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ANOTHER HARDY-HILBERT'S INTEGRAL INEQUALITY

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ABSTRACT. We give a new kind of Hardy-Hilbert integral inequality via homogeneous functions as well as some other generalizations. Special cases are also obtained.

1. INTRODUCTION

Let $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^2(t)\,dt < \infty \quad \text{and} \quad 0 < \int_0^\infty g^2(t)\,dt < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy < \pi \left(\int_0^\infty f^2(t)\,dt \int_0^\infty g^2(t)\,dt \right)^{1/2}, \tag{1}$$

where the constant factor π is the best possible (cf. Hardy et al. [2]). Inequality (1) is well known as Hilbert's integral inequality. This inequality had been extended by Hardy [1] as follows:

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^p(t)\,dt < \infty \quad \textit{and} \quad 0 < \int_0^\infty g^q(t)\,dt < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(t)\,dt \right)^{1/p} \left(\int_0^\infty g^q(t)\,dt \right)^{1/q}, \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (2) is called Hardy-Hilbert's integral inequality and is important in analysis and applications (cf. Mitrinovic et al. [3]).

B. Yang gave the following extension of (2) as follows:

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Theorem [4]. *If* $\lambda > 2 - \min\{p, q\}$, $f, g \geq 0$, *satisfy*

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)\, dt < \infty \quad \textit{and} \quad \int_0^\infty t^{1-\lambda} g^q(t)\, dt < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\, dxdy < k_\lambda(p) \left(\int_0^\infty t^{1-\lambda} f^p(t) dt \right)^{1/p} \left(\int_0^\infty t^{1-\lambda} g^q(t)\, dt \right)^{1/q}, \tag{3}$$

where the constant factor $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ *is the best possible,* B *is the beta function. The function* $f(x, y)$ *is said to be homogeneous of degree* λ, *if*

$$f(tx, ty) = t^\lambda f(x, y), \quad t > 0.$$

The object of this paper is that to give some new inequalities similar to that of Hardy-Hilbert's inequality.

2. New Results.

We state and prove the following:

Theorem 1. *Let* $f, g \geq 0, h$ *be a positive function of two variables and homogeneous of degree* $\lambda > 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\gamma > 0$. *Then*

$$\int_0^\gamma \int_0^\gamma \frac{f(u)g(v)}{h(u, v)}\, du\, dv \leq \frac{1}{p} \int_0^\gamma u^{\alpha(2-p)+1-\lambda} f^p(u) K_1(u)\, du + \frac{1}{q} \int_0^\gamma v^{2(2-q)+1-\lambda} g^q(v) K_2(v)\, dv, \tag{4}$$

where

$$K_1(u) = \int_0^{\gamma/u} \frac{y^\alpha}{h(1, y)}\, dy, \quad K_2(v) = \int_0^{\gamma/v} \frac{x^\alpha}{h(x, 1)}\, dx,$$

and

$$\int_0^\gamma v^{\frac{\alpha(2-q)+1-\lambda}{1-q}} K_2^{\frac{1}{1-q}}(v) \left(\int_0^\gamma \frac{f(u)}{h(u, v)}\, du \right)^p dv \leq \int_0^\gamma u^{\alpha(2-p)+1-\lambda} f^p(u) K_1(u)\, du. \tag{5}$$

Inequalities (4) and (5) are equivalent. In particular, for $\gamma = \infty$, $\alpha = \lambda/2 - 1$, we have

$$\int_0^\infty \int_0^\infty \frac{f(u)g(v)}{h(u,v)}\, du\, dv \leq K\left(\frac{1}{p}\int_0^\infty u^{\left(1-\frac{\lambda}{2}\right)p-1} f^p(u)\, du + \frac{1}{q}\int_0^\infty v^{\left(1-\frac{\lambda}{2}\right)q-1} g^q(v)\, dv\right) \quad (6)$$

where

$$K = \int_0^\infty \frac{z^{\lambda/2-1}}{h(1,z)}\, dz$$

and

$$\int_0^\infty v^{\frac{\left(1-\frac{\lambda}{2}\right)q-1}{1-q}} \left(\int_0^\infty \frac{f(u)}{h(u,v)}\, du\right)^p dv \leq \frac{K/p}{1-K/p}\int_0^\infty u^{\left(1-\frac{\lambda}{2}\right)p-1} f^p(u)\, du. \quad (7)$$

The inequalities (6) and (7) are equivalent.

