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TWO TYPES OF MULTILINEAR STIELTJES INTEGRALS IN THE HENSTOCK-KURZWEIL SENSE

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Dedicated to Professor Harry Miller on the occasion of his 70th birthday

Abstract. In this paper we examine and compare two types of multilinear integrals considering their Stieltjes sums. The convergence of the Stieltjes sums is considered in the Riemann, Moore-Pollard and Henstock-Kurzweil sense.

1. Introduction

Multilinear Stieltjes integrals are natural generalizations of bilinear and trilinear integrals. The aim of this article is to study how the existence and properties of the multilinear integrals depend on both the way we define the Stieltjes sums and the way we define the convergence of such sums. We regard the theory of stochastic processes as one of the possible fields of application for such integrals. We are concerned with multilinear Stieltjes-type integrals such as $\int_a^b A_1(dg_1, f_1, dg_2, f_2)$, $\int_a^b A_2(dg_1, f_1, f_2, dg_2, dg_3)$, $\int_a^b A_3(f_1, f_2, dg_1)$, where $[a, b]$ is a real interval, f_k, g_k, are vector-valued functions defined on $[a, b]$, and A_k, is a multilinear operator. Another notation for such integrals is, for instance, $\int_{[a,b]}^{A_1}(dg_1, f_1, dg_2, f_2)$.

Regarding the Stieltjes sums, two different kinds of definitions for a multilinear integral can be found in research papers on trilinear and, more generally, multilinear Stieltjes integrals. The basic difference in these definitions is in the number of associated points in the subintervals $[t_{i-1}, t_i]$ of the real interval $[a, b]$. If we have more than one "f-function", then in each subinterval $[t_{i-1}, t_i]$ we can choose associated points in one of the following ways:

a) For every "f-function" we choose an associated point, so that we have as many associated points in $[t_{i-1}, t_i]$ as "f-functions". These types of integrals have been considered in [1],. . . ,[5].

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b) We choose only one associated point in $[t_{i-1}, t_i]$ and evaluate all "f-functions" at that point. See, for example, [8], [11], [12].
Using different types of the limit procedure we get various kinds of multilinear Stieltjes integrals such as Riemann-Stieltjes, Moore-Pollard-Stieltjes and Henstock-Kurzweil-Stieltjes integrals of the first type (Type I) or of the second type (Type II).

For example, if we consider the multilinear Stieltjes integral

$$\int_a^b A(f_1, f_2, dg_1, f_3),$$

we divide $[a, b]$ into smaller intervals $[t_{i-1}, t_i]$, $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$ and then choose associated points (numbers) in each $[t_{i-1}, t_i]$. We consider two types of the Stieltjes sums and, consequently, two types of the multilinear integrals:
Type I. We choose three arbitrary points, s_1^i, s_2^i, s_3^i, in each interval $[t_{i-1}, t_i]$, which we substitute into f_1, f_2, and f_3. Then we build the Stieltjes sum of the first type,

$$S_I(P) = \sum_{i=1}^{n} A[f_1(s_1^i), f_2(s_2^i), g_1(t_i) - g_1(t_{i-1}), f_3(s_3^i)].$$

A Type I multilinear Stieltjes integral is the limit of the sums $S_I(P)$. For instance, if we consider the convergence in the Henstock-Kurzweil (HK) sense, then, if the limit exists and has the value J_I, we get the Type I Henstock-Kurzweil-Stieltjes integral,

$$J_I = (HKS_I) \int_a^b A(f_1, f_2, dg_1, f_3).$$

Type II. We choose only one point, s_i, in each interval $[t_{i-1}, t_i]$ and then evaluate all "f-function" using the same point, s_i, to build the Stieltjes sum of the second type,

$$S_{II}(P) = \sum_{i=1}^{n} A[f_1(s_i), f_2(s_i), g_1(t_i) - g_1(t_{i-1}), f_3(s_i)].$$

Then, if the HK-limit is J_{II} we denote

$$J_{II} = (HKS_{II}) \int_a^b A(f_1, f_2, dg_1, f_3).$$

The main results are Theorems 5.2 and 6.1.

When we consider a multilinear integral in the sense of Henstock-Kurtzweil, then (in contrast to the bilinear case) the existence on the intervals $[a, c]$ and $[c, b]$, $a < c < b$ does not imply the existence on $[a, b]$. In Theorem

5.2 we add a new condition (pseudo additivity) and get sufficient conditions for the existence of the multilinear integral on the hole interval $[a, b]$.

In Theorem 6.1 we show that a multilinear integral exists in the sense of Henstock-Kurtzweil on $[a, b]$ if the integral exists in the Moore-Pollard sense and the pseudo additivity condition holds at every point of the interval $[a, b]$.

2. Multilinear integrals

Definition 2.1. *Let $X_1, \dots, X_p$ and Y be linear normed spaces over the same field ($\mathbb{R}$ or $\mathbb{C}$) . Then $L(X_1, \dots, X_p; Y)$ denotes the linear normed space of bounded multilinear transformations $A : X_1 \times \cdots \times X_p \to Y$.*

When denoting functions in a specific case, using different letters for the "d" coordinates is convenient, such as in the integral $\int_a^b A_1(dg_1, f_1, f_2, dg_2)$, but in order to define multilinear integrals in the general case, we use an ordered set C, which indicates those functions and coordinates where we consider "df" (differences in the Stieltjes sums).

For example, $C = (1, 0, 1, 0, 1)$ means that we consider a multilinear Stieltjes integral of this type

$$\int_a^b A(df_1, f_2, df_3, f_4, df_5).$$

Let $C = (c_1, c_2, , \dots, c_p)$ be an ordered set of p elements where $c_j = 0$ or $c_j = 1$. We denote $I_s = \{j : c_j = 0\}$, $I_d = \{j : c_j = 1\}$, $K = card(I_s)$ and assume that $p \geq 2$, $I_s \neq \emptyset$ and $I_d \neq \emptyset$.

