the last estimate must be replaced by

$$|f(s, u)| \le a_r(s) \quad (|u| \le r, 0 < r < \infty)$$

where $a_r(s) \in l_1$.

Now we suppose that there exists a number c such that holds

$$(u - v)(f(s, u) - f(s, v)) \le c(u - v)^2 \quad (s \in \mathbb{N}, u \in \mathbb{R}). \tag{c}$$

If c_f is the smallest c for which (c) holds, we have

$$\begin{aligned}\langle Fh^* - Fh^{**}, h^* - h^{**}\rangle &= \sum_{s\in\mathbb{N}}[f(s, h^*(s)) - f(s, h^{**}(s))][h^*(s) - h^{**}(s)] \\ &\le c_f \sum_{s\in\mathbb{N}}[h^*(s) - h^{**}(s)][h^*(s) - h^{**}(s)] = c_f\|h^* - h^{**}\|^2.\end{aligned}$$

3. Solution of the system with positive defined kernels

Theorem 1. *Let the operator K defined by* (K) *be a $\mathbb{P}$-positive. Suppose that the generator $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ of the superposition operator F given by* (11)*, satisfies condition* (c) *for some $c_f > 0$ and $f(s, 0) = 0$ for all $s \in \mathbb{N}$. If*

$$c_f \mu_K < 1 \tag{13}$$

where μ_K is defined by (μ), then, for arbitrary $g \in N(l_2)$ the equation (2)*:*

$$x = KFx + g$$

has, a solution $\hat{x} \in N(l_2)$. If $g = Nl$ for some $l \in l_2$ then there exists $\hat{h} \in l_2$ such that $\hat{x} = N\hat{h}$, and

$$\|\hat{h}\| \le \frac{\|l\|}{1 - c_f\mu_K}; \tag{14}$$

moreover, the solution $\hat{x}$ is unique in the space l_p.

Proof. Let us put $\Pi h = M^*h - N^*FNh - M^*l$, and consider the operator equation

$$\Pi h = 0 \tag{15}$$

i.e. $M^*h = N^*FNh + M^*l$. If $\hat{h}$ is a solution of the equation $\Pi h = 0$, i.e. if $M^*\hat{h} = N^*FN\hat{h} + M^*l$, holds, then, by the relation (10), we get $M^*(\hat{h} - C^*FN\hat{h} - l) = 0$, hence $\hat{h} = C^*FN\hat{h} + l$ because the operator M^* has trivial null-space (see relation (8)).

On the other hand, applying the operator N to the last equation we get

$$N\hat{h} = NC^*FN\hat{h} + Nl = KFN\hat{h} + g \tag{16}$$

by the relation (9). From the equation (16) and (2) we conclude that $\hat{x} = N\hat{h}$ is a solution of the system (1). Thus, in order to prove the existence of a solution of the equation (2) we are going to study the equation $\Pi h = 0$, where Π is defined above, under the assumptions of the Theorem 1.

One can examine that operator

$$\Pi h = M^*h - N^*FNh - M^*l \tag{17}$$

which is monotone in the Minty-Browder sense (see, [13] or [7]), in fact, for any $h_1, h_2 \in l_2$ we have

$$\begin{aligned}
&\langle \Pi h_1 - \Pi h_2, h_1 - h_2\rangle = \langle M^*h_1 - N^*FNh_1 - M^*h_2 + N^*FNh_2, h_1 - h_2\rangle \\
&= \langle M^*(h_1 - h_2), h_1 - h_2\rangle - \langle N^*FNh_1 - N^*FNh_2, h_1 - h_2\rangle \\
&= \langle M^*(h_1 - h_2), h_1 - h_2\rangle - \langle FNh_1 - FNh_2, Nh_1 - Nh_2\rangle \\
&= \|h_1 - h_2\|^2 - \langle FNh_1 - FNh_2, Nh_1 - Nh_2\rangle \geq \|h_1 - h_2\|^2 \\
&\qquad - c_f\langle Nh_1 - Nh_2, Nh_1 - Nh_2\rangle \geq \|h_1 - h_2\|^2 - c_f\|N\|^2\|h_1 - h_2\|^2 \\
&= (1 - c_f\|N\|^2)\|h_1 - h_2\|^2 \geq (1 - c_f\mu_K)\|h_1 - h_2\|^2,
\end{aligned}$$

i.e.

$$\langle \Pi h_1 - \Pi h_2, h_1 - h_2\rangle \geq (1 - c_f\mu_K)\|h_1 - h_2\|^2.$$

From here, on the sphere $S = \{h \in l_2 | \|h\| = r\}$ it follows that

$$\begin{aligned}
\langle \Pi h, h\rangle &= \langle \Pi h - \Pi 0, h - 0\rangle + \langle \Pi 0, h\rangle \geq (1 - c_f\mu_K)\|h\|^2 - \|l\|\|h\| \\
&= (1 - c_f\mu_K)r^2 - \|l\|r,
\end{aligned}$$

since $F0 = 0$ and $M^*(h - C^*FNh - l) = \Pi h$, taking $h = 0$ we have $\Pi(0) = -l$. Consequently if we take a sphere S with

$$r \geq \frac{\|l\|}{1 - c_f\mu_K}$$

then $\langle \Pi h, h\rangle \geq 0$ holds for any $h \in S$.

