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ON A MIXED SUM-DIFFERENCE EQUATION OF VOLTERRA-FREDHOLM TYPE

B. G. PACHPATTE

ABSTRACT. The main objective of this paper is to study some basic properties of solutions of a mixed sum-difference equation of Volterra-Fredholm type. A variant of a certain finite difference inequality with explicit estimate is obtained and used to establish the results.

1. INTRODUCTION

Let $R_+ = [0, \infty), N_0 = \{0, 1, 2, \ldots\}$ be the given subsets of R, the set of real numbers. Let $N_i[\alpha_i, \beta_i] = \{\alpha_i, \alpha_i + 1, \ldots, \beta_i\}, \alpha_i, \beta_i \in N_0$, $(\alpha_i < \beta_i)$ $i = 1, 2, \ldots, m$ and $G = \prod_{i=1}^{m} N_i[\alpha_i, \beta_i]$. For any function $w : G \to R$, we denote the m-fold sum over G with respect to the variable $y = (y_1, \ldots, y_m) \in G$ by

$$\sum_G w(y) = \sum_{y_1=\alpha_1}^{\beta_1} \cdots \sum_{y_m=\alpha_m}^{\beta_m} w(y_1, \ldots, y_m).$$

Clearly, $\sum_G w(y) = \sum_G w(x)$ for $x, y \in G$. Let $E = G \times N_0$ and denote by $D(S_1, S_2)$ the class of discrete functions from the set S_1 to the set S_2. We use the usual convention that empty sums and products are taken to be 0 and 1 respectively and assume that all sums and products involved exist on the respective domains of their definitions and are finite.

Consider the sum-difference equation of the form:

$$u(x, n) = h(x, n) + \sum_{s=0}^{n-1} \sum_G F(x, n, y, s, u(y, s)), \tag{1.1}$$

for $(x, n) \in E$, where $h \in D(E, R)$, $F \in D(E^2 \times R, R)$. The integral analogue of equation (1.1) occur in a natural way while studying the parabolic equations which describe diffusion or heat transfer phenomena, see [1, p.

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18] and [4,Chapter VI]. For more recent results on Volterra-Fredholm type sum-difference equations, see [6]. The equation (1.1) appears to be Volterra type in n, and of Fredholm type with respect to x and hence we view it as a mixed Volterra-Fredholm type sum-difference equation. Here, we note that, one can formulate existence result for the solution of equation (1.1) by modifying the idea employed in [3], see also [2,8]. The main purpose of this paper is to study some basic properties of solutions of equation (1.1) under some suitable conditions on the functions involved therein. The analysis used in the proofs is based on the variant of a certain finite difference inequality given in [5].

2. A basic finite difference inequality

We need the following variant of the finite difference inequality given in [5, Theorem 1.2.3, p. 13].

Theorem 1. *Let* $u(x,n), a(x,n), b(x,n), p(x,n) \in D(E, R_+)$. *If*

$$u(x,n) \le a(x,n) + b(x,n) \sum_{s=0}^{n-1} \sum_{G} p(y,s)\, u(y,s), \tag{2.1}$$

for $(x,n) \in E$, *then*

$$u(x,n) \le a(x,n) + b(x,n) \sum_{s=0}^{n-1} \sum_{G} p(y,s)\, a(y,s) \times \prod_{\sigma=s+1}^{n-1} \left[1 + \sum_{G} p(y,\sigma)\, b(y,\sigma)\right], \tag{2.2}$$

for $(x,n) \in E$.

Proof. Introduce the notation

$$e(s) = \sum_{G} p(y,s)\, u(y,s). \tag{2.3}$$

Then the inequality (2.1) can be restated as

$$u(x,n) \le a(x,n) + b(x,n) \sum_{s=0}^{n-1} e(s), \tag{2.4}$$

for $(x,n) \in E$. Define

$$z(n) = \sum_{s=0}^{n-1} e(s), \tag{2.5}$$

for $n \in N_0$, then $z(0) = 0$ and from (2.4) we get

$$u(x,n) \le a(x,n) + b(x,n)\, z(n), \tag{2.6}$$

for $(x,n) \in E$. From (2.5), (2.3) and (2.6) we observe that

$$\begin{aligned} \Delta z(n) &= z(n+1) - z(n) = e(n) \\ &= \sum_G p(y,n)\, u(y,n) \\ &\le \sum_G p(y,n)\left\{a(y,n) + b(y,n)\, z(n)\right\} \\ &= z(n) \sum_G p(y,n)\, b(y,n) + \sum_G p(y,n)\, a(y,n). \end{aligned} \tag{2.7}$$

Now a suitable application of Theorem 1.2.1 given in [5, p.11] with $z(0) = 0$ to (2.7) yields

$$z(n) \le \sum_{s=0}^{n-1} \sum_G p(y,s) a(y,s) \prod_{\sigma=s+1}^{n-1} \left[1 + \sum_G p(y,\sigma)\, b(y,\sigma)\right]. \tag{2.8}$$

Using (2.8) in (2.6) we get the required inequality in (2.2). □

3. Properties of solutions

In this section we apply the inequality obtained in Theorem 1 to study some basic properties of solutions of equation (1.1) under some suitable conditions on the functions involved therein.

First, we shall give the following theorem concerning the estimate on the solution of equation (1.1).