Let $[a, b] \subset R$. A partition of $[a, b]$ is a finite set of numbers (points) $P = \{t_0, t_1, \dots, t_n\}$ such that $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$. The norm of the partition is defined as

$$|P| = \max\{|t_i - t_{i-1}|, \quad i = 1, \dots, n\}.$$

Let $f_j : [a, b] \to X_j, \quad j = 1, \dots, p$ and let $A \in L(X_1, \dots, X_p; Y)$. We define two types of Stieltjes sums, $S_I(P)$ and $S_{II}(P)$, and consider the convergence in the Riemann, Moore-Pollard or Henstock-Kurzweil sense. Consequently, we obtain two types of Stieltjes multilinear integrals:

Stieltjes sums, Type I

For every $i = 1, \dots, n$, let s_j^i be K points arbitrarily chosen in the interval $[t_{i-1}, t_i]$ of the partition P. Then, the Type I Stieltjes sum is

$$S_I(P) = \sum_{i=1}^{n} A[F_1^i, F_2^i, \dots, F_p^i]$$

where

$$F_j^i = \begin{cases} f_j(s_j^i), & \text{if} \quad c_j = 0; \\ f_j(t_i) - f_j(t_{i-1}), & \text{if} \quad c_j = 1. \end{cases}$$

Stieltjes sums, Type II
For every i, let s_i be an arbitrary point chosen in the interval $[t_{i-1}, t_i]$ of the partition P. Then the Type II Stieltjes sum is

$$S_{II}(P) = \sum_{i=1}^{n} A[F_1^i, F_2^i, \ldots, F_p^i]$$

where

$$F_j^i = \begin{cases} f_j(s_i), & \text{if} \quad c_j = 0; \\ f_j(t_i) - f_j(t_{i-1}), & \text{if} \quad c_j = 1. \end{cases}$$

Thus, in the case of the Type II Stieltjes sum, for an interval $[t_{i-1}, t_i]$ we choose only one point that we then substitute in each function $f_j, j \in I_s$.

3. Riemann-Stieltjes multilinear integrals (RS)

Definition 3.1. *We say that the Type I multilinear Riemann-Stieltjes integral of $f_j, j \in I_s$ with respect to A and $f_j, j \in I_d$, exists on $[a, b]$ if there exists a vector $J_I \in Y$ with the following property:*

For every $\epsilon > 0$, there is a constant $\delta > 0$, such that for any partition $P = \{t_0, t_1, \ldots, t_n\}$, $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$ and any choice of points s_i^j, $t_{i-1} \leq s_i^j \leq t_i$, $j \in I_s$, $i = 1, \ldots, n$, we have

$$|P| < \delta \Rightarrow |J_I - S_I(P)| < \epsilon.$$

We write

$$J_I = (RS_I) \int_a^b A(d_1 f_1, d_2 f_2, \ldots, d_p f_p)$$

where

$$d_j f_j = \begin{cases} f_j, & \text{if} \quad c_j = 0; \\ df_j, & \text{if} \quad c_j = 1. \end{cases}$$

Definition 3.2. *We say that the Type II multilinear Riemann-Stieltjes integral of $f_j, j \in I_s$ with respect to A and $f_j, j \in I_d$, exists on $[a, b]$ if there exists a vector $J_{II} \in Y$ with the following property:*

For every $\epsilon > 0$, there is a constant $\delta > 0$, such that for any partition $P = \{t_0, t_1, \ldots, t_n\}$, $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$ and any choice of points s_i, $t_{i-1} \leq s_i \leq t_i$, $j \in I_s$, $i = 1, \ldots, n$, we have

$$|P| < \delta \Rightarrow |J_{II} - S_{II}(P)| < \epsilon.$$

We write

$$J_{II} = (RS_{II}) \int_a^b A(d_1 f_1, d_2 f_2, \ldots, d_p f_p).$$

Remark 3.1. It is obvious that the existence of the integral

$$J_I = (RS_I)\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$$

implies the existence of

$$J_{II} = (RS_{II})\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p).$$

However, the converse conclusion is not valid as we see by the following examples:

Example 3.1. Suppose that $X_1 = X_2 = X_3 = Y = R$. We define A as the ordinary multiplication, $A(x_1, x_2, x_3) = x_1x_2x_3$.

Let $f_1(t) = \begin{cases} 0, & \text{for } 0 \le t < 1 \\ 1, & \text{for } 1 \le t \le 2 \end{cases}$, $f_2(t) = \begin{cases} 1, & \text{for } 0 \le t < 1 \\ 0, & \text{for } 1 \le t \le 2 \end{cases}$, and $g(t) = \begin{cases} 0, & \text{for } 0 \le t < 1 \\ 1, & \text{for } 1 \le t \le 2 \end{cases}$.

We shall show that the RS integral of the first type does not exist, but that $(RS_{II})\int_0^2 A(f_1, f_2, dg)$ does exist.

a) (*Type I*) Assume $0 < \delta < 1$. Let P be a subdivision of $[0, 2]$ which includes the interval $[t_{k-1}, t_k] = [1-\delta/2, 1+\delta/2]$. Note that $g(t_i)-g(t_{i-1}) = 0$ if $i \neq k$, since $g(t)$ is constant in any of the intervals $[0, 1)$ and $(1, 2]$. Thus

$$S_I(P) = \sum_{i=1}^{n} A[f_1(s_1^i), f_2(s_2^i), g(t_i)-g(t_{i-1})] = f_1(s_1^k)\cdot f_2(s_2^k)\cdot(g(t_k)-g(t_{k-1}))$$

$$= f_1(s_1^k) \cdot f_2(s_2^k) \cdot 1.$$

Therefore, we have

$$S_I(P) = \begin{cases} 1, & \text{if } s_1^k \ge 1 \text{ and } s_2^k < 1, \\ 0, & \text{if } s_1^k < 1 \text{ or } s_2^k \ge 1. \end{cases}$$

Thus, for any $\delta > 0$ we can build two Stieltjes sums, $S_I^1(P)$ and $S_I^2(P)$, that differ only in their associated points in the interval $[t_{k-1}, t_k]$, such that $|P| < \delta$ and $|S_I^1(P) - S_I^2(P)| = 1$. Therefore, the RS-integral of the first type (Type I) does not exist.

