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SOME NEW INEQUALITIES FOR THE STRICTLY MONOTONIC FUNCTIONS AND POSITIVE N-TUPLES

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Dedicated to the memory of Academician Fikret Vajzović

ABSTRACT. In this paper we give the generalization of the inequalities

$$n\sum_{i=1}^{n}a_i^n \geq \sum_{i=1}^{n}a_i^t\sum_{i=1}^{n}a_i^{n-t} \geq \sum_{i=1}^{n}a_i^s\sum_{i=1}^{n}a_i^{n-s} \geq (\sum_{i=1}^{n}a_i^{\frac{n}{2}})^2,$$
$$0 < t < s < \frac{n}{2},\ a_i > 0, \forall i \in \{1,\ldots,n\},$$

(see [3] and [1, p. 377]), and the new results for the strictly monotonic functions and positive n-tuples.

1. INTRODUCTION AND BASIC TERMS

Let $\overline{a} = (a_1,...,a_n), \overline{b} = (b_1,...,b_n)$ be positive n-tuples. Thus the positive n-tuples are:

$$\overline{a}^c = (a_1^c,...,a_n^c), (c \in \mathbb{R}), \overline{a}\overline{b} = (a_1b_1,...,a_nb_n),$$
$$\overline{a}_k = (a_1,...,a_{k-1},a_{k+1},...,a_n+1), 1 \leq k \leq n+1,$$

and $\overline{a}ln\overline{a} = (a_1lna_1,...,a_nlna_n)$ is real n-tuples.
Let $f(t)$ be a real function and let be

$$S_n(\overline{a}^f) := \sum_{i=1}^{n}a_i^{f(t)}.$$

Thus we have

$$S_n(\overline{a}_k^{f(t)}) = \sum_{\substack{i=1\\ i\neq k}}^{n+1}a_i^{f(t)} = S_{n+1}(\overline{a}^{f(t)}) - a_k^{f(t)}, \text{ i.e. } \sum_{i=1}^{n+1}S_n(\overline{a}_k^{f(t)}) = nS_{n+1}(\overline{a}^{f(t)})$$

Definition 1.1. *Let* $\mathbb{I} = (\alpha,\beta) \subset \mathbb{R},\ g\colon \mathbb{I} \to \mathbb{R}$. *The function g is strictly monotonically increasing (decreasing) on* $\mathbb{I}$ *if*

$$\alpha < u < v < \beta \Rightarrow g(v) - g(u) > (<)0. \tag{1.1}$$

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The fact that g is strictly monotonically increasing (decreasing) on $\mathbb{I}$, we will denote by $g\uparrow (g\downarrow)$.

Remark 1.1. In this paper, the features of strictly monotonically increasing (decreasing) functions, characterized by strict inequality on the right side of the implication (1.1), will be considered.

2. Some theorems about the strictly monotonic functions and positive n-tuples

Lemma 2.1. *Let g be a function that satisfies the conditions of definition 1.1. If the function g is also differentiable on $\mathbb{I}$, then we have $\frac{dg}{dt} \geq (\leq) 0, \forall t \in \mathbb{I}$ where the sign = is valid only for a discrete set of points in $\mathbb{I}$, and it does not exist any interval $\mathbb{J} \subset \mathbb{I}$ such that $\frac{dg}{dt} = 0, \forall t \in \mathbb{J}$.*

Proof. If we assume that the interval $\mathbb{J} \subset \mathbb{I}$, such that $\frac{dg}{dt} = 0, \forall t \in \mathbb{J}$, exists, then it would appear that $g(t) = const, \forall t \in \mathbb{J}$, which contradicts the assumption that g is strictly monotone on $\mathbb{I}$. □

Theorem 2.2. *Let $g\colon \mathbb{I} \to \mathbb{R}$, $\mathbb{I} = (\alpha, \beta)$, $\overline{a}$ positiv n-tuples and c a real number, then for a sequence of functions $F_n\colon \mathbb{I} \to \mathbb{R}$*

$$F_n(\overline{a}, g) := \sum_{i=1}^{n} a_i^{c+g(t)} \sum_{i=1}^{n} a_i^{c-g(t)} = S_n(\overline{a}^{c+g(t)}) S_n(\overline{a}^{c-g(t)}),$$

is valid

$$2.1.\quad if\ (g>0, g\uparrow)\ \ or\ \ (g<0, g\downarrow),\ \ then\ \ F_n\uparrow,$$
$$2.2.\quad if\ (g>0, g\downarrow)\ \ or\ \ (g<0, g\uparrow),\ \ then\ \ F_n\downarrow.$$

