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RIEMANN-LIOUVILLE AND CAPUTO FRACTIONAL APPROXIMATION OF CSISZAR'S f–DIVERGENCE

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Abstract. Here are established various tight probabilistic inequalities that give nearly best estimates for the Csiszar's f-divergence. These involve Riemann-Liouville and Caputo fractional derivatives of the directing function f. Also a lower bound is given for the Csiszar's distance. The Csiszar's discrimination is the most essential and general measure for the comparison between two probability measures. This is continuation of [4].

1. Preliminaries

Throughout this paper we use the following.

I) Let f be a convex function from $(0, +\infty)$ into $\mathbb{R}$ which is strictly convex at 1 with $f(1) = 0$. Let $(X, \mathcal{A}, \lambda)$ be a measure space, where λ is a finite or a σ-finite measure on $(X, \mathcal{A})$. And let μ_1, μ_2 be two probability measures on $(X, \mathcal{A})$ such that $\mu_1 \ll \lambda, \mu_2 \ll \lambda$ (absolutely continuous), e.g. $\lambda = \mu_1 + \mu_2$. Denote by $p = \frac{d\mu_1}{d\lambda}$, $q = \frac{d\mu_2}{d\lambda}$ the (densities) Radon-Nikodym derivatives of μ_1, μ_2 with respect to λ. Here we assume that

$$0 < a \leq \frac{p}{q} \leq b, \quad \text{a.e. on } X \text{ and } a \leq 1 \leq b.$$

The quantity

$$\Gamma_f(\mu_1, \mu_2) = \int_X q(x) f\left(\frac{p(x)}{q(x)}\right) d\lambda(x), \tag{1}$$

was introduced by I. Csiszar in 1967, see [7], and is called f-divergence of the probability measures μ_1 and μ_2. By Lemma 1.1 of [7], the integral (1) is well-defined and $\Gamma_f(\mu_1, \mu_2) \geq 0$ with equality only when $\mu_1 = \mu_2$. In [7] the author without proof mentions that $\Gamma_f(\mu_1, \mu_2)$ does not depend on the choice of λ.

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For a proof of the last see [4], Lemma 1.1.

The concept of f-divergence was introduced first in [6] as a generalization of Kullback's "information for discrimination" or I-divergence (generalized entropy) [11], [12] and of Rényi's "information gain" (I-divergence of order α) [13]. In fact the I-divergence of order 1 equals

$$\Gamma_{u \log_2 u} (\mu_1, \mu_2) .$$

The choice $f(u) = (u-1)^2$ produces again a known measure of difference of distributions that is called $\varkappa^2-$divergence, of course the total variation distance $|\mu_1 - \mu_2| = \int_X |p(x) - q(x)| \, d\lambda(x)$ equals $\Gamma_{|u-1|}(\mu_1, \mu_2)$.

Here by assuming $f(1) = 0$ we can consider $\Gamma_f(\mu_1, \mu_2)$ as a measure of the difference between the probability measures μ_1, μ_2. The f-divergence is in general asymmetric in μ_1 and μ_2. But since f is convex and strictly convex at 1 (see Lemma 2, [4]) so is

$$f^*(u) = uf\left(\frac{1}{u}\right) \tag{2}$$

and as in [7] we get

$$\Gamma_f(\mu_2, \mu_1) = \Gamma_{f^*}(\mu_1, \mu_2) . \tag{3}$$

In Information Theory and Statistics many other concrete divergences are used which are special cases of the above general Csiszar f-divergence, e.g. Hellinger distance D_H, α-divergence D_α, Bhattacharyya distance D_B, Harmonic distance D_{H_α}, Jeffrey's distance D_J, triangular discrimination D_Δ, for all these see, e.g. [5], [9]. The problem of finding and estimating the proper distance (or difference or discrimination) of two probability distributions is one of the major ones in Probability Theory.

The above f-divergence measures in their various forms have been also applied to Anthropology, Genetics, Finance, Economics, Political Science, Biology, Approximation of Probability distributions, Signal Processing and Pattern Recognition. A great inspiration for this article has been the very important monograph on the topic by S. Dragomir [9].