b) (*Type II*) Since $f_1(s_i) \cdot f_2(s_i) = 0$ for all i, we have that, for every partition P,

$$S_{II}(P) = \sum_{i=1}^{n} A[f_1(s_i), f_2(s_i), g(t_i) - g(t_{i-1})] = 0.$$

Therefore, the RS integral of the second type exists and

$$(RS_{II})\int_0^2 A(f_1, f_2, dg) = 0.$$

Example 3.2. Let $B([0,1],R)$ denote the normed space of real bounded functions on $[0,1]$ with the supremum norm, $|f| = \sup(|f(x)| : x \in [0,1])$. Suppose that $X_1 = X_2 = X_3 = X_4 = Y = B([0,1],R)$. We define A as the ordinary multiplication of functions, $A(f_1, f_2, f_3, f_4) = f_1 f_2 f_3 f_4$.
Let $f_1(t) = \chi_{[0,t]}$, $f_2(t) = \chi_{(t,1]}$, $g_1(t) = g_2(t) = \chi_{[0,t]}$ and C=(0,0,1,1), where χ_M is the indicator function of the set M, that is,

$$\chi_M(x) = \begin{cases} 1, & \text{if } x \in M, \\ 0, & \text{if } x \notin M. \end{cases}$$

Note that:
i) The functions f_1, f_2, g_1 and g_2 are discontinuous at each point of the interval $[0,1]$.
ii) $f_1(a)f_2(b) = \begin{cases} \chi_{(b,a]}, & \text{if } b < a \\ 0_Y, & \text{if } a \le b. \end{cases}$
iii) $(g_1(d) - g_1(c))(g_2(d) - g_2(c)) = \chi_{(c,d]}$ if $c < d$.
We now have the following:
a) The integral of the first type $(RS_I)\int_0^1 A(f_1, f_2, dg_1, dg_2)$ does not exist since for any $\delta > 0$ we can build two Stieltjes sums, $S_I^1(P)$ and $S_I^2(P)$, such that $|P| < \delta$ and $|S_I^1(P) - S_I^2(P)| = 1$.
To form such sums, consider any partition $P = \{t_0, t_1, \ldots, t_n\}$, $a = t_0 \le t_1 \le \cdots \le t_n = b$, such that $|P| < \delta$.
For the first sum, $S_I^1(P)$, we choose $s_1^i = s_2^i$ in each of the intervals $[t_{i-1}, t_i]$ so that $f_1(s_1^i)f_2(s_2^i) = 0_Y$. Therefore, for every i,

$$f_1(s_1^i)f_2(s_2^i)(g_1(t_i) - g_1(t_{i-1}))(g_2(t_i) - g_2(t_{i-1})) = 0_Y$$

and consequently $S_I^1(P) = 0_Y$.
For the second sum, $S_I^2(P)$, in the first interval $[t_0, t_1]$, we choose different associated values $s_1^1 > s_2^1$. In other intervals of the partition P, we choose the same values as those of the first sum, $S_I^1(P)$. Hence

$$S_I^2(P) = \chi_{(s_2^1, s_1^1]}.$$

Now,

$$|S_I^2(P) - S_I^1(P)| = sup|\chi_{(s_2^1, s_1^1]} - 0_Y| = 1.$$

Therefore, the integral of the first type $(RS_I)\int_0^1 A(f_1, f_2, dg_1, dg_2)$ does not exist.
b) Since, for every i,

$$f_1(s_i)f_2(s_i)(g_1(t_i) - g_1(t_{i-1}))(g_2(t_i) - g_2(t_{i-1})) = 0_Y,$$

we have $S_{II}(P) = 0_Y$. Therefore, the integral of the second type exists and $(RS_{II})\int_0^1 A(f_1, f_2, dg_1, dg_2) = 0_Y$.

4. Moore-Pollard-Stieltjes multilinear integrals (MPS)

Definition 4.1. *We say that the Type I multilinear Moore-Pollard-Stieltjes integral of $f_j, j \in I_s$ with respect to A and $f_j, j \in I_d$, exists on $[a, b]$ if there exists a vector $J_I \in Y$ with the following property:*

For every $\epsilon > 0$, there is a finite set $E \subset [a, b]$, such that for any partition $P \supseteq E$, $P = \{t_0, t_1, \dots, t_n\}$, $a = t_0 \le t_1 \le \cdots \le t_n = b$, and points s_i^j, $t_{i-1} \le s_i^j \le t_i$, $j \in I_s$, $i = 1, \dots, n$, we have

$$|J_I - S_I(P)| < \epsilon.$$

We write

$$J_I = (MPS_I)\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$$

where

$$d_jf_j = \begin{cases} f_j, & \text{if } c_j = 0; \\ df_j, & \text{if } c_j = 1. \end{cases}$$

In a similar way, when using $S_{II}(P)$, we define

$$(MPS_{II})\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p).$$

Remark 4.1. We see from the definitions of $RS_{I,II}$ and $MPS_{I,II}$ that the following implications are valid:
a) If $(RS_I)\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$ exists, $(RS_{II})\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$ also exists and has the same value.
b) If $(RS_I)\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$ exists, then $(MPS_I)\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$ exists and has the same value.
c) If $(RS_{II})\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$ exists, then $(MPS_{II})\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$, exists and has the same value.
d) If $(MPS_I)\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$ exists, then $(MPS_{II})\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$, exists and has the same value.