If the function g is differentiable on $\mathbb{I}$, we can relax the conditions 2.1. and 2.2. on the next way

$$2.1.^{*}\quad if\ g\frac{dg}{dt} \geq 0\ \ then\ \ F_n\uparrow; \qquad 2.2.^{*}\quad if\ g\frac{dg}{dt} \leq 0\ \ then\ \ F_n\downarrow.$$

According to the Lemma 2.1 *the sign = valid in 2.1.* and 2.2.* only for a discrete set of points in $\mathbb{I}$.*

Proof. For all integers $n \geq 2$ and $\alpha < u < v < \beta$ we will by mathematical induction prove that

$$2.1.(or\ \ 2.2.) \Rightarrow F_n(\overline{a}, g(v)) \geq (or \leq) F_n(\overline{a}, g(u)) \tag{2.1}$$

where the sign = is valid iff $a_1 = \cdots = a_n$.
Since

$$F_2(\overline{a}, g(v)) - F_2(\overline{a}, g(u)) = \sum_{i=1}^{2} a_i^{c+g(v)} \sum_{i=1}^{2} a_i^{c-g(v)} - \sum_{i=1}^{2} a_i^{c+g(u)} \sum_{i=1}^{2} a_i^{c-g(u)}$$
$$= a_1^{c-g(v))} a_2^{c-g(v))} (a_1^{g(v)-g(u)} - a_2^{g(v)-g(u)})(a_1^{g(v)+g(u)} - a_2^{g(v)+g(u)})$$

it comes out

$$sgn(F_2(\overline{a},g(v))-F_2(\overline{a},g(u)))=sgn(a_1-a_2)^2\geq 0,\ \text{if applicable 2.1.}$$

$$sgn(F_2(\overline{a},g(v))-F_2(\overline{a},g(u)))=-sgn(a_1-a_2)^2\leq 0,\text{if applicable 2.2.}$$

where the sign = is valid iff $a_1=a_2$
Thus, (2.1) is true for n = 2.
Suppose that (2.1) is valid for some $n(\geq 2)$ and the sign = in (2.1) holds iff $a_1=\cdots=a_n$. Then for $k\in\{1,\dots,n,n+1\}$ we obtain the inequality

$$F_n(\overline{a}_k,g(v))\geq F_n(\overline{a}_k,g(u)),\quad for\ \ \alpha<u<v<\beta \tag{2.2}$$

where the sign = is valid iff $a_1=\cdots=a_{k-1}=a_{k+1}=\cdots=a_{n+1}$.
By applying the conditions 2.2, instead of the conditions 2.1, in (2.2) the opposite sign of the inequality holds. So,

$$\begin{aligned}&\sum_{i=1,i\neq k}^{n+1}a_i^{c+g(v)}\sum_{i=1,i\neq k}^{n+1}a_i^{c-g(v)}\geq\sum_{i=1,i\neq k}^{n+1}a_i^{c+g(u)}\sum_{i=1,i\neq k}^{n+1}a_i^{c-g(u)},\\ &for\ \alpha<u<v<\beta,\end{aligned}$$

and summing these n + 1 inequalities we obtain

$$\begin{aligned}&\sum_{i=1}^{n+1}(S_{n+1}(\overline{a}^{c+g(v)})-a_k^{c+g(v)})\sum_{i=1}^{n+1}(S_{n+1}(\overline{a}^{c-g(v)})-a_k^{c-g(v)})\\ \geq&\sum_{i=1}^{n+1}(S_{n+1}(\overline{a}^{c+g(u)})-a_k^{c+g(u)})\sum_{i=1}^{n+1}(S_{n+1}(\overline{a}^{c-g(u)})-a_k^{c-g(u)}),\end{aligned}$$

$$(n-1)S_{n+1}(\overline{a}^{c+g(v)})S_{n+1}(\overline{a}^{c-g(v)})\geq(n-1)S_{n+1}(\overline{a}^{c+g(u)})S_{n+1}(\overline{a}^{c-g(u)}),\text{i.e.}$$

$$F_{n+1}(\overline{a},g(v))\geq F_{n+1}(\overline{a},g(u))\ \ for\ \alpha<u<v<\beta \tag{2.3}$$

where the sign = is valid iff $a_1=\cdots=a_{n+1}$.
By applying the conditions 2.2, instead of the conditions 2.1, in (2.3) the opposite sign of the inequality holds, which gives us the proof. □