Proof. Making use of the inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1,$$

$$\begin{aligned}
\int_0^\gamma \int_0^\gamma \frac{f(u)g(v)}{h(u,v)} du\, dv &= \int_0^\gamma \int_0^\gamma \frac{u^{-\frac{\alpha}{q}} v^{\frac{\alpha}{p}} f(u)}{h^{\frac{1}{p}}(u,v)} \times \frac{u^{\frac{\alpha}{q}} v^{-\frac{\alpha}{p}} g(v)}{h^{\frac{1}{q}}(u,v)} du\, dv \\
&\leq \frac{1}{p}\int_0^\gamma \int_0^\gamma \frac{u^{(1-p)\alpha} v^\alpha f^p(u)}{h(u,v)} du\, dv + \frac{1}{q}\int_0^\gamma \int_0^\gamma \frac{v^{(1-q)\alpha} u^\alpha g^q(v)}{h(u,v)} du\, dv \\
&= \frac{M}{p} + \frac{N}{q}.
\end{aligned}$$

$$M = \int_0^\gamma u^{(1-p)\alpha} f^p(u) du \int_0^\gamma \frac{v^\alpha}{h(u,v)}\, dv.$$

Observe that on putting $v = uy$, $dv = u\, dy$, $0 \leq y \leq \gamma/u$, we have

$$\int_0^\gamma \frac{v^\alpha}{h(u,v)}\, dv = \int_0^{\gamma/u} \frac{(uy)^\alpha u}{h(u,uy)}\, dy = u^{1+\alpha-\lambda}\int_0^{\gamma/u} \frac{y^\alpha}{h(1,y)}\, dy,$$

which implies

$$M = \int_0^\gamma u^{(2-p)\alpha+1-\lambda} f^p(u) K_1(u)\, du$$

$$N = \int_0^\gamma v^{(1-q)\alpha} g^q(v)\, dv \int_0^\gamma \frac{u^\alpha}{h(u,v)}\, du.$$

Now, on putting $u = vx$, $du = v\,dx$, $0 \le x \le \gamma/v$, we have

$$\begin{aligned} N &= \int_0^{\gamma} v^{(2-q)+1-\lambda\alpha} g^q(v)\,dv \int_0^{\gamma/v} \frac{x^{\alpha}}{h(x,1)}\,dx \\ &= \int_0^{\gamma} v^{(2-q)\alpha+1-\lambda} g^q(v) K_2(v)\,dv. \end{aligned}$$

Therefore (4) is satisfied. To prove the equivalence of (4) and (5), let (5) be satisfied. Then, we have

$$\begin{aligned} &\int_0^{\gamma} v^{\frac{(2-q)\alpha+1-\lambda}{1-q}} K_2^{\frac{1}{1-q}}(v) \left(\int_0^{\gamma} \frac{f(u)}{h(u,v)}du\right)^p dv \\ &\qquad = \int_0^{\gamma}\int_0^{\gamma} \frac{f(u) v^{\frac{(2-q)\alpha+1-\lambda}{1-q}} K_2^{\frac{1}{1-q}}(v) \left(\int_0^{\gamma} \frac{f(u)}{h(u,v)}du\right)^{p-1}}{h(u,v)} du\,dv \\ &\le \frac{1}{p}\int_0^{\gamma} u^{(2-p)\alpha+1-\lambda} f^p(u) K_1(u)\,du \\ &+ \frac{1}{q}\int_0^{\gamma} v^{(2-q)\alpha+1-\lambda} v^{\frac{q}{1-q}((2-q)\alpha+1-\lambda)} K_2(v) K_2^{\frac{q}{1-q}}(v) \left(\int_0^{\gamma} \frac{f(u)}{h(u,v)}du\right)^{q(p-1)} dv \\ &= \frac{1}{p}\int_0^{\gamma} u^{(2-p)\alpha+1-\lambda} f^p(u) K_1(u)\,du \\ &\qquad + \frac{1}{q}\int_0^{\gamma} v^{\frac{(2-q)\alpha+1-\lambda}{1-q}} K_2^{\frac{1}{1-q}}(v) \left(\int_0^{\gamma} \frac{f(u)}{h(u,v)}du\right)^p dv. \end{aligned}$$