These implications are illustrated in the following figure:

$$\begin{array}{ccc} RS_I & \Rightarrow & RS_{II} \\ \Downarrow & & \Downarrow \\ MPS_I & \Rightarrow & MPS_{II}. \end{array}$$

Remark 4.2. The existence of MPS_{II} does not imply the existence of MPS_I, as we show below by using the same functions as those used in Example 3.1.

a) Let E be an arbitrary partition of the interval $[a,b]$. For any refinement $P \supseteq E$, we can build two Stieltjes sums, $S_I^1(P)$ and $S_I^2(P)$, differing only in associated points in one interval which contains point 1, such that

$$|S_I^1(P) - S_I^2(P)| = 1.$$

To show this, we consider two cases, a1) $1 \notin P$ and a2) $1 \in P$.
a1) If 1 is an interior point of an interval defined by the partition P, we construct the sums $S_I^1(P)$ and $S_I^2(P)$ in the similar way as in Example 3.1.
a2) Let $1 \in P$. For this case, suppose that $t_k = 1$. In the interval $[t_{k-1}, t_k]$, we can choose points $s_1^k = 1$ and $s_2^k < 1$ to build the sum $S_I^1(P)$. In the same interval, for the sum $S_I^2(P)$ we can choose $s_1^k < 1$ and $s_2^k < 1$. In other subintervals, we can take the same associated points for both Stieltjes sums. Then we have

$$|S_I^1(P) - S_I^2(P)| = 1.$$

Therefore, the MPS-integral of the first type (Type I) does not exist.

b) (*Type II*) Since $f_1(s_i), f_2(s_i) = 0$ for all i, we have that, for every partition P,

$$S_{II}(P) = 0$$

This shows that the MPS integral of the second type exists and has the value

$$(MPS_{II}) \int_0^2 A(f_1, f_2, dg) = 0.$$

5. Henstock-Kurzweil-Stieltjes multilinear integrals (HKS)

The difference between Riemann-type integration and Henstock-Kurzweil (HK) integration is that δ in the definition of HK-integrals is not necessarily a constant but a positive function defined on $[a,b]$, see [6],[9] or [10].

The basic concept in the theory of the HK integral is that of a δ-fine division which we now extend to the case of the multilinear integration.

Stieltjes sums, Type I

Note that in this case we have K $(= card(I_s))$ associated points in each subinterval.

Let $P = \{t_0, t_1, \ldots, t_n\}$, $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$ and let

$$D_I = \{S_i, [t_{i-1}, t_i]\}$$

denote a collection of intervals $[t_{i-1}, t_i]$ and points $S_i = \{s_i^j\}$, where $t_{i-1} \leq s_i^j \leq t_i$, $j \in I_s$, $i = 1, \ldots, n$. We say that D_I is a division of the interval $[a,b]$.

Let $\delta : [a,b] \to R$ be a positive function. We say that a division D_I of the interval $[a,b]$ is δ-fine if

$$s_i^j - \delta(s_i^j) < t_{i-1} \leq s_i^j \leq t_i < s_i^j + \delta(s_i^j), \;\; j \in I_s, \;\; i = 1, \ldots, n.$$

The associated Stieltjes sum of the first type we define in the same way as in the definition of the RS-integral

$$S_I(D_I) = \sum_{i=1}^{n} A[F_1^i, F_2^i, \dots, F_p^i]$$

where

$$F_j^i = \begin{cases} f_j(s_j^i), & \text{if} \quad c_j = 0; \\ f_j(t_i) - f_j(t_{i-1}), & \text{if} \quad c_j = 1. \end{cases}$$

Stieltjes sums, Type II

In this case we choose only one associated point, s_i, in each interval $[t_{i-1}, t_i]$, and denote

$$D_{II} = \{s_i, [t_{i-1}, t_i]\}.$$

A division D_{II} in this case (Type II) is δ-fine if

$$s_i - \delta(s_i) < t_{i-1} \le s_i \le t_i < s_i + \delta(s_i), \;\; i = 1, \dots, n.$$

The second type Stieltjes sum is

$$S_{II}(D_{II}) = \sum_{i=1}^{n} A[F_1^i, F_2^i, \dots, F_p^i]$$

where

$$F_j^i = \begin{cases} f_j(s_i), & \text{if} \quad c_j = 0; \\ f_j(t_i) - f_j(t_{i-1}), & \text{if} \quad c_j = 1. \end{cases}$$

Lemma 5.1. *If δ is a positive function defined on $[a, b]$, then there exists at least one δ-fine division D_I.*

Proof. When we consider a HKS multilinear integral of the second type, we can prove this lemma in the same way as for the ordinary HKS integral, see [6],[9] or [10]. So, for any positive function δ defined on $[a, b]$, there is a division $D_{II} = \{s_i, [t_{i-1}, t_i]\}$, such that

$$s_i - \delta(s_i) < t_{i-1} \le s_i \le t_i < s_i + \delta(s_i), \;\; i = 1, \dots, n.$$

If we consider Stieltjes sums of the first type then we can simply take $s_i^j = s_i$ for $j \in I_s, \;\; i = 1, \dots, n$ and obtain at least one δ-fine division D_I. □

Definition 5.1. (Henstock-Kurzweil-Stieltjes multilinear integral) *We say that the Type I (Type II) multilinear Stieltjes integral of $f_j, j \in I_s$ with respect to A and $f_j, j \in I_d$, exists in the Henstock-Kurzweil sense on $[a, b]$, if there exists a vector $J_I \in Y$ $(J_{II} \in Y)$ with the following property:*

For every $\epsilon > 0$, there is a positive function $\delta : [a, b] \to R$ such that for any δ-fine division D_I (D_{II})we have

$$|J_I - S_I(D_I)| < \epsilon$$
$$(|J_{II} - S_{II}(D_{II})| < \epsilon).$$

We write

$$J_I = (HKS_I)\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$$

$$(J_{II} = (HKS_{II})\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p))$$

where

$$d_jf_j = \begin{cases} f_j, & \text{if} \;\; c_j = 0; \\ df_j, & \text{if} \;\; c_j = 1. \end{cases}$$

Remark 5.1. We see from the definitions of $RS_{I,II}$ and $HKS_{I,II}$ that the following implications are valid:
a) If $(RS_I)\int_a^b A(d1f_1, d_2f_2, \dots, d_pf_p)$ exists, then $(HKS_I)\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$ also exists and has the same value.
b) If $(RS_{II})\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$ exists, then $(HKS_{II})\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$, exists and has the same value.
c) If $(HKS_I)\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$ exists, then $(HKS_{II})\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$, exists and has the same value.

Note that the existence of RS_I implies the existence of RS_{II}. We can illustrate these implications in the following figure:

$$\begin{array}{ccc} RS_I & \Rightarrow & RS_{II} \\ \Downarrow & & \Downarrow \\ HKS_I & \Rightarrow & HKS_{II}. \end{array}$$

Since the theory of the multilinear HKS_{II} is very similar to that of the bilinear Henstock-Kurzweil-Stieltjes integral and moreover, the existence of HKS_I implies that of HKS_{II}, we shall mainly discuss the HKS-integral of the first type.