Now due the Minty-Browder existence principle, the equation (15) has the unique solution $\hat{h} \in S \subset l_2$. On the other hand, as it was shown above,

$\hat{x} = N\hat{h}$ is a solution of the Hammerstein nonlinear system (1). Moreover, if (c) holds, $\hat{x} \in l_2$ is the unique solution of the system (1). In order to prove it, let us suppose that $\hat{x}$ and $\hat{\hat{x}}$ are two solutions of the system (1), with $g = Nl$ for some $l \in l_2$. If we put

$$\hat{h} = C^*F\hat{x} + l, \quad \hat{\hat{h}} = C^*F\hat{\hat{x}} + l$$

then the elements $\hat{h}$ and $\hat{\hat{h}}$ belong to l_2, and since $\hat{x} = N\hat{h}$, $\hat{\hat{x}} = N\hat{\hat{h}}$, we have

$$\hat{h} = C^*FN\hat{h} + l, \quad \hat{\hat{h}} = C^*FN\hat{\hat{h}} + l.$$

Let us consider now the operator (17), and the above given relation (8), we get

$$\Pi\hat{h} = M^*\hat{h} - M^*C^*FN\hat{h} - M^*l = M^*(\hat{h} - C^*FN\hat{h} - l) = M^*(\hat{h} - \hat{h}) = 0,$$

and

$$\Pi\hat{\hat{h}} = M^*\hat{\hat{h}} - M^*C^*FN\hat{\hat{h}} - M^*l = M^*(\hat{\hat{h}} - C^*FN\hat{\hat{h}} - l) = M^*(\hat{\hat{h}} - \hat{\hat{h}}) = 0.$$

Since the equation $\Pi h = 0$ has only one solution, we conclude $\hat{h} = \hat{\hat{h}}$, hence $\hat{x} = \hat{\hat{x}}$, because the operator N has trivial null-space. □

4. Solution of the system with quasi-positive defined kernels

In above analysis the self-adjoint part (5) of the linear operator (K) was, by presumption, positive defined. However, we can apply the described methods to some classes of the operators which are not necessarily positive defined. Let the linear operator (K) acts in l_p but at the same time from $l_{p'}$ into l_p, where p and p' are as above. Moreover, let A and B be defined as in (5) and (6), where K satisfies inequality (7). We suppose now that $A = \frac{1}{2}(K + K^*)$ is a *quasi-positive defined operator*, i.e. operator A has at most a finite number of the negative eigenvalues of the multiplicity 1. If a matrix-kernel $a(s,t)$ has at most a finite number (for example the first n) of the negative eigenvalues each of the multiplicity 1, we can write this kernel in the form

$$a(s,t) = -\sum_{i=1}^{n} \alpha_i e_i(s) e_i(t) + \sum_{i=n+1}^{\infty} \alpha_i e_i(s) e_i(t) \quad (s,t \in \mathbb{N}) \tag{18}$$

where all $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}, \ldots$ are positive numbers. Moreover, the relation

$$l(s,t) = a(s,t) + 2\sum_{i=1}^{n} \alpha_i e_i(s) e_i(t) \quad (s,t \in \mathbb{N}) \tag{19}$$

gives an important connection between the kernel $a(s,t)$ and the kernel $l(s,t)$ which has all positive eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}, \ldots$.

In this situation we can consider the finite-dimensional orthogonal projection P of the l_2 into the subspace of the eigenvectors of A which correspond to the negative eigenvalues of A. The operator P acts at the same time in l_p and $l_{p'}$ and commutes with A. Moreover, in the polar decomposition $A = UL$ mentioned in the previous section allows us to take

$$A = (I - 2P)L \tag{20}$$

where $L = (I - 2P)A$ is now a positive operator. As the operator A in the previous analysis, the operator L from (20) can be represented in the form $L = DD^*$, where $D = L^{\frac{1}{2}}$ acts from l_2 into l_p and D^* acts from $l_{p'}$ into l_2. We will call the operator K - $\mathbb{P}$-quasi-positive if K satisfies condition (7), and its self-adjoint part is a quasi-positive defined operator. Now we need the next

