SOME NEW INEQUALITIES FOR THE STRICTLY MONOTONIC FUNCTIONS AND POSITIVE N-TUPLES

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Abstract. In this paper we give the generalization of the inequalities
\[ n \sum_{i=1}^{n} a_i^t \geq \sum_{i=1}^{n} a_i \sum_{i=1}^{n} a_i^{n-t} \geq \sum_{i=1}^{n} a_i^t \sum_{i=1}^{n} a_i^{n-t} \geq (\sum_{i=1}^{n} a_i^t)^2, \]
\[ 0 < t < n, a_i > 0, \forall i \in \{1, \ldots, n\}, \]
(see [3] and [1, p. 377]), and the new results for the strictly monotonic functions and positive n-tuples.

1. Introduction and Basic Terms

Let \( \bar{a} = (a_1, \ldots, a_n), \bar{b} = (b_1, \ldots, b_n) \) be positive n-tuples. Thus the positive n-tuples are:
\[ \bar{a}^c = (a_1^c, \ldots, a_n^c), (c \in \mathbb{R}), \bar{a} \bar{b} = (a_1 b_1, \ldots, a_n b_n), \]
\[ \bar{a}_k = (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n + 1), 1 \leq k \leq n + 1, \]
and \( \bar{a} \ln \bar{a} = (a_1 \ln a_1, \ldots, a_n \ln a_n) \) is real n-tuples.

Let \( f(t) \) be a real function and let be
\[ S_n(\bar{a}^f) := \sum_{i=1}^{n} a_i^f(i). \]

Thus we have
\[ S_n(\bar{a}_k^f(i)) = \sum_{i=1}^{n+1} a_i^f(i) = S_{n+1}(\bar{a}_k^f(i)) - a_k^f(i), \]
i.e. \( \sum_{i=1}^{n+1} S_n(\bar{a}_k^f(i)) = n S_{n+1}(\bar{a}_k^f(i)) \)

Definition 1.1. Let \( \mathbb{I} = (\alpha, \beta) \subset \mathbb{R}, g: \mathbb{I} \to \mathbb{R} \). The function \( g \) is strictly monotonic increasing (decreasing) on \( \mathbb{I} \) if
\[ \alpha < u < v < \beta \Rightarrow g(v) - g(u) > (\leq) 0. \]
The fact that \( g \) is strictly monotonically increasing (decreasing) on \( \mathbb{I} \), we will denote by \( g \uparrow (g \downarrow) \).

**Remark 1.1**. In this paper, the features of strictly monotonically increasing (decreasing) functions, characterized by strict inequality on the right side of the implication (1.1), will be considered.

2. **Some Theorems about the Strictly Monotonic Functions and Positive \( n \)-Tuples**

**Lemma 2.1.** Let \( g \) be a function that satisfies the conditions of definition 1.1. If the function \( g \) is also differentiable on \( \mathbb{I} \), then we have \( \frac{dg}{dt} \geq (\leq) 0, \forall t \in \mathbb{I} \) where the sign = is valid only for a discrete set of points in \( \mathbb{I} \), and it does not exist any interval \( \mathbb{J} \subset \mathbb{I} \) such that \( \frac{dg}{dt} = 0, \forall t \in \mathbb{J} \).

**Proof.** If we assume that the interval \( \mathbb{J} \subset \mathbb{I} \), such that \( \frac{dg}{dt} = 0, \forall t \in \mathbb{J} \), exists, then it would appear that \( g(t) = \text{const} \), \( \forall t \in \mathbb{J} \), which contradicts the assumption that \( g \) is strictly monotone on \( \mathbb{I} \).

