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CALCULATION OF THE MOMENTS OF THE CARDINAL B-SPLINE

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Dedicated to Professor Harry Miller on the occasion of his 70th birthday

ABSTRACT. In this paper we describe five methods for the calculation of the moments

$$\mathbb{M}_{n,m} = \int_0^m \varphi_m(t) t^n dt, n \in \mathbb{N}_0,$$

where weight function $\varphi_m(\cdot)$ is the cardinal B-spline of order $m, m \in \mathbb{N}$.

1. Introduction

Calculating moments of a given weight function has essential significance in the construction of the orthogonal polynomials and quadrature rules, as well as in other fields of the approximation theory. On the other hand, a very frequent weight function is the exactly cardinal B-spline (finite elements method, multiresolution approximation,...). In this paper we describe five methods for calculation of the moments of the cardinal B-spline. The first three methods are simple consequences of the basic properties of the cardinal B-spline. Their basic disadvantage is a recursive calculation. The fourth method can be used to calculate the first $m-1$ moments (if m is even) or the first m moments (if m is odd). This method is not recursive and uses only values of the cardinal B-spline at integer points. The basic disadvantage of this method is, of course, the limited number of moments which can be calculated. The last method is also a simple consequence of the basic properties of the cardinal B-spline. This method is the most valuable in practical realizations.

At the end of the introductory part we give the definition of the cardinal B-spline and a list of its basic properties.

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Definition 1. *The cardinal B-spline of the first order, denoted by $\varphi_1(\cdot)$, is the characteristic function of the unit interval, i.e.*

$$\varphi_1(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & \textit{otherwise.} \end{cases}$$

The Cardinal B-spline of order $m, m \in \mathbb{N}$, denoted by $\varphi_m(\cdot)$, is

$$\begin{aligned} \varphi_m(x) = (\varphi_{m-1} * \varphi_1)(x) &= \int_{\mathbb{R}} \varphi_{m-1}(x-t)\varphi_1(t)dt \\ &= \int_0^1 \varphi_{m-1}(x-t)dt. \end{aligned}$$

Theorem 1. *The cardinal B-spline of order $m, m \in \mathbb{N}$, has the following properties*

(1) *For every m times differentiable function $g(\cdot)$*

$$\int_{\mathbb{R}} \varphi_m(x) g^{(m)}(x)dx = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} g(k); \tag{1}$$

(2) $supp\varphi_m(\cdot) = [0, m]$;

(3) $(\forall t \in [0,m])\, \varphi_m(t) \geq 0$;

(4) $\int_{\mathbb{R}} \varphi_m(t)dt = 1$;

(5)

$$(\forall t \in [0,m])\, \varphi_m(t) = \frac{t}{m-1}\varphi_{m-1}(t) + \frac{m-t}{m-1}\varphi_{m-1}(t-1), m \geq 2; \tag{2}$$

(6)

$$(\forall t \in [0,m])\, \varphi'_m(t) = \varphi_{m-1}(t) - \varphi_{m-1}(t-1), m \geq 2; \tag{3}$$

(7) $\left(\forall t \in [0, \frac{m}{2}]\right) \varphi_m\left(\frac{m}{2} - t\right) = \varphi_m\left(\frac{m}{2} + t\right)$;

(8)

$$(\forall a \in \mathbb{R}) \sum_{i \in \mathbb{Z}} \varphi_m(i-a) = 1; \tag{4}$$

(9) *The cardinal B-spline satisfies the so called refinable equation*

$$(\forall t \in \mathbb{R})\, \varphi_m(t) = \frac{1}{2^{m-1}} \sum_{k=0}^{m} \binom{m}{k} \varphi_m(2t-k). \tag{5}$$

The proof of this theorem and many more details on the cardinal B-splines one can find in [1] or [2].

2. Calculation of the moments

The first method:

Using the refinable equation (5) we obtain

$$\begin{aligned}\mathbb{M}_{n,m} &= \frac{1}{2^{m-1}}\sum_{k=0}^{m}\binom{m}{k}\int_0^m \varphi_m(2t-k)t^n dt\\ &= \frac{1}{2^{m+n}}\sum_{k=0}^{m}\binom{m}{k}\int_0^m \varphi_m(x)(x+k)^n dx\\ &= \frac{1}{2^{m+n}}\sum_{k=0}^{m}\binom{m}{k}\left[\sum_{l=0}^{n-1}\binom{n}{l}k^{n-l}\mathbb{M}_{l,m}+\mathbb{M}_{n,m}\right]\\ &= \frac{1}{2^{m+n}}\sum_{k=1}^{m}\sum_{l=0}^{n-1}\binom{m}{k}\binom{n}{l}k^{n-l}\mathbb{M}_{l,m}+\frac{1}{2^n}\mathbb{M}_{n,m}.\end{aligned}$$