Lemma 2. *If the operator K is $\mathbb{P}$-quasi-positive then operator $M = D^{-1} K(D^*)^{-1}$ satisfies $\langle Mh, h\rangle = \|h\|^2 - 2\|Ph\|^2$ for all $h \in l_2$.*

Proof. For any $h \in l_2$, holds

$$\begin{aligned}\Delta = \langle Mh, h\rangle + \langle h, Mh\rangle - 2\langle (I-2P)h, h\rangle = \langle D^{-1}K(D^*)^{-1}h, h\rangle \\ + \langle D^{-1}K^*(D^*)^{-1}h, h\rangle - 2\langle (I-2P)D^*(D^*)^{-1}h, D^*(D^*)^{-1}h\rangle \\ = \langle K(D^*)^{-1}h, (D^*)^{-1}h\rangle + \langle K^*(D^*)^{-1}h, (D^*)^{-1}h\rangle \\ - 2\langle D(I-2P)D^*(D^*)^{-1}h, (D^*)^{-1}h\rangle \\ = \langle Kg, g\rangle + \langle K^*g, g\rangle - 2\langle (I-2P)Lg, g\rangle = \langle Kg, g\rangle + \langle K^*g, g\rangle - 2\langle Ag, g\rangle = 0.\end{aligned}$$

Since $\Delta = \langle M^*h, h\rangle + \langle h, M^*h\rangle - 2\langle (I-2P)h, h\rangle$, from $\Delta = 0$, one can easy get $\langle Mh, h\rangle = \|h\|^2 - 2\|Ph\|^2$. □

Now using Lemma 2 we have

$$\begin{aligned}|\langle D^{-1}B(D^*)^{-1}h_1, h_2\rangle| = \langle B(D^*)^{-1}h_1, (D^*)^{-1}h_2\rangle| = |\langle B\varphi, \psi\rangle| \\ = \frac{1}{2}|\langle K\varphi, \psi\rangle - \langle \varphi, K\psi\rangle| \leq \frac{1}{2}\beta\sqrt{\langle K\varphi, \varphi\rangle}\sqrt{\langle K\psi, \psi\rangle} \\ = \frac{1}{2}\beta\sqrt{\langle Mh_1, h_1\rangle}\sqrt{\langle Mh_2, h_2\rangle} \\ = \frac{1}{2}\beta\sqrt{\|h_1\|^2 - 2\|Ph_1\|^2}\sqrt{\|h_2\|^2 - 2\|Ph_2\|^2} \leq \frac{1}{2}\beta\|h_1\|\|h_2\|\end{aligned}$$

hence the operator M is bounded. Analogously we can prove the boundedness of the operator $N = K(D^*)^{-1}$; moreover, below we denote by M and N, as in the previous section, the closure in l_2 of the bounded operators $D^{-1}K(D^*)^{-1}$ and $K(D^*)^{-1}$.

The operators M and N are defined on the whole space l_2, and the relations

$$K = DMD^*, \quad K = ND^*, \quad N = DM, \quad N^* = M^*D^* \tag{21}$$

connect these operators. Below we use the number

$$\nu_K = sup\{\nu | \nu > 0, \|Nh\| \geq \sqrt{\nu}\|Ph\| \quad (h \in l_2)\}, \tag{ν}$$

which is, in the case $K = K^*$, in fact, the absolute value of the largest negative eigenvalues of $A = K$, ([11] or [5]).

Suppose again that function f generates the superposition operator (11) between l_p and $l_{p'}$ and satisfies the condition (c) with c_f as the smallest real number for which it holds.

Theorem 2. *Let the operator K be defined by* (K) *be $\mathbb{P}$-quasi-positive in l_2. Suppose that the generator $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ of the superposition operator F given by* (11)*, satisfies* (c)*, for some $c_f > 0$ and $f(s, 0) = 0$ for all $s \in \mathbb{N}$. If*

$$c_f \nu_K < -1 \tag{22}$$

where ν_K is defined above by (ν), then, for arbitrary $g \in N(l_2)$, equation (2)*:*

$$x = KFx + g$$

has a solution $\hat{x} \in N(l_2)$. If $g = Nl$ for some $l \in l_2$ then there exists $\hat{h} \in l_2$ such that $\hat{x} = N\hat{h}$, and

$$\|\hat{h}\| \leq -\frac{\|l\|}{1 + c_f \upsilon_K}; \tag{23}$$

moreover, the solution $\hat{x}$ is unique in the space l_p.