This implies

$$\begin{aligned} &\int_0^{\gamma} v^{\frac{(2-q)\alpha+1-\lambda}{1-q}} K_2^{\frac{1}{1-q}}(v) \left(\int_0^{\gamma} \frac{f(u)}{h(u,v)}du\right)^p dv \\ &\qquad \le \int_0^{\gamma} u^{(2-p)\alpha+1-\lambda} f^p(u) K_1(u)\,du. \end{aligned}$$

Now, let (4) be satisfied, we have

$$\begin{aligned} \int_0^{\gamma}\int_0^{\gamma} \frac{f(u)g(v)}{h(u,v)}\,du\,dv &= \int_0^{\gamma} v^{-\frac{(2-q)\alpha+1-\lambda}{(1-q)p}} g(v) K_2^{\frac{1}{q}}(v) v^{\frac{(2-q)\alpha+1-\lambda}{(1-q)p}}(v) \\ &\qquad \times K_2^{-\frac{1}{q}} \left(\int_0^{\gamma} \frac{f(u)}{h(u,v)}du\right) dv \\ \le \frac{1}{q}\int_0^{\gamma} v^{(2-q)\alpha+1-\lambda} g^q(v) K_2(v)\,dv& \\ &+ \frac{1}{p}\int_0^{\gamma} v^{\frac{(2-q)\alpha+1-\lambda}{1-q}} K_2^{\frac{1}{1-q}}(v) \left(\int_0^{\gamma} \frac{f(u)}{h(u,v)}du\right)^p dv \end{aligned}$$

$$\leq \frac{1}{q}\int_0^{\gamma} v^{(2-q)\alpha+1-\lambda} g^q(v) K_2(v)\, dv + \frac{1}{p}\int_0^{\gamma} u^{(2-p)\alpha+1-\lambda} f^p(u) K_1(u)\, du.$$

In the particular case, (6) and (7) follows from (4) and (5) respectively, noticing that for $\alpha = \lambda/2 - 1,\ \gamma = \infty$,

$$K_1 = K_2 = K = \int_0^{\infty} \frac{z^{\frac{\lambda}{2}-1}}{h(1,z)} dz,$$

follows as

$$\int_0^{\infty} \frac{x^{\frac{\lambda}{2}-1}}{h(x,1)} dx = \int_0^{\infty} \frac{x^{\frac{\lambda}{2}-1}}{h(x, xx^{-1})} dx = \int_0^{\infty} \frac{x^{-\frac{\lambda}{2}-1}}{h(1,x^{-1})} dx = \int_0^{\infty} \frac{z^{\frac{\lambda}{2}-1}}{h(1,z)} dz.$$

This completes the proof of the theorem. □

The following lemma is needed for the coming result.