Lemma 5.2. *If the integral* $(HKS_I)\int_a^b A(d_1f_1, d_2f_2, \dots, d_pf_p)$ *exists then it is uniquely determined.*

Proof. Suppose J_1 and J_2 are two values of this integral. Then, for any $\epsilon > 0$, there exist functions $\delta_1 > 0$ and $\delta_2 > 0$ such that $|J_1 - S_I(D_1)| < \epsilon$ for any δ_1-fine division D_1 and $|J_2 - S_I(D_2)| < \epsilon$ for any δ_2-fine division D_2.

Define $\delta = min(\delta_1, \delta_2)$ and consider δ-fine division $D = \{S_i, [t_{i-1}, t_i]\}$ where $S_i = \{s_i^j\}$. Since such division D satisfies

$$s_i^j - \delta(s_i^j) < t_{i-1} \le s_i^j \le t_i < s_i^j + \delta(s_i^j), \;\; j \in I_s, \;\; i = 1, \dots, n$$

we have that any δ-fine division D is also δ_1-fine and δ_2-fine.

Thus $|J_1 - S_I(D)| < \epsilon$ and $|J_2 - S_I(D)| < \epsilon$. Therefore, $|J_1 - J_2| < 2\epsilon$, which means that $J_1 = J_2$. □

Lemma 5.3. (Cauchy condition) *Let* $X_1, \ldots, X_p$ *and* Y *be linear normed spaces over the same field (* $\mathbb{R}$ *or* $\mathbb{C}$ *),* $A \in L(X_1, \ldots, X_p; Y)$ *and* $f_j : [a, b] \to X_j$*. If* Y *is a Banach space, then the integral* $(HKS_I) \int_a^b A(d_1 f_1, d_2 f_2, \ldots, d_p f_p)$ *exists if and only if for every* $\epsilon > 0$ *there exists a positive function* $\delta(s)$*, defined on* $[a, b]$*, such that for any* δ*-fine divisions* D_1 *and* D_2 *we have*

$$|S_I(D_1) - S_I(D_2)| < \epsilon.$$

Proof. Necessity. (Note that for this part we do not need the assumption that Y is complete.) Suppose that the integral exists and has the value of J. Then, for every $\epsilon > 0$, there exists a positive function $\delta(s)$, defined on $[a, b]$ such that

$$|J - S_I(D)| < \frac{\epsilon}{2}$$

for any δ-fine division $D = \{S_i, [t_{i-1}, t_i]\}$. Let D_1 and D_2 be two δ-fine divisions of $[a, b]$. Then

$$|S_I(D_1) - S_I(D_2)| \leq |S_I(D_1) - J| + |J - S_I(D_2)| < \epsilon.$$

Sufficiency. Assume that the Cauchy condition holds. For each $k \in N$ we choose a positive function δ_k such that for any δ_k-fine divisions D_1 and D_2 of $[a, b]$ we have

$$|S_I(D_1) - S_I(D_2)| < \frac{1}{k}.$$

We may assume that $\delta_{k+1}(s) \leq \delta_k(s)$; otherwise, we replace $\delta_{k+1}(s)$ by $min(\delta_{k+1}(s), \delta_k(s))$. Now, for each k, we fix a δ_k-fine division D_k and denote $J_k = S_I(D_k)$. For $m \geq k$, since $\delta_m(s) \leq \delta_k(s)$, it follows that D_m is a δ_k-fine division of $[a, b]$. Thus, for $m \geq k$,

$$|J_k - J_m| < \frac{1}{k}. \tag{5.1}$$

This implies that J_k, $k = 1, 2, \ldots$ is a Cauchy sequence in Y. Since Y is a Banach space, the sequence converges. Let J be the limit of the sequence. It remains to be shown that J is a HKS_I-integral, according to Definition 5.1. It follows from (5.1) that $|J_k - J| \leq \frac{1}{k}$ for $k \in N$. Now, for $\epsilon > 0$, we choose $K > \frac{1}{\epsilon}$. Let D be a δ_K-fine division of $[a, b]$. Then,

$$|S_I(D) - J| \leq |S_I(D) - J_K| + |J_K - J| = |S_I(D) - S_I(D_K)| + |J_K - J| \leq \frac{2}{K} < 2\epsilon.$$

Hence, the integral exists and $(HKS_I) \int_a^b A(d_1 f_1, d_2 f_2, \ldots, d_p f_p) = J$. □

Theorem 5.1. *If* Y *is a Banach space and the integral* $(HKS_I) \int_a^b A(d_1 f_1, d_2 f_2, \ldots, d_p f_p)$ *exists and if* $a \leq c \leq d \leq b$ *then* $(HKS_I) \int_c^d A(d_1 f_1, d_2 f_2, \ldots, d_p f_p)$ *also exists.*

Proof. Let $\epsilon > 0$. Since the integral exists on $[a,b]$, the Cauchy condition holds and there exists a positive function $\delta(s)$ on $[a,b]$ such that

$$|S_I(D_1) - S_I(D_2)| < \epsilon \tag{5.2}$$

for any δ-fine divisions D_1 and D_2 of $[a,b]$. Consider two δ-fine divisions D' and D'' of $[c,d]$. In order to show that

$$|S_I(D') - S_I(D'')| < \epsilon, \tag{5.3}$$

we shall construct two divisions D_1 and D_2 of the interval $[a,b]$ that include D' and D''. Let D_a be a δ-fine division of the interval $[a,c]$ and let D_b be a δ-fine division of the interval $[d,b]$. Now, we put $D_1 = D_a \cup D' \cup D_b$ and $D_2 = D_a \cup D'' \cup D_b$ and get two δ-fine divisions of the interval $[a,b]$. From (5.2), we have

$$|S_I(D_1) - S_I(D_2)| < \epsilon.$$

Since $S_I(D_1) - S_I(D_2) = S_I(D') - S_I(D'')$, we obtain

$$|S_I(D') - S_I(D'')| < \epsilon.$$

So, the Cauchy condition holds on the interval $[c,d]$. Consequently, since Y is a Banach space, the integral $(HKS_I)\int_c^d A(d_1f_1, d_2f_2, \ldots, d_pf_p)$ exists. □

As a consequence of this theorem we have that the existence of the HKS_I multilinear integral on $[a,b]$ implies the existence on the subintervals $[a,c]$ and $[c,b]$, where $a < c < b$.