Hence,

$$\mathbb{M}_{n,m} = \frac{1}{2^m(2^n-1)}\sum_{k=1}^{m}\sum_{l=0}^{n-1}\binom{m}{k}\binom{n}{l}k^{n-l}\mathbb{M}_{l,m}$$

The second method:

Using equality (2) we obtain

$$\begin{aligned}\mathbb{M}_{n,m} &= \frac{1}{m-1}\int_0^m \left[\varphi_{m-1}(t)t+\varphi_{m-1}(t-1)(m-t)\right]t^n dt\\ &= \frac{1}{m-1}\int_0^{m-1}\varphi_{m-1}(x)x^{n+1}dx\\ &\qquad+\frac{1}{m-1}\int_0^{m-1}\varphi_{m-1}(x)(m-1-x)(x+1)^n dx\\ &= \frac{1}{m-1}\mathbb{M}_{n+1,m-1}+\sum_{k=0}^{n}\binom{n}{k}\mathbb{M}_{k,m-1}-\frac{1}{m-1}\sum_{k=0}^{n}\binom{n}{k}\mathbb{M}_{k+1,m-1}\\ &= \sum_{k=0}^{n}\binom{n}{k}\mathbb{M}_{k,m-1}-\frac{1}{m-1}\sum_{k=0}^{n-1}\binom{n}{k}\mathbb{M}_{k+1,m-1}\\ &= 1+\sum_{k=1}^{n}\left[\binom{n}{k}-\frac{1}{m-1}\binom{n}{k-1}\right]\mathbb{M}_{k,m-1}.\end{aligned}$$

So, we have that

$$\mathbb{M}_{n,m} = \sum_{k=0}^{n} \binom{n}{k} \mathbb{M}_{k,m-1} - \frac{1}{m-1} \sum_{k=0}^{n-1} \binom{n}{k} \mathbb{M}_{k+1,m-1} \tag{6}$$
$$= 1 + \sum_{k=1}^{n} \left[\binom{n}{k} - \frac{1}{m-1} \binom{n}{k-1} \right] \mathbb{M}_{k,m-1}.$$

The third method:

Using equality (3) and the fact $\varphi_m(0) = \varphi_m(m) = 0$, after integration by parts, we obtain

$$\mathbb{M}_{n,m} = \frac{1}{n+1} \int_0^m \left[\varphi_{m-1}(t-1) - \varphi_{m-1}(t) \right] t^{n+1} dt$$
$$= \frac{1}{n+1} \left[\int_0^{m-1} \varphi_{m-1}(t)(t+1)^{n+1} dt - \int_0^{m-1} \varphi_{m-1}(t) t^{n+1} dt \right]$$
$$= \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} \mathbb{M}_{k,m-1}.$$

The fourth method is, of course, the most important result of this paper. This method generalizes equality (4) and we will formulate it as a theorem.

Theorem 2. *For any $a \in \mathbb{R}$ and every $m \in \mathbb{N}, m \geq 2$, the following equality holds*

$$\mathbb{M}_{n,m} = \sum_{i \in \mathbb{Z}} \varphi_m(i-a)(i-a)^n, 0 \leq n \leq m-1. \tag{7}$$

Proof. We will prove our statement by induction in $m \in \mathbb{N}, m \geq 2$. For $m = 2$ it can be checked directly.

Assume that

$$\mathbb{M}_{n,m} = \sum_{i \in \mathbb{Z}} \varphi_m(i-a)(i-a)^n,$$

for any $a \in \mathbb{R}$ and every n such that $0 \leq n \leq m-1$. Our aim is to prove that

$$\mathbb{M}_{n,m+1} = \sum_{i \in \mathbb{Z}} \varphi_{m+1}(i-a)(i-a)^n$$

for any $a \in \mathbb{R}$ and every n such that $0 \leq n \leq m$.

Let $0 \le n \le m-1$. Using the equality (6) we have

$$\begin{aligned}
\mathbb{M}_{n,m+1} &= \sum_{k=0}^{n} \binom{n}{k} \mathbb{M}_{k,m} - \frac{1}{m} \sum_{k=0}^{n-1} \binom{n}{k} \mathbb{M}_{k+1,m} \\
&= \sum_{k=0}^{n} \binom{n}{k} \sum_{i\in\mathbb{Z}} \varphi_m(i-a)(i-a)^k \\
&\qquad - \frac{1}{m} \sum_{k=0}^{n-1} \binom{n}{k} \sum_{i\in\mathbb{Z}} \varphi_m(i-a)(i-a)^{k+1} \\
&= \frac{1}{m} \sum_{i\in\mathbb{Z}} \varphi_m(i-a) \Big\{ m(i-a+1)^n \\
&\qquad - (i-a)\Big[(i-a+1)^n - (i-a)^n\Big]\Big\} \\
&= \frac{1}{m} \sum_{i\in\mathbb{Z}} \varphi_m(i-a)(i-a)^{n+1} \\
&\qquad + \frac{1}{m} \sum_{i\in\mathbb{Z}} \varphi_m(i-a)(m-i+a)(i-a+1)^n \\
&= \frac{1}{m} \sum_{i\in\mathbb{Z}} \varphi_m(i-a)(i-a)^{n+1} \\
&\qquad + \frac{1}{m} \sum_{i\in\mathbb{Z}} \varphi_m(i-a-1)(m+1-i+a)(i-a)^n \\
&= \frac{1}{m} \sum_{i\in\mathbb{Z}} \Big[\varphi_m(i-a)(i-a) \\
&\qquad + \varphi_m(i-a-1)(m+1-i+a)\Big](i-a)^n \\
&= \sum_{i\in\mathbb{Z}} \varphi_{m+1}(i-a)(i-a)^n.
\end{aligned}$$