II) Here we follow [8].

We start with

Definition 1. *Let $\nu \geq 0$, the operator J_a^ν, defined on $L_1(a,b)$ by*

$$J_a^\nu f(x) := \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t) \, dt \tag{4}$$

for $a \leq x \leq b$, is called the Riemann-Liouville fractional integral operator of order ν.

For $\nu = 0$, we set $J_a^0 := I$, the identity operator. Here Γ stands for the gamma function.

Let $\alpha > 0,\ f \in L_1(a,b),\ a,\ b \in \mathbb{R}$, see [8]. Here $[\cdot]$ stands for the integral part of the number.

We define the generalized Riemann-Liouville fractional derivative of f of order α by

$$D_a^{\alpha} f(s) := \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{ds}\right)^m \int_a^s (s-t)^{m-\alpha-1} f(t)\, dt,$$

where $m := [\alpha] + 1,\ s \in [a,b]$, see also [1], Remark 46 there.

In addition, we set

$$\begin{aligned} D_a^0 f &:= f, \\ J_a^{-\alpha} f &:= D_a^{\alpha} f, \quad \text{if } \alpha > 0, \\ D_a^{-\alpha} f &:= J_a^{\alpha} f, \quad \text{if } 0 < \alpha \leq 1, \\ D_a^n f &= f^{(n)}, \quad \text{for } n \in \mathbb{N}. \end{aligned} \tag{5}$$

We need

Definition 2. ([3]) *We say that $f \in L_1(a,b)$ has an L_∞ fractional derivative $D_a^{\alpha} f$ $(\alpha > 0)$ in $[a,b],\ a,\ b \in \mathbb{R}$, iff $D_a^{\alpha-k} f \in C([a,b]),\ k = 1,\ldots,m := [\alpha]+1$, and $D_a^{\alpha-1} f \in AC([a,b])$ (absolutely continuous functions) and $D_a^{\alpha} f \in L_\infty(a,b)$.*

Lemma 3. ([3]) *Let $\beta > \alpha \geq 0,\ f \in L_1(a,b),\ a,\ b \in \mathbb{R}$, have L_∞ fractional derivative $D_a^{\beta} f$ in $[a,b]$, let $D_a^{\beta-k} f(a) = 0$ for $k = 1,\ldots,[\beta]+1$. Then*

$$D_a^{\alpha} f(s) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^s (s-t)^{\beta-\alpha-1} D_a^{\beta} f(t)\, dt,\ \ \forall s \in [a,b]. \tag{6}$$

Here $D_a^{\alpha} f \in AC([a,b])$ for $\beta - \alpha \geq 1$, and $D_a^{\alpha} f \in C([a,b])$ for $\beta - \alpha \in (0,1)$.

Here $AC^n([a,b])$ is the space of functions with absolutely continuous $(n-1)$-st derivative.

We need to mention

Definition 4. ([8]) *Let $\nu \geq 0,\ n := \lceil \nu \rceil,\ \lceil \cdot \rceil$ is ceiling of the number, $f \in AC^n([a,b])$. We call Caputo fractional derivative*

$$D_{*a}^{\nu} f(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt,\ \ \forall x \in [a,b]. \tag{7}$$

The above function $D_{*a}^{\nu} f(x)$ exists almost everywhere for $x \in [a,b]$.

We need

Proposition 5. ([8]) *Let $\nu \geq 0,\ n := \lceil \nu \rceil,\ f \in AC^n([a,b])$. Then $D_{*a}^{\nu} f$ exists iff the generalized Riemann-Liouville fractional derivative $D_a^{\nu} f$ exists.*