**Theorem 2.2.** Let \( g : \mathbb{I} \rightarrow \mathbb{R}, \mathbb{I} = (\alpha, \beta) \), \( \alpha \) positively \( n \)-tuples and \( c \) a real number, then for a sequence of functions \( F_n : \mathbb{I} \rightarrow \mathbb{R} \)

\[
F_n(\bar{a}, g) := \frac{1}{a_i} \sum_{i=1}^{n} a_i c^{g(t)} + \sum_{i=1}^{n} a_i c^{-g(t)} = S_n(a_i c^{g(t)}) S_n(a_i c^{-g(t)}),
\]

is valid

1. if \( \alpha > 0, g \uparrow \) or \( \alpha < 0, g \downarrow \), then \( F_n \uparrow \),
2. if \( \alpha > 0, g \downarrow \) or \( \alpha < 0, g \uparrow \), then \( F_n \downarrow \).

If the function \( g \) is differentiable on \( \mathbb{I} \), we can relax the conditions 2.1 and 2.2. in the next way

1. \* if \( \frac{dg}{dt} \geq 0 \) then \( F_n \uparrow \); 2. \* if \( \frac{dg}{dt} \leq 0 \) then \( F_n \downarrow \).

According to the Lemma 2.1 the sign = valid in 2.1.\* and 2.2.\* only for a discrete set of points in \( \mathbb{I} \).

**Proof.** For all integers \( n \geq 2 \) and \( \alpha < u < v < \beta \) we will by mathematical induction prove that

\[
2.1. \text{(or 2.2.) } \Rightarrow F_n(\bar{a}, g(v)) \geq (\text{or } \leq) F_n(\bar{a}, g(u))
\]

(2.1)

where the sign = is valid iff \( a_1 = \cdots = a_n \). Since

\[
F_2(\bar{a}, g(v)) - F_2(\bar{a}, g(u)) = \sum_{i=1}^{2} a_i c^{g(v)} - \sum_{i=1}^{2} a_i c^{g(u)} = \left( a_1 c^{g(v)} - a_2 c^{g(u)} \right) \left( a_1 c^{g(v)} - a_2 c^{g(u)} \right)
\]
it comes out
\[ sgn(F_2(\tilde{a}, g(v)) - F_2(\tilde{a}, g(u))) = sgn(a_1 - a_2)^2 \geq 0, \text{ if applicable 2.1.} \]
\[ sgn(F_2(\tilde{a}, g(v)) - F_2(\tilde{a}, g(u))) = -sgn(a_1 - a_2)^2 \leq 0, \text{ if applicable 2.2.} \]
where the sign = is valid iff \( a_1 = a_2 \).

Thus, (2.1) is true for \( n = 2 \).

Suppose that (2.1) is valid for some \( n(\geq 2) \) and the sign = in (2.1) holds iff \( a_1 = \cdots = a_n \). Then for \( k \in \{1, \ldots, n, n+1\} \) we obtain the inequality
\[ F_n(\tilde{a}_k, g(v)) \geq F_n(\tilde{a}_k, g(u)), \text{ for } \alpha < u < v < \beta \] (2.2)
where the sign = is valid iff \( a_1 = \cdots = a_{k-1} = a_{k+1} = \cdots = a_{n+1} \).

By applying the conditions 2.2, instead of the conditions 2.1, in (2.2) the opposite sign of the inequality holds. So,
\[ \sum_{i=1, i \neq k}^{n+1} a_i^{c+g(v)} \sum_{i=1, i \neq k}^{n+1} a_i^{c-g(v)} \geq \sum_{i=1, i \neq k}^{n+1} a_i^{c+g(u)} \sum_{i=1, i \neq k}^{n+1} a_i^{c-g(u)}, \]
for \( \alpha < u < v < \beta \), and summing these \( n + 1 \) inequalities we obtain
\[ (n - 1)S_{n+1}(\tilde{a}^{c+g(v)})S_{n+1}(\tilde{a}^{c-g(v)}) \geq \]
\[ (n - 1)S_{n+1}(\tilde{a}^{c+g(u)})S_{n+1}(\tilde{a}^{c-g(u)}), \text{i.e.} \]
\[ F_{n+1}(\tilde{a}, g(v)) \geq F_{n+1}(\tilde{a}, g(u)), \text{ for } \alpha < u < v < \beta \] (2.3)
where the sign = is valid iff \( a_1 = \cdots = a_{n+1} \).