Theorem 2.3. *For every $c\in\mathbb{R},\ 0\leq t<s<\infty$, the following inequalities hold*

$$S_n(\overline{a}^c)^2\leq S_n(\overline{a}^{c+t})S_n(\overline{a}^{c-t})\leq S_n(\overline{a}^{c+s})S_n(\overline{a}^{c-s}), \tag{2.4}$$

whence for $c=n,\ t=k\in\mathbb{N}$ we get a series of the inequalities

$$\begin{aligned}S_n(\overline{a}^n)^2\leq S_n(\overline{a}^{n+1})S_n(\overline{a}^{n-1})\leq\cdots\leq S_n(\overline{a}^{n+k})S_n(\overline{a}^{n-k})\\ \leq S_n(\overline{a}^{n+k+1})S_n(\overline{a}^{n-k-1})\leq\cdots\end{aligned} \tag{2.5}$$

In (2.4), (2.5) *the sign = is valid iff $a_1=\cdots=a_n$.*

Proof. Let $g(t) = t$, i.e. $g(t)\frac{dg}{dt} = t$. By the condition 2.1.* (2.2.*) from the Theorem 2.2 we have the following continuous function

$$F_n(\overline{a},t) = \sum_{i=1}^{n} a_i^{c+t} \sum_{i=1}^{n} a_i^{c-t} = S_n(\overline{a}^{c+t})S_n(\overline{a}^{c-t}) \uparrow (\downarrow) \quad for\ t > (<) 0.$$

Therefore the function $F_n(\overline{a},t) = S_n(\overline{a}^{c+t})S_n(\overline{a}^{c-t})$ has a minimum for t = 0, i.e.

$$\min_{t\in\mathbb{R}} F_n(\overline{a},t) = S_n(\overline{a}^{c})^2 \leq S_n(\overline{a}^{c+t})S_n(\overline{a}^{c-t}) \uparrow (\downarrow) \quad for\ t > (<) 0. \tag{2.6}$$

From (2.6) we obtain the inequalities (2.4), (2.5). □

Remark 2.1. From (2.4), for c = 0, t = 1, we obtain

$$S_n(\overline{a}^{0})^2 = n^2 \leq S_n(\overline{a}^{1})S_n(\overline{a}^{-1}),$$

which is the inequality between the arithmetic mean $A_n(\overline{a})$ and the harmonic mean $H_n(\overline{a})$ of the positive n-tuple $\overline{a}$

$$A_n(\overline{a}) = \frac{S_n(\overline{a}^{1})}{n} \geq \frac{n}{S_n(\overline{a}^{-1})} = H_n(\overline{a})$$

Theorem 2.4. *Let* $g\colon \mathbb{I} \to \mathbb{R},\ \mathbb{I} = (\alpha, \beta)$. *Then for a finite series of functions* $G_n\colon \mathbb{I} \to \mathbb{R}$,

$$G_n(\overline{a},g) := \sum_{i=1}^{n} a_i^{g(t)} \sum_{i=1}^{n} a_i^{2c-g(t)} = S_n(\overline{a}^{g})S_n(\overline{a}^{2c-g}),$$

where c is any real number, holds the following conditions on $\mathbb{I}$

$$2.3.\ if\ (g > c,\ g\uparrow)\ or\ (g < c,\ g\downarrow),\ then\ G_n \uparrow,$$

$$2.4.\ if\ (g > c,\ g\downarrow)\ or\ (g < c,\ g\uparrow),\ then\ G_n \downarrow .$$

If the function g is differentiable on $\mathbb{I}$, *we can relax the conditions 2.3. and 2.4. on the next way:*

$$2.3.^{*}\ if\ (g-c)\frac{dg}{dt} \geq 0,\ then\ G_n \uparrow;\ 2.4.^{*}\ if\ (g-c)\frac{dg}{dt} \leq 0,\ then\ G_n \downarrow .$$

According to the Lemma 2.1 *the sign = valid in 2.3.* and 2.4.* only for a discrete set of points in* $\mathbb{I}$.