Proposition 6. ([8]) *Let $\nu \geq 0$, $n := \lceil \nu \rceil$. Assume that f is such that both $D_{*a}^{\nu} f$ and $D_a^{\nu} f$ exist. Suppose that $f^{(k)}(a) = 0$ for $k = 0, 1, \ldots, n-1$. Then*

$$D_{*a}^{\nu} f = D_a^{\nu} f. \tag{8}$$

In conclusion

Corollary 7. ([2]) *Let $\nu \geq 0$, $n := \lceil \nu \rceil$, $f \in AC^n([a,b])$, $D_{*a}^{\nu} f$ exists or $D_a^{\nu} f$ exists, and $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$. Then*

$$D_a^{\nu} f = D_{*a}^{\nu} f. \tag{9}$$

We need

Theorem 8. ([2]) *Let $\nu \geq 0$, $n := \lceil \nu \rceil$, $f \in AC^n([a,b])$ and $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$. Then*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} D_{*a}^{\nu} f(t)\, dt. \tag{10}$$

We also need

Theorem 9. ([2]) *Let $\nu \geq \gamma + 1$, $\gamma \geq 0$. Call $n := \lceil \nu \rceil$. Assume $f \in AC^n([a,b])$ such that $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$, and $D_{*a}^{\nu} f \in L_{\infty}(a,b)$. Then $D_{*a}^{\gamma} f \in AC([a,b])$, and*

$$D_{*a}^{\gamma} f(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-t)^{\nu-\gamma-1} D_{*a}^{\nu} f(t)\, dt, \ \forall x \in [a,b]. \tag{11}$$

Theorem 10. ([2])*Let $\nu \geq \gamma + 1$, $\gamma \geq 0$, $n := \lceil \nu \rceil$. Let $f \in AC^n([a,b])$ such that $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$. Assume $\exists D_a^{\nu} f(x) \in \mathbb{R}$, $\forall x \in [a,b]$, and $D_a^{\nu} f \in L_{\infty}(a,b)$. Then $D_a^{\gamma} f \in AC([a,b])$, and*

$$D_a^{\gamma} f(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-t)^{\nu-\gamma-1} D_a^{\nu} f(t)\, dt, \ \forall x \in [a,b]. \tag{12}$$

2. Results

Here f and the whole setting is as in 1. Preliminaries (I). We present first results regarding the Riemann-Liouville fractional derivative.

Theorem 11. *Let $\beta > 0$, $f \in L_1(a,b)$, have L_{∞} fractional derivative $D_a^{\beta} f$ in $[a,b]$, let $D_a^{\beta-k} f(a) = 0$ for $k = 1, \ldots, [\beta] + 1$. Also assume $0 < a \leq \frac{p(x)}{q(x)} \leq b$, a.e. on X, $a < b$. Then*

$$\Gamma_f(\mu_1, \mu_2) \leq \frac{\left\| D_a^{\beta} f \right\|_{\infty,[a,b]}}{\Gamma(\beta+1)} \int_X q(x)^{1-\beta} (p(x) - aq(x))^{\beta} \, d\lambda(x). \tag{13}$$

Proof. By (6) , $\alpha = 0$, we get

$$f(s) = \frac{1}{\Gamma(\beta)} \int_a^s (s-t)^{\beta-1} D_a^\beta f(t)\, dt, \text{ all } a \le s \le b. \tag{14}$$

Then

$$\begin{aligned} |f(s)| &\le \frac{1}{\Gamma(\beta)} \int_a^s (s-t)^{\beta-1} \left| D_a^\beta f(t) \right| dt \\ &\le \frac{\left\| D_a^\beta f \right\|_{\infty,[a,b]}}{\Gamma(\beta)} \int_a^s (s-t)^{\beta-1}\, dt \\ &= \frac{\left\| D_a^\beta f \right\|_{\infty,[a,b]}}{\Gamma(\beta)} \frac{(s-a)^\beta}{\beta} \\ &= \frac{\left\| D_a^\beta f \right\|_{\infty,[a,b]}}{\Gamma(\beta+1)} (s-a)^\beta, \quad \text{all } a \le s \le b. \end{aligned} \tag{15}$$

I.e. we have that

$$|f(s)| \le \frac{\left\| D_a^\beta f \right\|_{\infty,[a,b]}}{\Gamma(\beta+1)} (s-a)^\beta, \quad \text{all } a \le s \le b. \tag{16}$$