It remains to prove our statement for $n = m$. Again, using the equality (6) we have

$$\begin{aligned}
\mathbb{M}_{m,m+1} &= \sum_{k=0}^{m} \binom{m}{k} \mathbb{M}_{k,m} - \frac{1}{m} \sum_{k=0}^{m-1} \binom{m}{k} \mathbb{M}_{k+1,m} \\
&= \sum_{k=0}^{m-1} \binom{m}{k} \mathbb{M}_{k,m} - \frac{1}{m} \sum_{k=0}^{m-2} \binom{m}{k} \mathbb{M}_{k+1,m}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{m-1} \binom{m}{k} \sum_{i\in\mathbb{Z}} \varphi_m(i-a)(i-a)^k \\
&\qquad - \frac{1}{m} \sum_{k=0}^{m-2} \binom{m}{k} \sum_{i\in\mathbb{Z}} \varphi_m(i-a)(i-a)^{k+1} \\
&= \frac{1}{m} \sum_{i\in\mathbb{Z}} \varphi_m(i-a) \Big\{ m\Big[(i-a+1)^m - (i-a)^m\Big] \\
&\qquad - (i-a)\Big[(i-a+1)^m - (i-a)^m - m(i-a)^{m-1}\Big]\Big\} \\
&= \frac{1}{m} \sum_{i\in\mathbb{Z}} \varphi_m(i-a)(i-a)^{m+1} \\
&\qquad + \frac{1}{m} \sum_{i\in\mathbb{Z}} \varphi_m(i-a)(m-i+a)(i-a+1)^m \\
&= \frac{1}{m} \sum_{i\in\mathbb{Z}} \varphi_m(i-a)(i-a)^{m+1} \\
&\qquad + \frac{1}{m} \sum_{i\in\mathbb{Z}} \varphi_m(i-a-1)(m-i+1+a)(i-a)^m \\
&= \frac{1}{m} \sum_{i\in\mathbb{Z}} \Big[\varphi_m(i-a)(i-a) \\
&\qquad + \varphi_m(i-a-1)(m+1-i+a)\Big](i-a)^m \\
&= \sum_{i\in\mathbb{Z}} \varphi_{m+1}(i-a)(i-a)^m,
\end{aligned}
$$

which completes the proof. □

In particular, for $a = 0$ we have

$$
\mathbb{M}_{n,m} = \sum_{i\in\mathbb{Z}} \varphi_m(i) i^n = \sum_{i=1}^{m-1} \varphi_m(i) i^n,
$$

for every $m \in \mathbb{N}$ and every n, $0 \le n \le m-1$. This formula has also been obtained in [3], but in a different way.

Furthermore, the quadrature rule

$$
\int_0^m \varphi_m(x) f(x) dx \approx \sum_{i=1}^{m-1} \varphi_m(i) f(i).
$$

can be seen as a rule of Newton-Cotes type with the nodes $1, 2, \dots, m-1$, the coefficients $\varphi_m(1), \varphi_m(2), \dots, \varphi_m(m-1)$ and the weight function $\varphi_m(\cdot)$ (system of the nodes and coefficients can be extended by the nodes 0 and m, i.e. by the coefficients $\varphi_m(0)$ and $\varphi_m(m)$). In accordance with the previous result, this rule is exact for any polynomial of degree less than m. Since the weight function is even with respect to the midpoint of the interval of integration $[0, m]$ and the nodes are symmetric with respect to the midpoint of the same interval, this rule will be also exact for the polynomials of degree m, when m is odd. Finally, in the case when a is integer and m is odd, using the previous results, one can easily check that equality (7) also holds for $n = m$.

The fifth method:

Putting $g(x) = \dfrac{x^{m+n}}{(m+n)(m+n-1)\dots(n+1)}$ in (1) immediately gives

$$M_{n,m} = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \frac{k^{m+n}}{(m+n)(m+n-1)\dots(n+1)}.$$

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