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EXISTENCE OF WEAK SOLUTIONS FOR NONLINEAR SYSTEMS INVOLVING DEGENERATED P-LAPLACIAN OPERATORS

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Abstract. We study the existence of weak solutions for the nonlinear system

$$\left.\begin{aligned} -\Delta_{P,p} u &= a(x)|u|^{p-2}u - b(x)|u|^{\alpha}|v|^{\beta}v + f, \\ -\Delta_{Q,q} v &= -c(x)|u|^{\alpha}|v|^{\beta}u + d(x)|v|^{q-2}v + g, \end{aligned}\right\}$$

where, the degenerated p-Laplacian is defined as $\Delta_{P,p}u = div[P(x) |\nabla u|^{p-2}\nabla u]$. We prove the existence of weak solutions for this system defined on bounded domains using the theory of monotone operators. We also consider the case of an unbounded domain.

1. Introduction

The concept of weak (generalized) solution for boundary value problems for the equation $A(u) = f$, have their background in applications (namely, in the variational approach connected with the critical level of a certain energy functional as well as in numerical methods). This type of approach is closely related to the concept of Sobolev spaces and is well elaborated for both linear and nonlinear equations.

In various applications, we can meet boundary value problems for elliptic equations whose ellipticity is "disturbed" in the sense that some degeneration or singularity appears. This "bad" behavior can be caused by the coefficients of the corresponding differential operators as well as by the solution itself. The so-called p-Laplacian is a prototype of such an operator and its character can be interpreted as a degeneration or as a singularity of the classical (linear) Laplace operator (with $p = 2$). There are several very concrete problems from practice which lead to such differential equations, e.g. from glaciology, non-Newtonian fluid mechanics, flows through porous

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media, differential geometry, celestial mechanics, climatology, petroleum extraction, reaction-diffusion problems, etc. [2, 3, 6, 13].

In our work, we consider a nonlinear system involving degenerated p-Laplacian operators with model A of the form

$$A\{u,v\} = \{-\Delta_{p,P} u - a(x)|u|^{p-2}u + b(x)|u|^{\alpha}|v|^{\beta}v \\ - \Delta_{Q,q} v + c(x)|u|^{\alpha}|v|^{\beta}u - d(x)|v|^{q-2}v\}.$$

Here, we use the theory of monotone operators to prove the existence of weak solutions for the following nonlinear system involving different degenerated p-Laplacian operators with variable coefficients defined on a bounded domain Ω of $\Re^N$ with boundary $\partial\,\Omega$,

$$\left.\begin{aligned} -\Delta_{P,p} u = -div[P(x)|\nabla u|^{p-2}\nabla u] &= a(x)|u|^{p-2}u \\ &\quad - b(x)|u|^{\alpha}|v|^{\beta}v + f \qquad \text{in}\,\Omega, \\ -\Delta_{Q,q} v = -div[Q(x)|\nabla v|^{q-2}\nabla v] &= -c(x)|u|^{\alpha}|v|^{\beta}u \\ &\quad + d(x)|v|^{q-2}v + g \qquad \text{in}\ \Omega, \\ u = v &= 0 \qquad \text{on}\,\partial\Omega, \end{aligned}\right\} \quad \text{(S)}$$

Then, we generalize the discussion to system defined on the whole space $\Re^N$.

The existence of solutions for such system (when $P(x) = Q(x) = 1$) have been proved, using the method of sub and super solutions in [4, 5, 14] and using the method of the theory of monotone operators in [15].

In [12], Khafagy and Serag have been proved the existence of positive solutions for system likes (S) using the method of sub and super solutions.

In [13], a Degenerate Quasilinear Elliptic System (in a bounded domain) is studied by means of Bifurcation Theory. The system is of a form similar to (S) with more general driven terms f, g and different hypotheses on the coefficients and the exponents.

Our paper is organized as follow: In Section 2 we introduce some technical results and definitions concerning the theory of nonlinear monotone operators, also, the scalar case is discussed. We study the existence of weak solutions for nonlinear systems defined on a bounded domain in Section 3 and on unbounded domain in Section 4.

2. Technical results and scalar case

First, we introduce some technical results concerning the theory of nonlinear monotone operators [1, 9, 11, 16]

Definition 1. *Let $A : V \to V'$ be an operator on a Banach space V. We say that the operator A is:*

Coercive iff $\lim_{\|u\|\to\infty} \frac{\langle A(u),u\rangle}{\|u\|} = \infty$;
Monotone iff $\langle A(u_1) - A(u_2), u_1 - u_2\rangle \geq 0$ *for all* $u_1, u_2 \in V$;
Strictly monotone iff $\langle A(u_1) - A(u_2), u_1 - u_2\rangle > 0$ *for all* $u_1, u_2 \in V$, $u_1 \neq u_2$;
Strongly continuous if $u_n \xrightarrow{w} u$ *implies* $A(u_n) \longrightarrow A(u)$, *for all* u_n, $u \in V$;
Weakly continuous if $u_n \xrightarrow{w} u$ *implies* $A(u_n) \xrightarrow{w} A(u)$, *for all* $u_n, u \in V$;
Demicontinuous if $u_n \longrightarrow u$ *implies* $A(u_n) \xrightarrow{w} A(u)$, *for all* $u_n, u \in V$;

The operator A *is said to be satisfy the* M_0*-condition if* $u_n \xrightarrow{w} u$, $A(u_n) \xrightarrow{w} f$, *and* $[\langle A(u_n), u_n\rangle \longrightarrow \langle f, u\rangle]$ *imply* $A(u) = f$.