Consequently we obtain

$$\begin{aligned} \Gamma_f(\mu_1, \mu_2) &= \int_X q(x)\, f\left(\frac{p(x)}{q(x)}\right) d\lambda(x) \\ &\le \frac{\left\| D_a^\beta f \right\|_{\infty,[a,b]}}{\Gamma(\beta+1)} \int_X q(x) \left(\frac{p(x)}{q(x)} - a\right)^\beta d\lambda(x) \\ &= \frac{\left\| D_a^\beta f \right\|_{\infty,[a,b]}}{\Gamma(\beta+1)} \int_X q(x)^{1-\beta} (p(x) - a q(x))^\beta\, d\lambda(x), \end{aligned} \tag{17}$$

proving the claim. □

Next we give an L_δ result.

Theorem 12. *Same assumptions as in Theorem* 11*. Let* $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$ *and* $\gamma(\beta - 1) + 1 > 0$*. Then*

$$\Gamma_f(\mu_1, \mu_2) \le \frac{\left\| D_a^\beta f \right\|_{\delta,[a,b]}}{\Gamma(\beta) (\gamma(\beta-1)+1)^{1/\gamma}} \int_X q(x)^{2-\beta-\frac{1}{\gamma}} (p(x) - a q(x))^{\beta-1+\frac{1}{\gamma}}\, d\lambda(x). \tag{18}$$

Proof. By (6), $\alpha = 0$, we get again

$$f(s) = \frac{1}{\Gamma(\beta)} \int_a^s (s-t)^{\beta-1} D_a^\beta f(t)\, dt, \text{ all } a \le s \le b. \tag{19}$$

Hence

$$\begin{aligned} |f(s)| &\le \frac{1}{\Gamma(\beta)} \int_a^s (s-t)^{\beta-1} \left| D_a^\beta f(t) \right| dt \\ &\le \frac{1}{\Gamma(\beta)} \left(\int_a^s (s-t)^{\gamma(\beta-1)} dt \right)^{1/\gamma} \left(\int_a^s \left| D_a^\beta f(t) \right|^\delta dt \right)^{1/\delta} \\ &\le \frac{\left\| D_a^\beta f \right\|_{\delta,[a,b]}}{\Gamma(\beta)} \frac{(s-a)^{\beta-1+\frac{1}{\gamma}}}{(\gamma(\beta-1)+1)^{1/\gamma}}, \text{ all } a \le s \le b. \end{aligned} \tag{20}$$

That is

$$|f(s)| \le \frac{\left\| D_a^\beta f \right\|_{\delta,[a,b]}}{\Gamma(\beta)} \frac{(s-a)^{\beta-1+\frac{1}{\gamma}}}{(\gamma(\beta-1)+1)^{1/\gamma}}, \text{ all} a \le s \le b. \tag{21}$$

Consequently we obtain

$$\begin{aligned} \Gamma_f(\mu_1,\mu_2) &\le \int_X q \left| f\left(\frac{p}{q}\right) \right| d\lambda \\ &\le \frac{\left\| D_a^\beta f \right\|_{\delta,[a,b]}}{\Gamma(\beta)(\gamma(\beta-1)+1)^{1/\gamma}} \int_X q \left(\frac{p}{q} - a\right)^{\beta-1+\frac{1}{\gamma}} d\lambda \\ &= \frac{\left\| D_a^\beta f \right\|_{\delta,[a,b]}}{\Gamma(\beta)(\gamma(\beta-1)+1)^{1/\gamma}} \int_X q^{2-\beta-\frac{1}{\gamma}} (p-aq)^{\beta-1+\frac{1}{\gamma}} d\lambda, \end{aligned} \tag{22}$$

proving the claim. □

An L_1 estimate follows.