Proof. The essential point of difference from the proof of Theorem 1, here is how to use the operator $\Pi h = M^*h - N^*FNh - M^*l$ ((17)), and the equation (15) in order to provide application of the Minty's theorem. In fact, for all $h_1, h_2 \in l_2$, we have now

$$\begin{aligned}
&\langle \Pi h_1 - \Pi h_2, h_1 - h_2 \rangle = \langle M^*h_1 - N^*FNh_1 - M^*h_2 + N^*FNh_2, h_1 - h_2 \rangle \\
&= \langle M^*(h_1 - h_2), h_1 - h_2 \rangle - \langle N^*FNh_1 - N^*FNh_2, h_1 - h_2 \rangle \\
&= \langle M^*(h_1 - h_2), h_1 - h_2 \rangle - \langle FNh_1 - FNh_2, Nh_1 - Nh_2 \rangle \\
&\geq \|h_1 - h_2\|^2 - 2\|P(h_1 - h_2)\|^2 - c_f \langle Nh_1 - Nh_2, Nh_1 - Nh_2 \rangle \\
&\geq \|h_1 - h_2\|^2 - 2\|P(h_1 - h_2)\|^2 - c_f \upsilon_K \|P(h_1 - h_2)\|^2 \\
&= \|h_1 - h_2\|^2 - (2 + c_f \upsilon_K)\|P(h_1 - h_2)\|^2 \geq -(1 + c_f \upsilon_K)\|h_1 - h_2\|^2.
\end{aligned}$$

We remark that for all $h \in S \subset l_2$, since $\Pi 0 = -l$, we get

$$\begin{aligned}
\langle \Pi h, h \rangle &= \langle \Pi h - \Pi 0, h - 0 \rangle + \langle \Pi 0, h \rangle \geq -(1 + c_f \upsilon_K)\|h\|^2 - \|l\|\|h\| \\
&= -(1 + c_f \upsilon_K) r^2 - \|l\| r.
\end{aligned}$$

Consequently if we chose a sphere $S = \{h \in l_2 | \|h\| = r\}$ with

$$r \geq -\frac{\|l\|}{1 + c_f \upsilon_K}$$

then we can see that $\langle \Pi h, h \rangle \geq 0$ holds for any $h \in S$.

The last part of the proof of Theorem 2 is literally the same as the corresponding part of the proof in Theorem 1, so that the Theorem 2 is proved. □

5. Conclusion

In conclusion, we note that the results obtained in this article on the existence of unique solution of the system (1), allows an easy application. One can carry over our argument to the case when operators (K) and (11) act in weighted spaces $l_{p,\sigma}$, where σ is a weight function.

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References

[1] N. I. Ahiezer ans I. M. Glazman, *Linear Operator Theory in Hilbert Spaces*, Engl.transl. Pitman, Boston, 1981.

[2] J. Appell and P. P. Zabreiko, *Nonlinear superposition operators*, Cambridge University Press, Cambridge 1990.

[3] J. Appell, E. De Pascal and P.P. Zabreiko, *On the Unique Solvability of Hammerstein Integral Equations with Non-Symmetric Kernels*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 40, 27-34, Birkhauser, Basel-Boston- Berlin, 2000.

[4] S.N. Ashabov, *Investigation of nonlinear singular integral equations by the method of monotone operators*,(in Russian), Izv.Severo-Kavkaz. Nauchn. Tsentra Vyssh. Shkoly ser. Estestv. Nauk., no. 2,3,... 5,108, 1980.

[5] F. Dedagich and P. P. Zabreiko, *On the superposition operators in l_p spaces.* Sib. Mat. Zh., XXVIII, N^o1. Engl.transl., Plenum Publishing Corporation, 1987.

[6] M. Golomb, *Zur Theorie der nichtlinearen Integralgleichungen, Integralgleichungssysteme und Functionalgleichungen*, Math. Z., 39 (1934), 45–75.

[7] O. Hadzich, *Introduction in the fixpoint theory* , Institut for mathematics Novi Sad, 1978.

[8] A. Hammerstein, *Nichtlinear Integralgleichungen nebst Anwendungen*, Acta Math., 54 (1930),117–176.

[9] G. H. Hardy, J. E. Littlewood and G. Polija, *Inequalities*, At the University Press – Cambridge, 1934.

[10] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, Engl.transl., Pergamon Press, Oxford, 1982.

[11] M. A. Krasnosel'skij, *Topological Methods in the Theory of Nonlinear Integral Equations*, Engl.tansl., Macmillan, New York, 1964.

[12] M. A. Krasnosel'skij and P. P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*, Engl. transl.: Springer, Berlin, 1984.

[13] G. Minty, *Monotone nonlinear operators in Hilbert spaces*, Duke Math. J., 29 (1962), 311–316.

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