Lemma 2. *Let* $a_i \geq 0,\ p_i > 1,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} \frac{1}{p_i} = 1.$ *Then*

$$\prod_{i=1}^{n} a_i \leq \sum_{i=1}^{n} \frac{a_i^{p_i}}{p_i}.$$

Proof. We shall prove the lemma using induction. Obviously it is true for $n = 1, 2$. Suppose it is true for $n - 1$. Since

$$\frac{1}{1/\sum_{i=1}^{n-1}(1/p_i)} + \frac{1}{p_n} = 1, \sum_{i=1}^{n-1} \left(\frac{1}{p_i \left(\sum_{i=1}^{n-1} 1/p_i \right)} \right) = 1,$$

then

$$\prod_{i=1}^{n} a_i = \left(\prod_{i=1}^{n-1} a_i\right) a_n \leq \left(\sum_{i=1}^{n-1} 1/p_i\right) \left(\prod_{i=1}^{n-1} a_i\right)^{1/\left(\sum_{i=1}^{n-1} 1/p_i\right)} + \frac{1}{p_n} a_n^{p_n}$$
$$= \left(\sum_{i=1}^{n-1} 1/p_i\right) \left(\prod_{i=1}^{n-1} a_i^{1/\left(\sum_{i=1}^{n-1} 1/p_i\right)}\right) + \frac{1}{p_n} a_n^{\rho_n}$$
$$\leq \left(\sum_{i=1}^{n-1} 1/p_i\right) \sum_{i=1}^{n-1} \frac{1}{\left(\sum_{i=1}^{n-1} 1/p_i\right) p_i} \left(a_i^{1/\left(\sum_{i=1}^{n-1} 1/p_i\right)}\right)^{\left(\sum_{i=1}^{n-1} 1/p_i\right) p_i}$$
$$= \sum_{i=1}^{n-1} \frac{1}{p_i} a_i^{p_i} + \frac{1}{p_n} a_n^{p_n} = \sum_{i=1}^{n} \frac{a_i^{p_i}}{p_i}.$$

□

Theorem 3. *Let* $f_i \geq 0,\ p_i > 1,\ \lambda > p_i - 1,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} \frac{1}{p_i} = 1.$ *Then*

$$\int_0^\infty \cdots \int_0^\infty \frac{f_1(t_1) \ldots f_n(t_n)}{(t_1 + \cdots + t_n)^\lambda}\, dt_1 \ldots dt_n \leq \frac{1}{\Gamma(\lambda)} \sum_{i=1}^{n} \frac{1}{p_i} \Gamma^{p_i}\left(\frac{\lambda}{p_i}\right) \int_0^\infty t^{p_i - \lambda - 2} f_i^{p_i}(t)\, dt. \quad (8)$$

Proof. Define, for $i = 1, \ldots, n$

$$F_i(x) = \int_0^\infty e^{-tx} f_i(t)\, dt.$$

Observe that, via Lemma 2,

$$\int_0^\infty s^{\lambda-1} F_1(s) \ldots F_n(s)\, ds = \int_0^\infty s^{\frac{\lambda-1}{p_1}} F_1(s) \ldots s^{\frac{\lambda-1}{p_n}} F_n(s)\, ds$$
$$\leq \int_0^\infty \sum_{i=1}^{n} \frac{s^{\lambda-1} F_i^{p_i}(s)}{p_i}\, ds = \sum_{i=1}^{n} \frac{1}{p_i} \int_0^\infty s^{\lambda-1} F_i^{p_i}(s)\, ds.$$

Since

$$F_i^{p_i}(s) = \left(\int_0^\infty e^{-ts} f_i(t)\, dt\right)^{p_i} \leq \int_0^\infty e^{-ts} f_i^{p_i}(t) dt \left(\int_0^\infty e^{-ts}\, dt\right)^{p_i - 1}$$
$$= s^{1-p_i} \int_0^\infty e^{-ts} f_i^{p_i}(t)\, dt,$$

we obtain

$$\int_0^\infty s^{\lambda-1} F_1(s) \ldots F_n(s)\, ds \leq \sum_{i=1}^{n} \frac{1}{p_i} \int_0^\infty s^{\lambda-p_i} \int_0^\infty e^{-ts} f_i^{p_i}(t)\, dt ds$$
$$= \sum_{i=1}^{n} \frac{1}{p_i} \int_0^\infty f_i^{p_i}(t) dt \int_0^\infty s^{\lambda-p_i} e^{-ts}\, ds$$
$$= \sum_{i=1}^{n} \frac{1}{p_i} \int_0^\infty t^{p_i-\lambda-1} f_i^{p_i}(t) dt \int_0^\infty u^{\lambda-p_i} e^{-u}\, du$$
$$= \sum_{i=1}^{n} \frac{\Gamma(1+\lambda-p_i)}{p_i} \int_0^\infty t^{p_i-\lambda-1} f_i^{p_i}(t)\, dt.$$