Theorem 1. *Let* V *be a separable reflexive Banach space and* $A : V \to V'$ *an operator which is: coercive, bounded, demicontinuous and satisfying* M_0-*condition. Then the equation* $A(u) = f$ *admits a solution for each* $f \in V'$.

Second, we also introduce some technical results concerning the degenerated homogeneous eigenvalue problem

$$\left.\begin{array}{rl} -\Delta_{H,p} u = -div[H(x)|\nabla u|^{p-2}\nabla u] = \lambda G(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{2.1}$$

where $H(x)$ and $G(x)$ are measurable functions satisfying

$$\frac{\nu(x)}{c_1} \leq H(x) \leq c_1\nu(x), \tag{2.2}$$

for a.e. $x \in \Omega$ with some constant $c_1 \geq 1$, where $\nu(x)$ is a weight function in Ω satisfying the conditions

$$\nu \in L^1_{Loc}(\Omega), \quad \nu^{-\frac{1}{p-1}} \in L^1_{Loc}(\Omega), \quad \nu^{-s} \in L^1(\Omega), \tag{2.3}$$

with

$$s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty), \tag{2.4}$$

and

$$G(x) \in L^{\frac{k}{k-p}}(\Omega), \tag{2.5}$$

with some k satisfies $p < k < p_s^*$ where $p_s^* = \frac{NP_s}{N-P_s}$ with $p_s = \frac{sp}{s+1} < p < p_s^*$.

Lemma 2. *There exists the least (i.e. the first or principal) eigenvalue* $\lambda = \lambda_G(p, \Omega) > 0$ *and at least one corresponding eigenfunction* $u = u_G \geq 0$ *a.e. in*Ω *of the eigenvalue problem* (2.1).

Theorem 3. *Let* $H(x)$ *satisfies* (2.2) *and* $G(x)$ *satisfies* (2.5), *then* (2.1) *admits a positive principal eigenvalue* $\lambda_G(p)$. *Moreover, it is characterized by*

$$\lambda_G(p) \int_\Omega G(x)|u|^p \leq \int_\Omega H(x)|\nabla u|^p. \tag{2.6}$$

Now, let us introduce the weighted Sobolve space $W^{1,p}(\nu,\Omega)$ which is the set of all real valued functions u defined in Ω for which (see [9])

$$\|u\|_{W^{1,p}(\nu,\Omega)} = \left[\int_\Omega |u|^p + \int_\Omega \nu(x)|\nabla u|^p\right]^{\frac{1}{p}} < \infty. \tag{2.7}$$

Since we are dealing with the Dirichlet problem, we define also the space $W_0^{1,p}(\nu,\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\nu,\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,p}(\nu,\Omega)} = \left[\int_\Omega \nu(x)|\nabla u|^p\right]^{\frac{1}{p}} < \infty, \tag{2.8}$$

which is equivalent to the norm given by (2.7). Both spaces $W^{1,p}(\nu,\Omega)$ and $W_0^{1,p}(\nu,\Omega)$ are well defined reflexive Banach Spaces. The space $W_0^{1,p}(\nu,\Omega)$ is compactly imbedded in the space $L^p(\Omega)$, under the conditions given by (2.3) and (2.4), i.e.

$$W_0^{1,p}(\nu,\Omega) \hookrightarrow\hookrightarrow L^p(\Omega), \tag{2.9}$$

which means that

$$\int_\Omega |u|^p \le c_2 \int_\Omega \nu(x)|\nabla u|^p, \text{ i.e., } \|u\|_{L^p(\Omega)} \le c\, \|u\|_{W_0^{1,p}(\nu,\Omega)}. \tag{2.10}$$

for every $u \in W_0^{1,p}(\nu,\Omega)$ with a constant $c_2 > 0$ independent of u.

In this paper, we shall take $c_1 = 1$ in (2.2) i. e. $\nu(x) = H(x)$.

Now, we prove the existence of a weak solution $u \in W_0^{1,p}(P,\Omega)$ for the following scalar case:

$$\left.\begin{array}{rl} -\Delta_{P,p} u = -div[P(x)|\nabla u|^{p-2}\nabla u] = a(x)|u|^{p-2}u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{2.11}$$

where $P(x)$ and $a(x)$ are satisfying the conditions (2.2) and (2.5) respectively. The scalar case (2.11) can be written in the form:

$$Au = -\Delta_{P,P} u - a(x)|u|^{p-2}u = f. \tag{2.12}$$

Now we are in a position which enables us to prove that, according to Theorem 1, the scalar case given by (2.11) admits a weak solution $u \in W_0^{1,p}(P,\Omega)$ if $\lambda_a(p) > 1$.

First, we prove that A is a coercive operator.

We have from (2.6),

$$\begin{aligned}(Au,u) &= \int_\Omega P(x)|\nabla u|^p - \int_\Omega a(x)|u|^p \\ &\ge \int_\Omega P(x)|\nabla u|^p - \frac{1}{\lambda_a(p)}\int_\Omega P(x)|\nabla u|^p = (1 - \frac{1}{\lambda_a(p)})\, \|u\|^p_{W_0^{1,p}(P,\Omega)},\end{aligned}$$

and hence,

$$\frac{(Au,u)}{\|u\|_{W_0^{1,p}(P,\Omega)}} = \|u\|^{p-1}_{W_0^{1,p}(P,\Omega)} \longrightarrow \infty \quad \text{as} \quad \|u\|_{W_0^{1,p}(P,\Omega)} \longrightarrow \infty.$$