Theorem 13. *Same assumptions as in Theorem* 11. *Let* $\beta \ge 1$. *Then*

$$\Gamma_f(\mu_1,\mu_2) \le \frac{\left\| D_a^\beta f \right\|_{1,[a,b]}}{\Gamma(\beta)} \left(\int_X (q(x))^{2-\beta} (p(x) - aq(x))^{\beta-1} d\lambda(x) \right) \tag{23}$$

Proof. By (19) we have

$$\begin{aligned} |f(s)| &\le \frac{1}{\Gamma(\beta)} \int_a^s (s-t)^{\beta-1} \left| D_a^\beta f(t) \right| dt \\ &\le \frac{(s-a)^{\beta-1}}{\Gamma(\beta)} \int_a^b \left| D_a^\beta f(t) \right| dt = \frac{(s-a)^{\beta-1}}{\Gamma(\beta)} \left\| D_a^\beta f \right\|_{1,[a,b]}. \end{aligned} \tag{24}$$

I.e.

$$|f(s)| \leq \frac{(s-a)^{\beta-1}}{\Gamma(\beta)} \left\| D_a^{\beta} f \right\|_{1,[a,b]}, \tag{25}$$

for all s in $[a,b]$. Therefore

$$\Gamma_f(\mu_1,\mu_2) \leq \int_X q \left| f\left(\frac{p}{q}\right) \right| d\lambda \leq \frac{\left\| D_a^{\beta} f \right\|_{1,[a,b]}}{\Gamma(\beta)} \int_X q \left(\frac{p}{q} - a\right)^{\beta-1} d\lambda$$
$$= \frac{\left\| D_a^{\beta} f \right\|_{1,[a,b]}}{\Gamma(\beta)} \left(\int_X q^{2-\beta} (p-aq)^{\beta-1} d\lambda \right), \tag{26}$$

proving the claim. □

We continue with results regarding the Caputo fractional derivative.

Theorem 14. *Let* $\nu > 0$, $n := \lceil \nu \rceil$, $f \in AC^n([a,b])$ *and* $f^{(k)}(a) = 0$, $k = 0,1,\ldots,n-1$. *Assume* $D_{*a}^{\nu} f \in L_{\infty}(a,b)$, $0 < a \leq \frac{p(x)}{q(x)} \leq b$, *a.e. on* X, $a < b$. *Then*

$$\Gamma_f(\mu_1,\mu_2) \leq \frac{\|D_{*a}^{\nu} f\|_{\infty,[a,b]}}{\Gamma(\nu+1)} \int_X q(x)^{1-\nu} (p(x) - aq(x))^{\nu} d\lambda(x). \tag{27}$$

Proof. Similar to Theorem 11, using Theorem 8. □

Next we give an L_{δ} result.

Theorem 15. *Assume all as in Theorem* 14. *Let* γ, $\delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$ *and* $\gamma(\nu-1)+1 > 0$. *Then*

$$\Gamma_f(\mu_1,\mu_2) \leq \frac{\|D_{*a}^{\nu} f\|_{\delta,[a,b]}}{\Gamma(\nu)(\gamma(\nu-1)+1)^{1/\gamma}} \int_X q(x)^{2-\nu-\frac{1}{\gamma}} (p(x) - aq(x))^{\nu-1+\frac{1}{\gamma}} d\lambda(x). \tag{28}$$

Proof. Similar to Theorem 12, using Theorem 8. □

It follows an L_1 estimate.

Theorem 16. *Assume all as in Theorem* 14. *Let* $\nu \geq 1$. *Then*

$$\Gamma_f(\mu_1,\mu_2) \leq \frac{\|D_{*a}^{\nu} f\|_{1,[a,b]}}{\Gamma(\nu)} \left(\int_X (q(x))^{2-\nu} (p(x) - aq(x))^{\nu-1} d\lambda(x) \right). \tag{29}$$

Proof. Similar to Theorem 13, using Theorem 8. □

Regarding again the Riemann-Liouville fractional derivative we need:

Corollary 17. *Let* $\nu \geq 0$, $n := \lceil \nu \rceil$, $f \in AC^n([a,b])$, $\exists D_a^\nu f(x) \in \mathbb{R}$, $\forall x \in [a,b]$, $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$. *Then*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} D_a^\nu f(t)\, dt. \tag{30}$$

Proof. By Corollary 7 and Theorem 8. □

We continue with results again regarding the Riemann-Liouville fractional derivative.