The converse statement is true in the case of bilinear HKS integral. This is also true in a special case of HKS multilinear integrals when I_d has only one element (see remark 5.2 below).

But, in the general case when I_d has more than one element, the existence on the subintervals $[a,c]$ and $[c,b]$, $a < c < b$ does not imply the existence of the HKS multilinear integral on $[a,b]$, as we see in the following example.

Example 5.1. Suppose that $X_1 = X_2 = X_3 = Y = R$. We define A as the ordinary multiplication, $A(x_1,x_2,x_3) = x_1x_2x_3$.

Let $f_1(t) = 1$ for $0 \le t \le 2$, and $f_2(t) = f_3(t) = \begin{cases} 0, & \text{for } 0 \le t < 1 \\ 1, & \text{for } t = 1 \\ 2, & \text{for } 1 < t < 2. \end{cases}$

Then, the HKS_I integral exists on the intervals $[0,1]$ and $[1,2]$; (HKS_I) $\int_0^1 A(f_1, df_2, df_3) = 1$ and $(HKS_I)\int_1^2 A(f_1, df_2, df_3) = 1$. But, the integral does not exist on the interval $[0,2]$. To show this, we consider a positive function $\delta : [0,2] \to R$. We can always choose a δ-fine division, D_1, that includes 1 as an associated number (point) which is in the interior of a subinterval. Let $[t_{k-1}, t_k]$ be the interval which contains the number 1 and

$t_{k-1} < 1 < t_k$. Since functions f_2 and f_3 are constant in other intervals, we have

$$S_I(D_1) = 1 \cdot (f_2(t_k) - f_2(t_{k-1}))(f_3(t_k) - f_3(t_{k-1})) = 1 \cdot 2 \cdot 2 = 4.$$

Now, we construct another δ-fine division, D_2, by replacing $[t_{k-1}, t_k]$ with two intervals $[t_{k-1}, 1]$ and $[1, t_k]$. In both intervals we use 1 as an associated number (point). So, we have

$$\begin{aligned} S_I(D_2) &= 1 \cdot (f_2(1) - f_2(t_{k-1}))(f_3(1) - f_3(t_{k-1})) \\ &\quad + 1 \cdot (f_2(t_k) - f_2(1))(f_3(t_k) - f_3(1)) = 1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 1 = 2. \end{aligned}$$

Thus, $S_I(D_1) - S_I(D_2) = 2$. Since δ is an arbitrary positive function on $[0, 2]$, the Cauchy condition does not hold. Consequently, the HKS_I integral of the function f_1, with respect to functions f_2 and f_3, and the operator A, does not exist on $[0, 2]$ even though it exists on $[0, 1]$ and $[1, 2]$.

Note that there is no difference between HKS_I and HKS_{II} in Example 5.1, since I_s has only one element.

In the next theorem, assuming that the HKS_I integral exists on both $[a, c]$ and $[c, b]$, we add a new condition at the point c and obtain the existence of the integral on $[a, b]$. In the case of ordinary Stieltjes integrals, a similar condition is called pseudoadditivity, see [7])

Theorem 5.2. *Let $a < c < b$. Assume that:*
a) *The integrals $(HKS_I) \int_a^c A(d_1f_1, d_2f_2, \ldots, d_pf_p)$ and $(HKS_I) \int_c^b A(d_1f_1, d_2f_2, \ldots, d_pf_p)$ exist.*
b) *For any $\epsilon > 0$, there exists $d_\epsilon > 0$ such that $c - d_\epsilon < t' \le c \le t'' < c + d_\epsilon$ implies*

$$|A[F_1, F_2, \ldots, F_p] + A[G_1, G_2, \ldots, G_p] - A[H_1, H_2, \ldots, H_p]| < \epsilon,$$

where

$$F_j = \begin{cases} f_j(c), & \text{if } c_j = 0 \\ f_j(c) - f_j(t'), & \text{if } c_j = 1 \end{cases},$$

$$G_j = \begin{cases} f_j(c), & \text{if } c_j = 0 \\ f_j(t'') - f_j(c), & \text{if } c_j = 1 \end{cases} \quad \text{and}$$

$$H_j = \begin{cases} f_j(c), & \text{if } c_j = 0 \\ f_j(t'') - f_j(t'), & \text{if } c_j = 1 \end{cases}.$$

Then, the integral $(HKS_I) \int_a^b A(d_1f_1, d_2f_2, \ldots, d_pf_p)$ exists.

Proof. Let us denote the integrals on $[a, c]$ and $[c, b]$ by J_1 and J_2. Fix $\epsilon > 0$. There exists a positive function δ_1, defined on $[a, c]$ such that $|J_1 - S_I(D_1)| < \epsilon$ for any δ_1-fine division D_1 of $[a, c]$. Similarly, there exists a positive function δ_2, defined on $[c, b]$ such that $|J_2 - S_I(D_2)| < \epsilon$ for any

δ_2-fine division D_2 of $[c, b]$. Now, we shall construct a positive function δ, defined on $[a, b]$, which forces the point c to be an associated point for every δ-fine division of $[a, b]$. We denote the distance between s and c by $d(s, c)$ and define δ as follows:

$$\delta(s) = \begin{cases} min(\delta_1(s), d(s, c)), & \text{if} \quad s \in [a, c) \\ min(\delta_2(s), d(s, c)), & \text{if} \quad s \in (c, d] \\ min(d_\epsilon, \delta_1(c), \delta_2(c)), & \text{if} \quad s = c. \end{cases}$$