Proof. Since

$$G_n(\overline{a},g) = S_n(\overline{a}^{c+(g-c)})S_n(\overline{a}^{c-(g-c)}) = F_n(\overline{a},g-c),$$

from Theorem 2.2 we obtain Theorem 2.4 □

Theorem 2.5. *Let* $n \in \mathbb{N}$ *and* $c > 0$.

i) *For* $-\infty < t < s \leq 0$, *the following inequalities hold*

$$S_n(\overline{a}^{t})S_n(\overline{a}^{2c-t}) \geq S_n(\overline{a}^{s})S_n(\overline{a}^{2c-s}) \geq nS_n(\overline{a}^{2c}). \tag{2.7}$$

ii) *For* $0 \le t < s \le c$ *the following inequalities hold*

$$nS_n(\overline{a}^{2c}) \ge S_n(\overline{a}^t)S_n(\overline{a}^{2c-t}) \ge S_n(\overline{a}^s)S_n(\overline{a}^{2c-s}) \ge S_n(\overline{a}^c)^2. \tag{2.8}$$

In (2.7) *and* (2.8) *the sign = is valid iff* $a_1 = \cdots = a_n$.

Proof. If we take $g(t) = t$ in the Theorem 2.4, we obtain

$$(g(t)-c)\frac{dg}{dt} = t - c \le (\ge)\ 0,\ if\ t \le (\ge)\ c.$$

Therefore, a continuous function $G_n : \mathbb{I} \to \mathbb{R}$ has a minimum for $t = c$ and we have

$$\begin{aligned} \min_{t\in\mathbb{R}} G_n(\overline{a},t) =& G_n(\overline{a},c) \\ =& S_n(\overline{a}^c)^2 \le S_n(\overline{a}^t)S_n(\overline{a}^{2c-t}) \downarrow (\uparrow)\ if\ t < (>)\ c \end{aligned} \tag{2.9}$$

For $c > 0$, the nequalities (2.7) end (2.8) follow from (2.9). □

Remark 2.2. If we take $2c = n$, then

a) for $-\infty < t < s \le 0$, from the inequality (2.7), the following inequalities hold

$$S_n(\overline{a}^t)S_n(\overline{a}^{n-t}) \ge S_n(\overline{a}^s)S_n(\overline{a}^{n-s}) \ge nS_n(\overline{a}^n), \tag{2.10}$$

b) for $0 \le t < s \le \frac{n}{2}$, from the inequality (2.8), the following inequalities hold

$$nS_n(\overline{a}^n) \ge S_n(\overline{a}^t)S_n(\overline{a}^{n-t}) \ge S_n(\overline{a}^s)S_n(\overline{a}^{n-s}) \ge S_n(\overline{a}^{\frac{n}{2}})^2. \tag{2.11}$$

In (2.10) and (2.11) the sign = is valid iff $a_1 = \ldots a_n$

Remark 2.3. Inequalities (2.11) have been otherwise proved in [3].

3. Main results

Theorem 3.1. *Suppose that all assumptions of the Theorem 2.2 hold and the condition 2.1.* (or 2.2.*) of the Theorem 2.2 is satisfied. Then for* $g > (<)\ 0$ *on* $\mathbb{I}$ *we have the following inequality*

$$S_n(\overline{a}^{c+g}log\overline{a})S_n(\overline{a}^{c-g}) \ge (\le) S_n(\overline{a}^{c+g})S_n(\overline{a}^{c-g}log\overline{a}), \tag{3.1}$$

Proof. From the Theorem 2.2 follows if function g satisfies condition 2.1* (or 2.2*), the function F_n is strictly increasing (decreasing), differentiable on $\mathbb{I}$ and it holds

$$\begin{aligned} \frac{dF_n}{dt} =& [S_n(\overline{a}^{c+g}log\overline{a})S_n(\overline{a}^{c-g}) - S_n(\overline{a}^{c+g})S_n(\overline{a}^{c-g}log\overline{a})]\frac{dg}{dt} \\ &\ge (\le) 0. \end{aligned} \tag{3.2}$$