On the other hand,

$$\begin{aligned}\int_0^\infty s^{\lambda-1}F_1(s)\dots F_n(s)\,ds &= \int_0^\infty s^{\lambda-1}\int_0^\infty e^{-st_1}f_1(t_1)\,dt_1\dots\int_0^\infty e^{-st_n}f_n(t_n)\,dt_n\\ &= \int_0^\infty\dots\int_0^\infty f_1(t_1)\dots f_n(t_n)dt_1\dots dt_n\int_0^\infty s^{\lambda-1}e^{-s(t_1+\dots+t_n)}ds\\ &= \int_0^\infty\dots\int_0^\infty \frac{f_1(t_1)\dots f_n(t_n)}{(t_1+\dots+t_n)^\lambda}dt_1\dots dt_n\int_0^\infty z^{\lambda-1}e^{-z}dz\\ &= \Gamma(\lambda)\int_0^\infty\dots\int_0^\infty \frac{f_1(t_1)\dots f_n(t_n)}{(t_1+\dots+t_n)^\lambda}dt_1\dots dt_n.\end{aligned}$$

Summarizing, we have

$$\int_0^\infty\dots\int_0^\infty \frac{f_1(t_1)\dots f_n(t_n)}{(t_1+\dots+t_n)^\lambda}dt_1\dots dt_n \le \frac{1}{\Gamma(\lambda)}\sum_{i=1}^n\frac{\Gamma(1+\lambda-p_i)}{p_i}\int_0^\infty t^{p_i-\lambda-1}f_i^{p_i}(t)\,dt.$$

□

3. Applications

Corollary 4. *On putting $h(x,y)=(x+y)^\lambda$, which is homogeneous of degree λ in Theorem 1, inequality (6), we obtain*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dx\,dy \le B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\frac{1}{p}\int_0^\infty x^{\left(1-\frac{\lambda}{2}\right)p-1}f^p(x)\,dx+\frac{1}{q}\int_0^\infty y^{\left(1-\frac{\lambda}{2}\right)q-1}g^q(y)\,dy\right).$$

Corollary 5. *On putting $f(x,y)=x^\lambda+y^\lambda$, which is homogeneous of degree λ, in Theorem 1, inequality (6), we obtain*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}\,dx\,dy \le \frac{\pi}{\lambda}\left(\frac{1}{p}\int_0^\infty x^{\left(1-\frac{\lambda}{2}\right)p-1}f^p(x)\,dx+\frac{1}{q}\int_0^\infty y^{\left(1-\frac{\lambda}{2}\right)q-1}g^q(y)\,dy\right).$$

Corollary 6. *On putting* $f(x,y)=(x^{\sqrt{\lambda}}+y^{\sqrt{\lambda}})^{\sqrt{\lambda}}$, *which is homogeneous of degree* λ, *in Theorem* 1, *inequality* (6), *we get*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x^{\sqrt{\lambda}}+y^{\sqrt{\lambda}})^{\sqrt{\lambda}}}\,dx\,dy$$
$$\le \frac{1}{\sqrt{\lambda}}B\left(\frac{\sqrt{\lambda}}{2},\frac{\sqrt{\lambda}}{2}\right)\left(\frac{1}{p}\int_0^\infty x^{\left(1-\frac{\lambda}{2}\right)p-1}f^p(x)\,dx+\frac{1}{q}\int_0^\infty y^{\left(1-\frac{\lambda}{2}\right)q-1}g^q(y)\,dy\right).$$

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