To prove that A is a bounded operator, we have

$$(Au,v) = \int_\Omega P(x)|\nabla u|^{p-2}\nabla u \nabla v - \int_\Omega a(x)|u|^{p-2}u\, v,$$

using Hölder's inequality, we obtain

$$\begin{aligned} |A(u,v)| \leq & \left[\int_\Omega P(x)|\nabla u|^p\right]^{\frac{p-1}{p}} \left[\int_\Omega P(x)|\nabla v|^p\right]^{\frac{1}{p}} \\ & - \left[\int_\Omega a(x)|u|^p\right]^{\frac{p-1}{p}} \left[\int_\Omega a(x)|v|^p\right]^{\frac{1}{p}} \\ & \leq \|u\|^{p-1}_{W_0^{1,p}(P,\Omega)} \|v\|_{W_0^{1,p}(P,\Omega)}. \end{aligned}$$

To prove that A is continuous, let us assume that $u_n \longrightarrow u$ in $W_0^{1,p}(P,\Omega)$. Then $\|u_n - u\|_{W_0^{1,p}(P,\Omega)} \longrightarrow 0$, so that

$$\|\nabla u_n - \nabla u\|_{L^p(\Omega)} \longrightarrow 0.$$

Applying the Dominated Convergence Theorem, we obtain

$$\left\|P(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\right\|_{L^p(\Omega)} \longrightarrow 0,$$

and hence

$$\begin{aligned} \|Au_n - Au\|_{L^p(\Omega)} \leq & \left\|P(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\right\|_{L^p(\Omega)} \\ & + \left\||u_n|^{p-2}u_n - |u|^{p-2}u\right\|_{L^p(\Omega)} \longrightarrow 0. \end{aligned}$$

Finally, A is strictly monotone:

$$\begin{aligned} (Au_1 - Au_2, u_1 - u_2) = & \int_\Omega P(x)|\nabla u_1|^{p-2}\nabla u_1 \nabla u_1 \\ & + \int_\Omega P(x)|\nabla u_2|^{p-2}\nabla u_2 \nabla u_2 - \int_\Omega P(x)|\nabla u_1|^{p-2}\nabla u_1 \nabla u_2 \\ & - \int_\Omega P(x)|\nabla u_2|^{p-2}\nabla u_2 \nabla u_1, \end{aligned}$$

using Hölder's inequality, we obtain

$$(Au_1 - Au_2, u_1 - u_2) \geq \int_\Omega P(x)|\nabla u_1|^p + \int_\Omega P(x)|\nabla u_2|^p - \left[\int_\Omega P(x)|\nabla u_1|^p\right]^{\frac{p-1}{p}}$$

$$\left[\int_\Omega P(x)|\nabla u_2|^p\right]^{\frac{1}{p}} - \left[\int_\Omega P(x)|\nabla u_2|^p\right]^{\frac{p-1}{p}}\left[\int_\Omega P(x)|\nabla u_1|^p\right]^{\frac{1}{p}}$$
$$= \|u_1\|^p_{W_0^{1,p}(P,\Omega)} + \|u_2\|^p_{W_0^{1,p}(P,\Omega)} - \|u_1\|^{p-1}_{W_0^{1,p}(P,\Omega)}\|u_2\|^p_{W_0^{1,p}(P,\Omega)} - \|u_2\|^{p-1}_{W_0^{1,p}(P,\Omega)}\|u_1\|^p_{W_0^{1,p}(P,\Omega)},$$

and hence,

$$(Au_1 - Au_2, u_1 - u_2) \geq (\|u_1\|^{p-1}_{W_0^{1,p}(P,\Omega)} - \|u_2\|^{p-1}_{W_0^{1,p}(P,\Omega)})(\|u_1\|^p_{W_0^{1,p}(P,\Omega)} - \|u_2\|^p_{W_0^{1,p}(P,\Omega)}) > 0.$$

This proves the strictly monotone condition and so, the existence of a weak solution for (2.11)

3. Nonlinear system defined on bounded domain

Let us consider the nonlinear system

$$\left.\begin{aligned} -\Delta_{P,p}u &= a(x)|u|^{p-2}u - b(x)|u|^\alpha|v|^\beta v + f && \text{in } \Omega,\\ -\Delta_{Q,q}v &= -c(x)|u|^\alpha|v|^\beta u + d(x)|v|^{q-2}v + g && \text{in } \Omega,\\ u &= v = 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.1}$$

where Ω is a bounded domain of $\Re^N$.

Let us assume that

$$\left.\begin{aligned} &\alpha,\beta \geq 0;\ p,q > 1,\ \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1,\ \alpha+\beta+2 < N\\ &f \in L^{p^*}(\Omega),\ g \in L^{q^*}(\Omega),\ \frac{1}{p} + \frac{1}{p^*} = 1 \text{ and } \frac{1}{q} + \frac{1}{q^*} = 1, \end{aligned}\right\} \tag{3.2}$$

and

$$\left.\begin{aligned} &P(x) \in L^1_{Loc}(\Omega),\ (P(x))^{-\frac{1}{p-1}} \in L^1_{Loc}(\Omega),\ (P(x))^{-s} \in L^1(\Omega),\\ &\text{with } s \in (\frac{N}{p},\infty) \cap [\frac{1}{p-1},\infty),\\ &Q(x) \in L^1_{Loc}(\Omega),\ (Q(x))^{-\frac{1}{q-1}} \in L^1_{Loc}(\Omega),\ (Q(x))^{-t} \in L^1(\Omega),\\ &\text{with } t \in (\frac{N}{q},\infty) \cap [\frac{1}{q-1},\infty). \end{aligned}\right\} \tag{3.3}$$