The sign = is valid in (3.2) if $a_1 = \cdots = a_n$.
For $g > (<) 0$ on $\mathbb{I}$, (3.1) follows from (3.2). □

Example 3.1. *If we take $g(t) = \log t,\ t \in (1, \infty)$ in the Theorem 3.1 we obtain the following inequality*

$$S_n(\overline{a}^{c+logt} log\overline{a})S_n(\overline{a}^{c-logt}) \geq S_n(\overline{a}^{c+logt})S_n(\overline{a}^{c-logt} log\overline{a}), \tag{3.3}$$

and for $t \in (0,1)$ the opposite inequality holds in (3.3).
The sign = is valid in (3.3) *if $a_1 = \cdots = a_n$.*

Example 3.2. *If we take $g(t) = \sin t (> 0),\ t \in \cup_{k\in\mathbb{Z}}[2k\pi, (2k+1)\pi]$ in the Theorem 3.1. the following inequality holds*

$$S_n(\overline{a}^{c+\sin t} \log\overline{a})S_n(\overline{a}^{c-\sin t}) \geq S_n(\overline{a}^{c+logt})S_n(\overline{a}^{c-logt} log\overline{a}), \tag{3.4}$$

and for $t \in \cup_{k\in\mathbb{Z}}[(2k+1)\pi, (2(k+1)\pi]$, the opposite inequality holds in (3.4).
The sign = is valid in (3.4) *if $a_1 = \cdots = a_n$.*

Theorem 3.2. *Suppose that all assumptions of the Theorem 2.4 hold and the condition 2.3.* (or 2.4.*) of the Theorem 2.4. is satisfied. Then for $g > (<)c$ on $\mathbb{I}$ we have the following inequality:*

$$S_n(\overline{a}^{g} \log\overline{a})S_n(\overline{a}^{2c-g}) \geq (\leq) S_n(\overline{a}^{g})S_n(\overline{a}^{2c-g} \log\overline{a}). \tag{3.5}$$

The sign = is valid in (3.5) *if $a_1 = \cdots = a_n$.*

Proof. From the Theorem 2.4 follows:
If the function g satisfies condition 2.3.* (or 2.4.*), then G_n is strictly increasing (decreasing) and differentiable on $\mathbb{I}$, and we have

$$\begin{aligned} \frac{dG_n}{dt} &= [S_n(\overline{a}^{g} \log\overline{a})S_n(\overline{a}^{2c-g}) - S_n(\overline{a}^{g})S_n(\overline{a}^{2c-g} \log\overline{a})]\frac{dg}{dt} \\ &\geq (\leq) 0, \end{aligned} \tag{3.6}$$

where the sign = is valid if $a_1 = \cdots = a_n$.
For $g > (<)c$ on $\mathbb{I}$, (3.5) follows from (3.6). □

Example 3.3. *If we take $g(t) = e^{1/t} (< 1),\ t \in (-\infty, 0)$ in the Theorem 3.2 the following inequality holds*

$$S_n(\overline{a}^{e^{1/t}} \log\overline{a})S_n(\overline{a}^{2-e^{1/t}}) \leq S_n(\overline{a}^{e^{1/t}})S_n(\overline{a}^{2-e^{1/t}} \log\overline{a}). \tag{3.7}$$

For $t \in (0, \infty)$ is $g(t) = e^{1/t} (> 1)$, and in (3.7) *the opposite inequality holds.*
The sign = is valid in (3.7) if $a_1 = \cdots = a_n$.

Example 3.4. *Let $g(t) = \log t$. Then for $t \in (e^c, \infty)$ we have $g(t) > c$, and by the Theorem 3.2 we have*

$$S_n(\overline{a}^{\log t} \log\overline{a})S_n(\overline{a}^{2c-\log t}) \geq S_n(\overline{a}^{\log t})S_n(\overline{a}^{2c-\log t} \log\overline{a}) \tag{3.8}$$

For $t \in (0, e^c)$ we have $g(t) < c$, and be the Theorem 3.2 the opposite inequality in (3.8) *holds.*
The sign = is valid in (3.8) *if $a_1 = \cdots = a_n$.*

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