Theorem 18. *Let* $\nu > 0$, $n := \lceil \nu \rceil$, $f \in AC^n([a,b])$, $\exists D_a^\nu f(x) \in \mathbb{R}$, $\forall x \in [a,b]$, $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$. *Assume* $D_a^\nu f \in L_\infty(a,b)$, $0 < a \leq \frac{p(x)}{q(x)} \leq b$, *a.e. on* X, $a < b$. *Then*

$$\Gamma_f(\mu_1, \mu_2) \leq \frac{\|D_a^\nu f\|_{\infty,[a,b]}}{\Gamma(\nu+1)} \int_X q(x)^{1-\nu} (p(x) - aq(x))^\nu \, d\lambda(x). \tag{31}$$

Proof. Similar to Theorem 11, using Corollary 17. □

Next we give the corresponding L_δ result.

Theorem 19. *Assume all as in Theorem 18. Let* γ, $\delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$ *and* $\gamma(\nu - 1) + 1 > 0$. *Then*

$$\Gamma_f(\mu_1, \mu_2) \leq \frac{\|D_a^\nu f\|_{\delta,[a,b]}}{\Gamma(\nu)(\gamma(\nu-1)+1)^{1/\gamma}} \int_X q(x)^{2-\nu-\frac{1}{\gamma}} (p(x) - aq(x))^{\nu-1+\frac{1}{\gamma}} d\lambda(x). \tag{32}$$

Proof. Similar to Theorem 12, using Corollary 17. □

It follows the L_1 estimate.

Theorem 20. *Assume all as in Theorem* 18. *Let* $\nu \geq 1$. *Then*

$$\Gamma_f(\mu_1, \mu_2) \leq \frac{\|D_a^\nu f\|_{1,[a,b]}}{\Gamma(\nu)} \left(\int_X (q(x))^{2-\nu} (p(x) - aq(x))^{\nu-1} \, d\lambda(x) \right). \tag{33}$$

Proof. Similar to Theorem 13, using Corollary 17. □

We need

Theorem 21. (Taylor expansion for Caputo derivatives, [8], p. 40) *Assume* $\nu \geq 0$, $n = \lceil \nu \rceil$, *and* $f \in AC^n([a,b])$. *Then*

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} D_{*a}^\nu f(t)\, dt, \ \ \forall x \in [a,b]. \tag{34}$$

We make

Remark 22. Let $\nu > 0$, $n = \lceil \nu \rceil$, and $f \in AC^n([a,b])$.

If $D_{*a}^{\nu} f \underset{(\leq 0)}{\geq} 0$ over $[a,b]$, then

$$\int_a^x (x-t)^{\nu-1} D_{*a}^{\nu} f(t)\, dt \underset{(\leq 0)}{\geq} 0 \text{ on } [a,b].$$

By (34) then we obtain

$$f(x) \geq \ (\leq) \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k, \tag{35}$$

$\forall x \in [a,b]$. Hence

$$qf\left(\frac{p}{q}\right) \geq \ (\leq) \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} q \left(\frac{p}{q} - a\right)^k, \text{ a.e. on } X. \tag{36}$$

Consequently we get

$$\Gamma_f(\mu_1, \mu_2) \geq \ (\leq) \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \int_X q^{1-k} (p - aq)^k \, d\lambda. \tag{37}$$

We have established

Theorem 23. *Let $\nu > 0$, $n = \lceil \nu \rceil$, and $f \in AC^n([a,b])$. If $D_{*a}^{\nu} f \underset{(\leq 0)}{\geq} 0$ on $[a,b]$, then*

$$\Gamma_f(\mu_1, \mu_2) \geq \ (\leq) \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \left(\int_X (q(x))^{1-k} (p(x) - aq(x))^k \, d\lambda(x)\right). \tag{38}$$

We finish with

Remark 24. Using Lemma 3, Theorem 9 and Theorem 10 and in their settings, for g any of $D_a^{\alpha} f$, $D_{*a}^{\gamma} f$, $D_a^{\gamma} f$, which fulfill the conditions and assumptions of 1. Preliminaries (I), we can find as above similar estimates for $\Gamma_g(\mu_1, \mu_2)$.

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