We also assume that the variable coefficients $a(x), b(x), c(x)$, and $d(x)$ are bounded smooth positive functions such that

$$\left.\begin{aligned} a(x) &\in L^{\frac{k}{k-p}}(\Omega) \cap L^p(\Omega),\ \text{with } p < k < p^*_s,\\ d(x) &\in L^{\frac{l}{l-q}}(\Omega) \cap L^q(\Omega),\ \text{with } q < l < q^*_t \end{aligned}\right\} \tag{3.4}$$

Theorem 4. *For* $(f,g) \in L^{p^*}(\Omega) \times L^{q^*}(\Omega)$, *there exists a weak solution* $(u,v) \in W_0^{1,p}(P,\Omega) \times W_0^{1,q}(Q,\Omega)$ *for the system* (3.1) *if the following condition is satisfied:*

$$\lambda_a(p) > 1, \;\; \lambda_d(q) > 1. \tag{3.5}$$

Proof. We transform the weak formulation of the system (3.1) to the following operator form

$$A(u,v) - B(u,v) = F,$$

where, A, B and F are operators defined on $W_0^{1,p}(P,\Omega) \times W_0^{1,q}(Q,\Omega)$ by

$$(A(u,v),(\Phi_1,\Phi_2)) = \int_\Omega P(x)|\nabla u|^{p-2}\nabla u \nabla \Phi_1 + \int_\Omega Q(x)|\nabla v|^{q-2}\nabla v \nabla \Phi_2, \tag{3.6}$$

$$(B(u,v),(\Phi_1,\Phi_2)) = \int_\Omega a(x)|u|^{p-2}u\Phi_1 + \int_\Omega d(x)|v|^{q-2}v\Phi_2$$
$$- \int_\Omega b(x)|u|^{\alpha}|v|^{\beta}v\Phi_1 - \int_\Omega c(x)|v|^{\beta}|u|^{\alpha}u\Phi_2, \tag{3.7}$$

$$(F,\Phi) = ((f,g),(\Phi_1,\Phi_2) = \int_\Omega f\Phi_1 + \int_\Omega g\Phi_2. \tag{3.8}$$

The operator $A(u,v)$ can be written as the sum of the two operators $J_1(u), J_2(v)$, where

$$(J_1(u),(\Phi_1)) = \int_\Omega P(x)|\nabla u|^{p-2}\nabla u \nabla \Phi_1 \;\; \text{and}$$
$$(J_2(v),(\Phi_2)) = \int_\Omega Q(x)|\nabla v|^{q-2}\nabla v \nabla \Phi_2.$$

As in the scalar case, operators $J_1(u)$ and $J_2(v)$ are bounded, continuous and strictly monotone; so their sum, the operator A, will be the same.

For the operator $B(u,v)$,

$$B(u,v)\colon W_0^{1,p}(P,\Omega) \times W_0^{1,q}(Q,\Omega) \longrightarrow L^p(\Omega) \times L^q(\Omega),$$

we can prove that it is a strongly continuous operator. To prove that, let us assume that $u_n \xrightarrow{w} u$ in $W_0^{1,p}(P,\Omega)$ and $v_n \xrightarrow{w} v$ in $W_0^{1,q}(Q,\Omega)$. Then, using (2.9), $(u_n,v_n) \longrightarrow (u,v)$ in $L^p(\Omega) \times L^q(\Omega)$. Also, $(\bigtriangledown u_n, \bigtriangledown v_n) \longrightarrow (\bigtriangledown u, \bigtriangledown v)$ in $L^p(\Omega) \times L^q(\Omega)$. By the the Dominated Convergence Theorem, we get

$$\begin{aligned}
&a(x)\,|u_n|^{p-2}\,u_n \longrightarrow a(x)\,|u|^{p-2}\,u, && \text{in } L^p(\Omega),\\
&d(x)\,|v_n|^{q-2}\,v_n \longrightarrow d(x)\,|v|^{q-2}\,v, && \text{in } L^q(\Omega),\\
&-b(x)\,|u_n|^{\alpha}\,|v_n|^{\beta}\,v_n \longrightarrow -b(x)\,|u|^{\alpha}\,|v|^{\beta}\,v, && \text{in } L^p(\Omega),\\
&-c(x)\,|v_n|^{\beta}\,|u_n|^{\alpha}\,u_n \longrightarrow -c(x)\,|v|^{\beta}\,|u|^{\alpha}\,u, && \text{in } L^q(\Omega).
\end{aligned}$$

Since

$$(B(u_n,v_n)-B(u,v),(w_1,w_2))=\int_\Omega a(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)w_1$$
$$+\int_\Omega d(x)(|v_n|^{q-2}v_n-|v|^{q-2}v)w_2-\int_\Omega b(x)(|u_n|^{\alpha}|v_n|^{\beta}v_n-|u|^{\alpha}|v|^{\beta}v)w_1$$
$$-\int_\Omega c(x)(|v_n|^{\beta}|u_n|^{\alpha}u_n-|v|^{\beta}|u|^{\alpha}u)w_2,$$

it follows that

$$\|B(u_n,v_n)-B(u,v))\|\le\left\|a(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)\right\|_{L^p(\Omega)}$$
$$+\left\|d(x)(|v_n|^{q-2}v_n-|v|^{q-2}v)\right\|_{L^q(\Omega)}+\left\|b(x)(|u_n|^{\alpha}|v_n|^{\beta+1}-|u|^{\alpha}|v|^{\beta+1})\right\|_{L^p(\Omega)}$$
$$+\left\|c(x)(|v_n|^{\beta}|u_n|^{\alpha+1}-|v|^{\beta}|u|^{\alpha+1})\right\|_{L^q(\Omega)}\longrightarrow 0.$$