By applying the conditions 2.2, instead of the conditions 2.1, in (2.3) the opposite sign of the inequality holds, which gives us the proof.

**Theorem 2.3.** For every \( c \in \mathbb{R}, \ 0 \leq t < s < \infty \), the following inequalities hold
\[ S_n(\tilde{a}^c) \leq S_n(\tilde{a}^{c+t})S_n(\tilde{a}^{c-s}) \leq S_n(\tilde{a}^{c+s})S_n(\tilde{a}^{c-t}), \] (2.4)
whence for \( c = n, t = k \in \mathbb{N} \) we get a series of the inequalities
\[ S_n(\tilde{a}^n) \leq S_n(\tilde{a}^{n+1})S_n(\tilde{a}^{n-1}) \leq \cdots \leq S_n(\tilde{a}^{n+k})S_n(\tilde{a}^{n-k}) \leq S_n(\tilde{a}^{n+k+1})S_n(\tilde{a}^{n-k-1}) \leq \cdots \] (2.5)
In (2.4), (2.5) the sign = is valid iff \( a_1 = \cdots = a_n \).
Proof. Let \( g(t) = t \), i.e. \( g(t) \frac{dt}{dt} = t \). By the condition 2.1.* (2.2.*) from the Theorem 2.2 we have the following continuous function

\[
F_n(\bar{a}, t) = \sum_{i=1}^{n} a_i^{c+i-t} \sum_{i=1}^{n} a_i^{c-i} = S_n(\bar{a}^{c+i}) S_n(\bar{a}^{c-i}) \uparrow (\downarrow) \text{ for } t > (<) 0.
\]

Therefore the function \( F_n(\bar{a}, t) = S_n(\bar{a}^{c+i}) S_n(\bar{a}^{c-i}) \) has a minimum for \( t = 0 \), i.e.

\[
\min_{t \in \mathbb{R}} F_n(\bar{a}, t) = S_n(\bar{a}^{c})^2 \leq S_n(\bar{a}^{c+i}) S_n(\bar{a}^{c-i}) \uparrow (\downarrow) \text{ for } t > (<) 0. \tag{2.6}
\]

From (2.6) we obtain the inequalities (2.4), (2.5).

Remark 2.1. From (2.4), for \( c = 0 \), \( t = 1 \), we obtain

\[
S_n(\bar{a})^2 = n^2 \leq S_n(\bar{a}^1) S_n(\bar{a}^{-1}),
\]

which is the inequality between the arithmetic mean \( A_n(\bar{a}) \) and the harmonic mean \( H_n(\bar{a}) \) of the positive n-tuple \( \bar{a} \)

\[
A_n(\bar{a}) = \frac{S_n(\bar{a}^1)}{n} \geq \frac{n}{S_n(\bar{a}^{-1})} = H_n(\bar{a})
\]

Theorem 2.4. Let \( g: \mathbb{I} \to \mathbb{R}, \mathbb{I} = (\alpha, \beta) \). Then for a finite series of functions \( G_n: \mathbb{I} \to \mathbb{R}, \)

\[
G_n(\bar{a}, g) := \sum_{i=1}^{n} a_i^{g(t)} \sum_{i=1}^{n} a_i^{2^c-g(t)} = S_n(\bar{a}^g) S_n(\bar{a}^{2^c-g}),
\]

where \( c \) is any real number, holds the following conditions on \( \mathbb{I} \)

1. if \( (g > c, g \uparrow) \) or \( (g < c, g \downarrow) \), then \( G_n \uparrow \),
2. if \( (g > c, g \downarrow) \) or \( (g < c, g \uparrow) \), then \( G_n \downarrow \).

If the function \( g \) is differentiable on \( \mathbb{I} \), we can relax the conditions 2.3. and 2.4. on the next way:

2.3. if \( (g - c) \frac{dg}{dt} \geq 0 \), then \( G_n \uparrow \); 2.4. if \( (g - c) \frac{dg}{dt} \leq 0 \), then \( G_n \downarrow \).

According to the Lemma 2.1 the sign = valid in 2.3.* and 2.4.* only for a discrete set of points in \( \mathbb{I} \).