Suppose that D is a δ-fine division of $[a, b]$ and that $c \in [t', t'']$. We consider $|J_1 + J_2 - S_I(D)|$. In the Stieltjes sum $S_I(D)$ we "approximate" the c-term $A[H_1, H_2, \ldots, H_p]$ by $A[F_1, F_2, \ldots, F_p] + A[G_1, G_2, \ldots, G_p]$ and obtain the following:

$$S_I(D) = S_1(D_1) + S_2(D_2) + A[H_1, H_2, \ldots, H_p] \\ - A[F_1, F_2, \ldots, F_p] - A[G_1, G_2, \ldots, G_p].$$

Hence, using the conditions a) and b), we have

$$|J_1 + J_2 - S_I(D)| \leq |J_1 - S_I(D_1)| + |J_2 - S_I(D_2)| + |A[H_1, H_2, \ldots, H_p] \\ - A[F_1, F_2, \ldots, F_p] - A[G_1, G_2, \ldots, G_p] \leq \epsilon + \epsilon + \epsilon.$$

So, the integral $(HKS_I) \int_a^b A(d_1 f_1, d_2 f_2, \ldots, d_p f_p)$ exists and has the value of $J_1 + J_2$. □

Remark 5.2. The condition b) in Theorem 5.2 is identically fulfilled if $I_d = \{j : c_j = 1\}$ has only one element (i. e., we "differentiate" only one function). For example, if we consider $(HKS_I) \int_a^b A(f_1, df_2, f_3)$, we have:

$$A[f_1(c), f_2(c) - f_2(t'), f_3(c)] + A[f_1(c), f_2(t'') - f_2(c), f_3(c)] \\ - A[f_1(c), f_2(t'') - f_2(t'), f_3(c)] \equiv 0_Y.$$

Thus, if $(HKS_I) \int_a^c A(f_1, df_2, f_3)$ and $(HKS_I) \int_c^b A(f_1, df_2, f_3)$ exist, where $a < c < b$, then $(HKS_I) \int_a^b A(f_1, df_2, f_3)$ also exists.

So, when I_d has only one element, the existence of the HKS_I (HKS_{II}) multilinear integral on the subintervals $[a, c]$ and $[c, b]$ implies the existence of the integral on $[a, b]$, without new conditions, as we state in the following corollary.

Corollary 5.1. *Let $a < c < b$. If the integrals*

$$J_1 = (HKS_I) \int_a^c A(f_1, \ldots, f_{k-1}, df_k, f_{k+1} \ldots, f_p)$$

and

$$J_2 = (HKS_I) \int_c^b A(f_1, \ldots, f_{k-1}, df_k, f_{k+1} \ldots, f_p)$$

exist, then the integral

$$J = (HKS_I) \int_c^b A(f_1, \dots, f_{k-1}, df_k, f_{k+1} \dots, f_p)$$

exists and $J = J_1 + J_2$.

Proof. Assume that the HKS_I integral exists on the intervals $[a, c]$ and $[c, b]$. It is enough to prove that condition b) in Theorem 5.2 is fulfilled. Assume that $a \leq t' \leq c \leq t'' \leq< b$. Then, we can write

$$f_k(t'') - f_k(t') = f_k(t'') - f_k(c) + f_k(c) - f_k(t').$$

Since A is a multilinear operator, we have

$$A[F_1, F_2, \dots, F_p] + A[G_1, G_2, \dots, G_p] - A[H_1, H_2, \dots, H_p] = 0_Y,$$

where

$$F_j = \begin{cases} f_j(c), & \text{if} \quad j \neq k \\ f_j(c) - f_j(t'), & \text{if} \quad j = k \end{cases},$$

$$G_j = \begin{cases} f_j(c), & \text{if} \quad j \neq k \\ f_j(t'') - f_j(c), & \text{if} \quad j = k \end{cases} \quad \text{and}$$

$$H_j = \begin{cases} f_j(c), & \text{if} \quad j \neq k \\ f_j(t'') - f_j(t'), & \text{if} \quad j = k \end{cases}.$$

Thus, the condition b) in Theorem 5.2 is identically fulfilled. Hence, the integral $J = (HKS_I) \int_c^b A(f_1, \dots, f_{k-1}, df_k, f_{k+1} \dots, f_p)$ exists and $J = J_1 + J_2$. □

6. *MPS* and *HKS* multilinear integrals

In this section we shall discuss the relationship between the existence of the multilinear Stieltjes integrals in the Moore-Pollard and Henstock-Kurzweil sense. If we consider ordinary, bilinear (or multilinear integral where I_d has only one element), the existence of the MPS integral implies the existence of the HKS integral. But, in general case, the existence of the MPS_I (MPS_{II}) integral does not imply the existence of the integral HKS_I (HKS_{II}). To show this, we again use Example 5.1. When we discus the existence of an MPS integral, we can always include any given point in a subdivision. We consider a division D of the interval $[0, 2]$ which includes t=1 as a division point. For any such division we have: $S_I(D) = 2$. Hence, the MPS_I integral exists and $(MPS_I) \int_0^2 A(f_1, df_2, df_3) = 2$.
(Note again, since I_s has only one element, that there is no difference between integrals of the first and second types in this example.)
We have already shown in Example 5.1 that HKS_I (HKS_{II}) does not exist over [0,2]. So, in the general case, the existence of the MPS_I (MPS_{II})

integral does not imply the existence of the HKS_I (HKS_{II}) multilinear integral.

In the next theorem we assume the existence of the MPS_I multilinear integral. Then, to obtain the existence of the HKS_I integral, we add the pseudoadditivity condition at every point $x \in [a, b]$).