This proves that $-B(u,v)$ is a strongly continuous operators. Similarly, as above, we can prove that $A(u,v)-B(u,v)$ satisfies the M_0-condition. Now, to apply Theorem 1, it remains to prove that $A(u,v)-B(u,v)$ is a coercive operator

$$(A(u,v)-B(u,v),(u,v))=\int_\Omega P(x)|\nabla u|^p+\int_\Omega Q(x)|\nabla v|^q-\int_\Omega a(x)|u|^p$$
$$-\int_\Omega d(x)|v|^q+\int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}+\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}$$
$$\ge\int_\Omega P(x)|\nabla u|^p+\int_\Omega Q(x)|\nabla v|^q-\int_\Omega a(x)|u|^p-\int_\Omega d(x)|v|^q.$$

Using (2.6), we get

$$(A(u,v)-B(u,v),(u,v))\ge\int_\Omega P(x)|\nabla u|^p+\int_\Omega Q(x)|\nabla v|^q$$
$$-\frac{1}{\lambda_a(p)}\int_\Omega|\bigtriangledown u|^p-\frac{1}{\lambda_d(q)}\int_\Omega|\bigtriangledown v|^q$$
$$=(1-\frac{1}{\lambda_a(p)})\int_\Omega P(x)|\nabla u|^p+(1-\frac{1}{\lambda_d(q)})\int_\Omega Q(x)|\nabla v|^q.$$

From (3.5), we deduce

$$(A(u,v)-B(u,v),(u,v))\ge k(\|u\|^p_{W_0^{1,p}(P,\Omega)}+\|v\|^q_{W_0^{1,q}(Q,\Omega)})$$
$$=k\,\|(u,v)\|_{W_0^{1,p}(P,\Omega)\times W_0^{1,q}(Q,\Omega)}\,.$$

So that

$$(A(u,v) - B(u,v),(u,v)) \longrightarrow \infty \text{ as } \|(u,v)\|_{W_0^{1,p}(P,\Omega)\times W_0^{1,p}(Q,\Omega)} \longrightarrow \infty.$$

This proves the coercivity condition and so, the existence of a weak solution for system (3.1). □

4. Nonlinear system defined on $\Re^N$

We consider the nonlinear system,

$$\left.\begin{aligned} -\Delta_{P,p}u &= a(x)|u|^{p-2}u - b(x)|u|^{\alpha}|v|^{\beta}v + f, && x\in\Re^N,\\ -\Delta_{Q,q}v &= -c(x)|u|^{\alpha}|v|^{\beta}u + d(x)|v|^{q-2}v + g, && x\in\Re^N,\\ \lim_{|x|\to\infty}u(x) &= \lim_{|x|\to\infty}v(x) = 0,\ u,v>0 && \text{in } \Re^N. \end{aligned}\right\} \tag{4.1}$$

We assume that $1 < p,q < N$ and the coefficients $a(x), b(x), c(x)$ and $d(x)$ are bounded smooth positive functions.

To discuss our problem, we need the following lemma [7, 8, 9, 10].

Lemma 5. *The degenerated homogeneous eigenvalue problem*

$$-\Delta_{H,p}u = -div[H(x)|\nabla u|^{p-2}\nabla u] = \lambda G(x)|u|^{p-2}u \text{ in } \Re^N,$$

$$\int_{\Re^N} G(x)|u|^p > 0, \tag{4.2}$$

has a pair $(\lambda_G(p), u_1)$ of a principal eigenvalue $\lambda_G(p)$ and an eigenfunction u_1 with $\lambda_G(p) > 0$ and $u_1 > 0$. Moreover, the principal eigenvalue $\lambda_G(p)$ is characterized by

$$\lambda_G(p)\int_{\Re^N} G^+(x)|u|^p \le \int_{\Re^N} H(x)|\nabla u|^p, \tag{4.3}$$

with $G(x) = G^+(x) - G^-(x)$, $G^+(x) > 0, G^-(x) > 0$, where the positive part of $G(x) = G^+(x) = \max(G,0)$ and the negative part of $G(x) = G^-(x) = \max(-G,0)$ are satisfying

$$G^+(x),\ G^-(x) \in L^{\frac{N}{p}}(\Re^N)\cap L^\infty(\Re^N). \tag{4.4}$$

For this section, the basic function space will be the separable uniformly convex Banach space [9]

$$D_H^{1,p}(\Re^N) = \{u \in L^{\frac{Np}{N-p}}(\Re^N) : |\nabla u| \in L^p(\Re^N)\},$$

which is defined as the completion of $C_0^\infty(\Re^N)$ with respect to the norm

$$\|u\|_{D_H^{1,p}(\Re^N)} = \left[\int_\Omega H(x)|\nabla u|^p\right]^{\frac{1}{p}} < \infty.$$

Moreover $D_H^{1,p}(\Re^N)$ is embedded continuously in the space $L^{\frac{Np}{N-p}}(\Re^N)$, which means that

$$\|u\|_{L^{\frac{Np}{N-p}}(\Re^N)} \leq k \, \|u\|_{D_H^{1,p}(\Re^N)}. \tag{4.5}$$

We will write the function $a = a(x)$ in the form $a(x) = a^+(x) - a^-(x)$ and the function $d = d(x)$ in the form $d(x) = d^+(x) - d^-(x)$ such that

$$\begin{aligned} &a^+(x),\ a^-(x) \in L^{\frac{N}{p}}(\Re^N) \cap L^\infty(\Re^N) \text{ and} \\ &d^+(x),\ d^-(x) \in L^{\frac{N}{q}}(\Re^N) \cap L^\infty(\Re^N). \end{aligned} \tag{4.6}$$