Proof. Since

\[
G_n(\bar{a}, g) = S_n(\bar{a}^{c+(g-c)}) S_n(\bar{a}^{c-(g-c)}) = F_n(\bar{a}, g - c),
\]

from Theorem 2.2 we obtain Theorem 2.4.

Theorem 2.5. Let \( n \in \mathbb{N} \) and \( c > 0 \).

1. For \( -\infty < t < s \leq 0 \), the following inequalities hold

\[
S_n(\bar{a}^t) S_n(\bar{a}^{2^c-t}) \geq S_n(\bar{a}^s) S_n(\bar{a}^{2^c-s}) \geq n S_n(\bar{a}^c). \tag{2.7}
\]
ii) For $0 \leq t < s \leq c$ the following inequalities hold
$$nS_n(\bar{a}^{2c}) \geq S_n(\bar{a}^{t})S_n(\bar{a}^{2c-t}) \geq S_n(\bar{a}^{s})S_n(\bar{a}^{2c-s}) \geq S_n(\bar{a}^c)^2.$$  (2.8)

In (2.7) and (2.8) the sign $=$ is valid iff $a_1 = \cdots = a_n$.

Proof. If we take $g(t) = t$ in the Theorem 2.4, we obtain
$$(g(t) - c)\frac{dg}{dt} = t - c \leq (\geq) 0, \text{ if } t \leq (\geq) c.$$ Therefore, a continuous function $G_n : \mathbb{I} \rightarrow \mathbb{R}$ has a minimum for $t = c$ and we have
$$\min_{t \in \mathbb{I}} G_n(\bar{a}, t) = G_n(\bar{a}, c)$$
$$= S_n(\bar{a}^c)^2 \leq S_n(\bar{a}^-)S_n(\bar{a}^{2c-t}) \downarrow (\uparrow) \text{ if } t < (>) c.$$  (2.9)

For $c > 0$, the inequalities (2.7) end (2.8) follow from (2.9).

Remark 2.2. If we take $2c = n$, then
a) for $-\infty < t < s \leq 0$, from the inequality (2.7), the following inequalities hold
$$S_n(\bar{a}^t)S_n(\bar{a}^{n-t}) \geq S_n(\bar{a}^s)S_n(\bar{a}^{n-s}) \geq nS_n(\bar{a}^n),$$  (2.10)
b) for $0 \leq t < s \leq \frac{n}{2}$, from the inequality (2.8), the following inequalities hold
$$nS_n(\bar{a}^n) \geq S_n(\bar{a}^t)S_n(\bar{a}^{n-t}) \geq S_n(\bar{a}^s)S_n(\bar{a}^{n-s}) \geq S_n(\bar{a}^s)^2.$$  (2.11)

In (2.10) and (2.11) the sign $=$ is valid iff $a_1 = \cdots a_n$.

Remark 2.3. Inequalities (2.11) have been otherwise proved in [3].

3. Main results

Theorem 3.1. Suppose that all assumptions of the Theorem 2.2 hold and the condition 2.1* (or 2.2*) of the Theorem 2.2 is satisfied. Then for $g > (\leq) 0$ on $\mathbb{I}$ we have the following inequality
$$S_n(\bar{a}^{c+g} \log \bar{a})S_n(\bar{a}^{-g}) \geq (\leq) S_n(\bar{a}^{c+g})S_n(\bar{a}^{-g} \log \bar{a}),$$  (3.1)

Proof. From the Theorem 2.2 follows if function $g$ satisfies condition 2.1* (or 2.2*), the function $F_n$ is strictly increasing (decreasing), differentiable on $\mathbb{I}$ and it holds
$$\frac{dF_n}{dt} = \left[ S_n(\bar{a}^{c+g} \log \bar{a})S_n(\bar{a}^{-g}) - S_n(\bar{a}^{c+g})S_n(\bar{a}^{-g} \log \bar{a}) \right] \frac{dg}{dt}$$
$$\geq (\leq) 0.$$  (3.2)

The sign $=$ is valid in (3.2) if $a_1 = \cdots = a_n$.