Theorem 6.1. *Assume that:*
a) $(MPS_I) \int_a^b A(d_1 f_1, d_2 f_2, \ldots, d_p f_p)$ *exists and*
b) *for any* $x \in [a, b]$ *and* $\epsilon > 0$*, there exists* $d_\epsilon > 0$ *such that* $t', t'' \in [a, b]$ *and* $x - d_\epsilon < t' \leq x \leq t'' < x + d_\epsilon$ *implies*

$$|A[F_1, F_2, \ldots, F_p] + A[G_1, G_2, \ldots, G_p] - A[H_1, H_2, \ldots, H_p]| < \epsilon,$$

where

$$F_j = \begin{cases} f_j(x), & \text{if} \quad c_j = 0 \\ f_j(x) - f_j(t'), & \text{if} \quad c_j = 1 \end{cases},$$

$$G_j = \begin{cases} f_j(x), & \text{if} \quad c_j = 0 \\ f_j(t'') - f_j(x), & \text{if} \quad c_j = 1 \end{cases} \quad \text{and}$$

$$H_j = \begin{cases} f_j(x), & \text{if} \quad c_j = 0 \\ f_j(t'') - f_j(t'), & \text{if} \quad c_j = 1 \end{cases}.$$

Then $(HKS_I) \int_a^b A(d_1 f_1, d_2 f_2, \ldots, d_p f_p)$ *also exists and has the same value as* $(MPS_I) \int_a^b A(d_1 f_1, d_2 f_2, \ldots, d_p f_p)$*.*

Proof. Fix $\epsilon > 0$ and denote $(MPS_I) \int_a^b A(d_1 f_1, d_2 f_2, \ldots, d_p f_p)$ by J_M. Then, there exists a finite set

$$E = \{x_0, x_1, \ldots, x_m\} \subset [a, b], \ \ a = x_0 < x_1 < \cdots < x_m = b,$$

such that for any partition

$$P = \{t_0, t_1, \ldots, t_n\}, \ \ a = t_0 \leq t_1 < \cdots \leq t_n = b,$$

which contains E, and for points s_j^i, $t_{i-1} \leq s_j^i \leq t_i$, where $j \in I_s$ and $i = 1, 2, \ldots, n$, we have

$$|J_M - S_I(P)| < \epsilon. \tag{6.1}$$

We apply condition b) on the points of the finite set $E = \{x_0, x_1, \ldots, x_m\}$. So, there exists $d_\epsilon > 0$ such that $x_k - d_\epsilon < t' \leq x_k \leq t'' < x_k + d_\epsilon$ implies

$$|A[F_1, F_2, \ldots, F_p] + A[G_1, G_2, \ldots, G_p] - A[H_1, H_2, \ldots, H_p]| < \frac{\epsilon}{m},$$

where

$$F_j = \begin{cases} f_j(x_k), & \text{if } \ c_j = 0 \\ f_j(x_k) - f_j(t'), & \text{if } \ c_j = 1 \end{cases},$$

$$G_j = \begin{cases} f_j(x_k), & \text{if } \ c_j = 0 \\ f_j(t'') - f_j(x_k), & \text{if } \ c_j = 1 \end{cases} \text{ and}$$

$$H_j = \begin{cases} f_j(x_k), & \text{if } \ c_j = 0 \\ f_j(t'') - f_j(t'), & \text{if } \ c_j = 1 \end{cases}.$$

Now, using a distance function, we construct a positive function δ on the interval $[a, b]$ that forces the points of the finite set E to be associated points for every δ-fine division D.

Let $d(s, E)$ denote the distance between s and E, that is,

$$d(s, E) = min\{|s - x_k|, \ \ k = 0, \dots, m\}.$$

So, if $s \notin E$ then $d(s, E) > 0$. Let

$$\delta_1 = min\{|x_k - x_{k-1}|, \quad k = 1, \dots, m\}.$$

We define $\delta(s)$ to be

$$\delta(s) = \begin{cases} min(d_\epsilon, \delta_1/2), & \text{if } \ s \in E \\ d(s, E), & \text{if } \ s \notin E \end{cases}.$$

Let $D = \{\{s_i^j\}, [t_{i-1}, t_i]\}$ be a δ-fine division of $[a, b]$. Due to the definition of $\delta(s)$, the points $\{x_0, x_1, \dots, x_m\}$ are associated points. Furthermore if x_k lies in the interval $[t_{i-1}, t_i]$, then the relation

$$s_i^j - \delta(s_i^j) < t_{i-1} \le s_i^j \le t_i < s_i^j + \delta(s_i^j),$$

is possible only if $s_i^j = x_k$ for all j.

Consider the Stieltjes sum $S_I(D)$. In order to obtain a partition P which includes E , we divide every interval $[t_{i-1}, t_i]$ which contains x_k as an interior point into two parts, $[t_{i-1}, x_k]$ and $[x_k, t_i]$. Then, we "approximate" the term

$$A[H_1, H_2, \dots, H_p]$$

by the sum

$$A[F_1, F_2, \dots, F_p] + A[G_1, G_2, \dots, G_p],$$

where

$$F_j = \begin{cases} f_j(x_k), & \text{if } \ c_j = 0 \\ f_j(x_k) - f_j(t_{i-1}), & \text{if } \ c_j = 1 \end{cases},$$

$$G_j = \begin{cases} f_j(x_k), & \text{if } \ c_j = 0 \\ f_j(t_i) - f_j(x_k), & \text{if } \ c_j = 1 \end{cases} \text{ and}$$

$$H_j = \begin{cases} f_j(x_k), & \text{if } \ c_j = 0 \\ f_j(t_i) - f_j(t_{i-1}), & \text{if } \ c_j = 1. \end{cases}.$$

Let us denote $L_k = A[H_1, H_2, \dots, H_p] - A[F_1, F_2, \dots, F_p] - A[G_1, G_2, \dots, G_p]$. Then using (6.1) and condition b) we have

$$|J_M - S_I(D)| \leq |J_M - S_I(P)| + \sum_{k=1}^{m} |L_k| < \epsilon + m\frac{\epsilon}{m} = 2\epsilon.$$

This means that $(HKS_I) \int_a^b A(d_1 f_1, d_2 f_2, \dots, d_p f_p)$ exists and has the value $J_M = (MPS_I) \int_a^b A(d_1 f_1, d_2 f_2, \dots, d_p f_p)$. □

Note again, that in the case when I_d has only one element (we "differentiate" only one function for example f_k), condition b) is identically fulfilled and we have the following:

Corollary 6.1. *If*

$$(MPS_I) \int_a^b A(f_1, \dots, f_{k-1}, df_k, f_{k+1} \dots, f_p)$$

exists, then

$$(HKS_I) \int_a^b A(f_1, \dots, f_{k-1}, df_k, f_{k+1} \dots, f_p)$$

also exists.

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