Theorem 6. *For* $(f, g) \in L^{\frac{Np}{N(p-1)+p}}(\Re^N) \times L^{\frac{Nq}{N(q-1)+q}}(\Re^N)$, *there exists a weak solution* $(u, v) \in D_P^{1,p}(\Re^N) \times D_Q^{1,q}(\Re^N)$ *for system* (4.1) *if the following condition is satisfied:*

$$\lambda_a(p) > 1, \quad \lambda_d(q) > 1 \tag{4.7}$$

Proof. As in Section 3, we transform the weak formulation of the system (4.1) to the operator form

$$A(u, v) - B(u, v) = F,$$

where, A, B and F are operators defined on $D_P^{1,p}(\Re^N) \times D_Q^{1,q}(\Re^N)$ by

$$(A(u, v), (\Phi_1, \Phi_2)) = \int_{\Re^N} P(x)|\nabla u|^{p-2}\nabla u \nabla \Phi_1 + \int_{\Re^N} Q(x)|\nabla v|^{q-2}\nabla v \nabla \Phi_2, \tag{4.8}$$

$$\begin{aligned} (B(u, v), (\Phi_1, \Phi_2)) = &\int_{\Re^N} a(x)|u|^{p-2}u\Phi_1 + \int_{\Re^N} d(x)|v|^{q-2}v\Phi_2 \\ &- \int_{\Re^N} b(x)|u|^\alpha |v|^\beta v\Phi_1 - \int_{\Re^N} c(x)|v|^\beta |u|^\alpha u\Phi_2, \end{aligned} \tag{4.9}$$

$$(F, \Phi) = ((f, g), (\Phi_1, \Phi_2) = \int_{\Re^N} f\Phi_1 + \int_{\Re^N} g\Phi_2. \tag{4.10}$$

First, we prove that A, B and F are bounded operators on $D_P^{1,p}(\Re^N) \times D_Q^{1,q}(\Re^N)$.

For the operator A, by using (4.8) and applying Holder inequality, we have

$$\begin{aligned} |(A(u, v), (\Phi_1, \Phi_2))| &\leq \int_{\Re^N} P(x)|\nabla u|^{p-1}|\nabla \Phi_1| + \int_{\Re^N} Q(x)|\nabla v|^{q-1}|\nabla \Phi_2| \\ &\leq \left[\int_{\Re^N} P(x)|u|^p\right]^{\frac{p-1}{p}} \left[\int_{\Re^N} P(x)|\Phi_1|^p\right]^{\frac{1}{p}} \\ &+ \left[\int_{\Re^N} Q(x)|v|^q\right]^{\frac{q-1}{q}} \left[\int_{\Re^N} Q(x)|\Phi_2|^q\right]^{\frac{1}{q}} \end{aligned}$$

$$
\begin{aligned}
&= \|u\|^{p-1}_{D^{1,p}_P(\Re^N)} \|\Phi_1\|_{D^{1,p}_P(\Re^N)} + \|v\|^{q-1}_{D^{1,q}_Q(\Re^N)} \|\Phi_2\|_{D^{1,q}_Q(\Re^N)} \\
&\leq (\|u\|^{p-1}_{D^{1,p}_P(\Re^N)} + \|v\|^{q-1}_{D^{1,q}_Q(\Re^N)})(\|\Phi_1\|_{D^{1,p}_P(\Re^N)} + \|\Phi_2\|_{D^{1,q}_Q(\Re^N)}) \\
&= (\|u\|^{p-1}_{D^{1,p}_P(\Re^N)} + \|v\|^{q-1}_{D^{1,q}_Q(\Re^N)}) \|(\Phi_1, \Phi_2)\|_{D^{1,p}_P(\Re^N)\times D^{1,q}_Q(\Re^N)} .
\end{aligned}
$$

This proves the boundedness of the operator $A(u,v)$.