For $g > (\leq) 0$ on $\mathbb{I}$, (3.1) follows from (3.2).
Example 3.1. If we take $g(t) = \log t$, $t \in (1, \infty)$ in the Theorem 3.1 we obtain the following inequality
\[ S_n(\bar{a}^{\log t} \log \bar{a})S_n(\bar{a}^{\log t} \log \bar{a}) \geq S_n(\bar{a}^{\log t} \log \bar{a}), \tag{3.3} \]
and for $t \in (0, 1)$ the opposite inequality holds in (3.3). The sign $=$ is valid in (3.3) if $a_1 = \cdots = a_n$.

Example 3.2. If we take $g(t) = \sin t (> 0)$, $t \in \cup_{k \in \mathbb{Z}} [2k\pi, (2k+1)\pi]$ in the Theorem 3.1 the following inequality holds
\[ S_n(\bar{a}^{\sin t} \log \bar{a})S_n(\bar{a}^{\sin t} \log \bar{a}) \geq S_n(\bar{a}^{\log t} \log \bar{a}), \tag{3.4} \]
and for $t \in \cup_{k \in \mathbb{Z}} [(2k+1)\pi, (2k+1)\pi]$, the opposite inequality holds in (3.4). The sign $=$ is valid in (3.4) if $a_1 = \cdots = a_n$.

Theorem 3.2. Suppose that all assumptions of the Theorem 2.4 hold and the condition 2.3.* (or 2.4.*) of the Theorem 2.4. is satisfied. Then for $g > (\leq) c$ on $\mathbb{I}$ we have the following inequality:
\[ S_n(\bar{a}^g \log \bar{a})S_n(\bar{a}^{2c-g} \log \bar{a}) \geq (\leq) S_n(\bar{a}^g \log \bar{a})S_n(\bar{a}^{2c-g} \log \bar{a}). \tag{3.5} \]
The sign $=$ is valid in (3.5) if $a_1 = \cdots = a_n$.

Proof. From the Theorem 2.4 follows:
If the function $g$ satisfies condition 2.3.* (or 2.4.*), then $G_n$ is strictly increasing (decreasing) and differentiable on $\mathbb{I}$, and we have
\[ \frac{dG_n}{dt} = [S_n(\bar{a}^g \log \bar{a})S_n(\bar{a}^{2c-g}) - S_n(\bar{a}^g \log \bar{a})S_n(\bar{a}^{2c-g} \log \bar{a})] \frac{dg}{dt} \geq (\leq) 0, \tag{3.6} \]
where the sign $=$ is valid if $a_1 = \cdots = a_n$.
For $g > (\leq) c$ on $\mathbb{I}$, (3.5) follows from (3.6).

Example 3.3. If we take $g(t) = e^{1/t} (< 1)$, $t \in (-\infty, 0)$ in the Theorem 3.2 the following inequality holds
\[ S_n(\bar{a}^{1/t} \log \bar{a})S_n(\bar{a}^{2c-e^t} \log \bar{a}) \leq S_n(\bar{a}^{1/t} \log \bar{a}), \tag{3.7} \]
For $t \in (0, \infty)$ is $g(t) = e^{1/t} (> 1)$, and in (3.7) the opposite inequality holds. The sign $=$ is valid in (3.7) if $a_1 = \cdots = a_n$.

Example 3.4. Let $g(t) = \log t$. Then for $t \in (e^t, \infty)$ we have $g(t) > c$, and by the Theorem 3.2 we have
\[ S_n(\bar{a}^{\log t} \log \bar{a})S_n(\bar{a}^{2c-\log t} \log \bar{a}) \geq S_n(\bar{a}^{\log t} \log \bar{a}), S_n(\bar{a}^{2c-\log t} \log \bar{a}) \tag{3.8} \]
For $t \in (0, e^t)$ we have $g(t) < c$, and be the Theorem 3.2 the opposite inequality in (3.8) holds. The sign $=$ is valid in (3.8) if $a_1 = \cdots = a_n$. 

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