For the operator B, we have

$$
\begin{aligned}
&|(B(u,v),(\Phi_1,\Phi_2))| \leq \left[\left(\int_{\Re^N} (a^+(x))^{\frac{N}{p}}\right)^{\frac{p}{N}} - \left(\int_{\Re^N} (a^-(x))^{\frac{N}{p}}\right)^{\frac{p}{N}}\right] \\
&\times \left[\int_{\Re^N} |u(x)|^{\frac{Np}{N-p}}\right]^{\frac{(p-1)(N-p)}{Np}} \left[\int_{\Re^N} |\Phi_1|^{\frac{Np}{N-p}}\right]^{\frac{N-p}{Np}} + \left[\left(\int_{\Re^N} (d^+(x))^{\frac{N}{q}}\right)^{\frac{q}{N}}\right. \\
&\left. - \left(\int_{\Re^N} (d^-(x))^{\frac{N}{q}}\right)^{\frac{q}{N}}\right] \left[\int_{\Re^N} |v(x)|^{\frac{Nq}{N-q}}\right]^{\frac{(q-1)(N-q)}{Nq}} \left[\int_{\Re^N} |\Phi_2|^{\frac{Nq}{N-q}}\right]^{\frac{N-q}{Nq}} \\
&+ \left[\int_{\Re^N} (b(x))^{\frac{N}{\alpha+\beta+2}}\right]^{\frac{\alpha+\beta+2}{N}} \left[\int_{\Re^N} |u|^{\frac{Np}{N-p}}\right]^{\frac{\alpha(N-p)}{Np}} \left[\int_{\Re^N} |v|^{\frac{Nq}{N-q}}\right]^{\frac{(\beta+1)(N-q)}{Nq}} \\
&\times \left[\int_{\Re^N} |\Phi_1|^{\frac{Np}{N-p}}\right]^{\frac{N-p}{Np}} + \left[\int_{\Re^N} (c(x))^{\frac{N}{\alpha+\beta+2}}\right]^{\frac{\alpha+\beta+2}{N}} \left[\int_{\Re^N} |u|^{\frac{Np}{N-p}}\right]^{\frac{(\alpha+1)(N-p)}{Np}} \\
&\times \left[\int_{\Re^N} |v|^{\frac{Nq}{N-q}}\right]^{\frac{\beta(N-q)}{Nq}} \left[\int_{\Re^N} |\Phi_2|^{\frac{Nq}{N-q}}\right]^{\frac{N-q}{Nq}}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&|(B(u,v),(\Phi_1,\Phi_2))| \\
&\leq k_1 \|u\|^{p-1}_{D^{1,p}_P(\Re^N)} \|\Phi_1\|_{D^{1,p}_P(\Re^N)} + k_2 \|v\|^{q-1}_{D^{1,q}_Q(\Re^N)} \|\Phi_2\|_{D^{1,q}_Q(\Re^N)} \\
&+ k_3 \|u\|^{\alpha}_{D^{1,p}_P(\Re^N)} \|v\|^{\beta+1}_{D^{1,q}_Q(\Re^N)} \|\Phi_1\|_{D^{1,p}_P(\Re^N)} \\
&+ k_4 \|u\|^{\alpha+1}_{D^{1,p}_P(\Re^N)} \|v\|^{\beta}_{D^{1,q}_Q(\Re^N)} \|\Phi_2\|_{D^{1,q}_Q(\Re^N)} \\
&\leq [k_1 \|u\|^{p-1}_{D^{1,p}_P(\Re^N)} + k_2 \|v\|^{q-1}_{D^{1,q}_Q(\Re^N)} + k_3 \|u\|^{\alpha}_{D^{1,p}_P(\Re^N)} \|v\|^{\beta+1}_{D^{1,q}_Q(\Re^N)} \\
&+ k_4 \|u\|^{\alpha+1}_{D^{1,p}_P(\Re^N)} \|v\|^{\beta}_{D^{1,q}_Q(\Re^N)}] \times \|(\Phi_1,\Phi_2)\|_{D^{1,p}_P(\Re^N)\times D^{1,q}_Q(\Re^N)} ,
\end{aligned}
$$

For the operator F, we have

$$
\begin{aligned}
|(F,\Phi)| &= |((f,g),(\Phi_1,\Phi_2))| \\
&\leq \left[\int_{\Re^N} |f|^{\frac{Np}{N(p-1)+p}}\right]^{\frac{N(p-1)+p}{Np}} \left[\int_{\Re^N} |\Phi_1|^{\frac{Np}{N-p}}\right]^{\frac{N-p}{Np}}
\end{aligned}
$$

$$+\left[\int_{\Re^N}|g|^{\frac{Nq}{N(q-1)+q}}\right]^{\frac{N(q-1)+q}{Nq}}\left[\int_{\Re^N}|\Phi_2|^{\frac{Nq}{N-q}}\right]^{\frac{N-q}{Nq}}$$
$$\leq\left(\|f_1\|_{L^{\frac{Np}{N(p-1)+p}}(\Re^N)}+\|f_2\|_{L^{\frac{Nq}{N(q-1)+q}}(\Re^N)}\right)$$
$$\times\|(\Phi_1,\Phi_2)\|_{D_P^{1,p}(\Re^N)\times D_Q^{1,q}(\Re^N)}.$$

The operator $A(u,v)=J_1(u)+J_2(v)$ is continuous and strictly monotone on $D_P^{1,p}(\Re^N)\times D_Q^{1,q}(\Re^N)$, since

$$(J_1(u_1)-J_1(u_2),u_1-u_2)\geq\left(\|u_1\|^{p-1}_{D_P^{1,p}(\Re^N)}-\|u_2\|^{p-1}_{D_P^{1,p}(\Re^N)}\right)$$
$$\left(\|u_1\|_{D_P^{1,p}(\Re^N)}-\|u_2\|_{D_P^{1,p}(\Re^N)}\right)>0,$$
$$(J_2(v_1)-J_2(v_2),v_1-v_2)\geq\left(\|v_1\|^{q-1}_{D_Q^{1,q}(\Re^N)}-\|v_2\|^{q-1}_{D_Q^{1,q}(\Re^N)}\right)$$
$$\left(\|v_1\|_{D_Q^{1,q}(\Re^N)}-\|v_2\|_{D_Q^{1,q}(\Re^N)}\right)>0.$$

For the operator $B(u,v)$, we can prove that it is a strongly continuous operator by using the Dominated Convergence Theorem and continuous imbedding property for the space $D_P^{1,p}(\Re^N)\times D_Q^{1,q}(\Re^N)$ into $L^{\frac{Np}{N-p}}(\Re^N)\times L^{\frac{Nq}{N-q}}(\Re^N)$. To prove that, let us assume that $u_n\xrightarrow{w}u$ in $D_P^{1,p}(\Re^N)$ and $v_n\xrightarrow{w}v$ in $D_Q^{1,q}(\Re^N)$.Then $(u_n,v_n)\to(u,v)$ in $L^{\frac{Np}{N-p}}(\Re^N)\times L^{\frac{Nq}{N-q}}(\Re^N)$ and $(\bigtriangledown u_n,\bigtriangledown v_n)\to(\bigtriangledown u,\bigtriangledown v)$ in $L^p(\Re^N)\times L^q(\Re^N)$. Now, the sequence (u_n) is bounded in $D_P^{1,p}(\Re^N)$, and hence contains a subsequence again denoted by (u_n) converging strongly to u in $L^{\frac{Np}{N-p}}(B_{r_0})$ for any bounded ball $B_{r_0}=\{x\in\Re^N:\|x\|\leq r_0\}$. Similarly (v_n) converges strongly to v in $L^{\frac{Nq}{N-q}}(B_{r_0})$. Since $u_n,u\in L^{\frac{Np}{N-p}}(B_{r_0})$ and $v_n,v\in L^{\frac{Nq}{N-q}}(B_{r_0})$. Then using the Dominated Convergence Theorem, we have

$$\left\|a(x)(|u_n|^{p-2}u_n\longrightarrow|u|^{p-2}u)\right\|_{\frac{Np}{N(p-1)+p}}\to 0,$$
$$\left\|d(x)(|v_n|^{q-2}v_n\longrightarrow|v|^{q-2}v)\right\|_{\frac{Nq}{N(q-1)+q}}\to 0,$$
$$\left\|b(x)(|u_n|^{\alpha-1}|v_n|^{\beta+1}v_n-|u|^{\alpha-1}|v|^{\beta+1}v)\right\|_{\frac{Np}{N(p-1)+p}}\to 0,$$
$$\left\|c(x)(|v_n|^{\beta-1}|u_n|^{\alpha+1}u_n-|v|^{\beta-1}|u|^{\alpha+1}u)\right\|_{\frac{Nq}{N(q-1)+q}}\to 0.$$

Then

$$\|B(u_n,v_n)-B(u,v)\|_{D_P^{1,p}(B_{r_0})\times D_Q^{1,q}(B_{r_0})}$$

$$\leq \left\| a(x)(|u_n|^{p-2}\, u_n \longrightarrow |u|^{p-2}\, u) \right\|_{\frac{Np}{N(p-1)+p}}$$
$$+ \left\| d(x)(|v_n|^{q-2}\, v_n \longrightarrow |v|^{q-2}\, v) \right\|_{\frac{Nq}{N(q-1)+q}}$$
$$+ \left\| b(x)(|u_n|^{\alpha-1}\, |v_n|^{\beta+1}\, v_n - |u|^{\alpha-1}\, |v|^{\beta+1}\, v) \right\|_{\frac{Np}{N(p-1)+p}}$$
$$+ \left\| c(x)(|v_n|^{\beta-1}\, |u_n|^{\alpha+1}\, u_n - |v|^{\beta-1}\, |u|^{\alpha+1}\, u) \right\|_{\frac{Nq}{N(q-1)+q}}$$

In [15], it is proved that, the norm

$$\|B(u_n, v_n) - B(u, v)\|_{D_P^{1,p}(\Re^N)\times D_Q^{1,q}(\Re^N)}\,,$$

tends strongly to zero and then the operator $B(u, v)$ is strongly continuous.

Finally, it remains to prove that the operator $A(u, v)-B(u, v)$ is a coercive operator

$$\begin{aligned}(A(u,v) - B(u,v),(u,v)) &= \int_{\Re^N} P(x)|\nabla u|^p + \int_{\Re^N} Q(x)|\nabla v|^q - \int_{\Re^N} a(x)|u|^p\\
&- \int_{\Re^N} d(x)|v|^q + \int_v b(x)|u|^{\alpha+1}|v|^{\beta+1} + \int_{\Re^N} c(x)|u|^{\alpha+1}|v|^{\beta+1}\\
&= \int_{\Re^N} P(x)|\nabla u|^p + \int_{\Re^N} Q(x)|\nabla v|^q - \int_{\Re^N} a^+(x)|u|^p\\
&- \int_{\Re^N} d^+(x)|v|^q + \int_{\Re^N} a^-(x)|u|^p + \int_{\Re^N} d^-(x)|v|^q\\
&+ \int_v b(x)|u|^{\alpha+1}|v|^{\beta+1} + \int_{\Re^N} c(x)|u|^{\alpha+1}|v|^{\beta+1}\\
&\geq \int_{\Re^N} P(x)|\nabla u|^p + \int_{\Re^N} Q(x)|\nabla v|^q - \int_{\Re^N} a^+(x)|u|^p - \int_{\Re^N} d^+(x)|v|^q.\end{aligned}$$

Using (4.3), and (4.8), we have

$$\begin{aligned}(A(u,v) - B(u,v),(u,v)) &\geq \int_\Omega P(x)|\nabla u|^p + \int_\Omega Q(x)|\nabla v|^q\\
&\quad - \frac{1}{\lambda_a(p)}\int_\Omega P(x)|\bigtriangledown u|^p - \frac{1}{\lambda_d(q)}\int_\Omega Q(x)|\bigtriangledown v|^q\\
&= (1-\frac{1}{\lambda_a(p)})\int_\Omega P(x)|\nabla u|^p + (1-\frac{1}{\lambda_d(q)})\int_\Omega Q(x)|\nabla v|^q\\
&\geq k(\|u\|^p_{W_0^{1,p}(P,\Omega)} + \|v\|^q_{W_0^{1,q}(Q,\Omega)}) = k\,\|(u,v)\|_{W_0^{1,p}(P,\Omega)\times W_0^{1,q}(Q,\Omega)}\,.\end{aligned}$$

So that:

$$(A(u,v) - B(u,v),(u,v)) \longrightarrow \infty \text{ as } \|(u,v)\|_{W_0^{1,p}(P,\Omega)\times W_0^{1,q}(Q,\Omega)} \longrightarrow \infty.$$

This proves the coercivity condition and so, the existence of a weak solution for system (4.1). □

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