Theorem 2. *Suppose that the function F in equation (1.1) satisfies the condition*

$$|F(x,n,y,s,u)| \le b(x,n)\, p(y,s)\, |u|, \tag{3.1}$$

where $b, p \in D(E, R_+)$. If $u(x,n)$ is any solution of equation (1.1) on E, then

$$\begin{aligned} |u(x,n)| \le |h(x,n)| + b(x,n) \sum_{s=0}^{n-1} \sum_G p(y,s)\, |h(y,s)| \\ \times \prod_{\sigma=s+1}^{n-1} \left[1 + \sum_G p(y,\sigma)\, b(y,\sigma)\right], \end{aligned} \tag{3.2}$$

for $(x,n) \in E$.

Proof. Using the fact that $u(x,n)$ is a solution of equation (1.1) and the condition (3.1) we have

$$\begin{aligned} |u(x,n)| &\le |h(x,n)| + \sum_{s=0}^{n-1}\sum_{G} |F(x,n,y,s,u(y,s))| \\ &\le |h(x,n)| + b(x,n)\sum_{s=0}^{n-1}\sum_{G} p(y,s)\,|u(y,s)|. \end{aligned} \tag{3.3}$$

Now an application of Theorem 1 to (3.3) yields (3.2). □

In the following theorem we obtain estimate on the solution of equation (1.1) by assuming that the function F satisfies the Lipschitz type condition.

Theorem 3. *Suppose that the function F in equation* (1.1) *satisfies the condition*

$$|F(x,n,y,s,u) - F(x,n,y,s,v)| \le b(x,n)\,p(y,s)\,|u-v|, \tag{3.4}$$

where $b, p \in D(E, R_+)$. If $u(x,n)$ is any solution of equation (1.1) *on E, then*

$$\begin{aligned} |u(x,n) - h(x,n)| \le a(x,n) &+ b(x,n)\sum_{s=0}^{n-1}\sum_{G} p(y,s)\,a(y,s) \\ &\times \prod_{\sigma=s+1}^{n-1}\left[1 + \sum_{G} p(y,\sigma)\,b(y,\sigma)\right], \end{aligned} \tag{3.5}$$

for $(x,n) \in E$, where

$$a(x,n) = \sum_{s=0}^{n-1}\sum_{G} |F(x,n,y,s,h(y,s))|, \tag{3.6}$$

for $(x,n) \in E$.

Proof. Using the fact that $u(x,n)$ is any solution of equation (1.1) and the condition (3.4) we have

$$\begin{aligned} |u(x,n) - h(x,n)| &\le \sum_{s=0}^{n-1}\sum_{G} |F(x,n,y,s,u(y,s)) - F(x,n,y,s,h(y,s))| \\ &\quad + \sum_{s=0}^{n-1}\sum_{G} |F(x,n,y,s,h(y,s))| \\ &\le a(x,n) + b(x,n)\sum_{s=0}^{n-1}\sum_{G} p(y,s)\,|u(y,s) - h(y,s)|, \end{aligned} \tag{3.7}$$

for $(x,n) \in E$. Now an application of Theorem 1 to (3.7), we get the required estimation in (3.5). □

Next theorem deals with the uniqueness of solutions of equation (1.1).

Theorem 4. *Suppose that the function F in equation* (1.1) *satisfies the condition* (3.4). *Then the equation* (1.1) *has at most one solution on E.*

Proof. Let $u_1(x,n)$ and $u_2(x,n)$ be two solutions of equation (1.1) on E. Using this fact and the condition (3.4) we have

$$|u_1(x,n)-u_2(x,n)| \le \sum_{s=0}^{n-1}\sum_{G} |F(x,n,y,s,u_1(y,s))-F(x,n,y,s,u_2(y,s))|$$
$$\le b(x,n)\sum_{s=0}^{n-1}\sum_{G} p(y,s)\,|u_1(y,s)-u_2(y,s)|. \tag{3.8}$$

Now a suitable application of Theorem 1 to (3.8) yields $|u_1(x,n)-u_2(x,n)| \le 0$, which implies $u_1(x,n) = u_2(x,n)$. Thus there is at most one solution to equation (1.1) on E. □

The following theorem deals with the continuous dependence of solution of equation (1.1) on the functions involved therein. Consider the equation (1.1) and the corresponding equation

$$v(x,n) = \bar{h}(x,n) + \sum_{s=0}^{n-1}\sum_{G} \bar{F}(x,n,y,s,v(y,s)), \tag{3.9}$$

for $(x,n) \in E$, where $\bar{h} \in D(E,R)$, $\bar{F} \in D\left(E^2 \times R, R\right)$.

Theorem 5. *Suppose that the function F in equation* (1.1) *satisfies the condition* (3.4). *Furthermore, suppose that*

$$\left|h(x,n)-\bar{h}(x,n)\right| + \sum_{s=0}^{n-1}\sum_{G} \left|F(x,n,y,s,v(y,s))-\bar{F}(x,n,y,s,v(y,s))\right| \le \varepsilon, \tag{3.10}$$

where h, F and $\bar{h}, \bar{F}$ are as in equations (1.1) *and* (3.9) *respectively, $\varepsilon > 0$ is an arbitrary small constant and $v(x,n)$ is a solution of equation* (3.9). *Then the solution $u(x,n)$, $(x,n) \in E$ of equation* (1.1) *depends continuously on the functions involved in equation* (1.1).

Proof. Let $w(x,n) = |u(x,n) - v(x,n)|, (x,n) \in E$. Using the facts that $u(x,n)$ and $v(x,n)$ are the solutions of equations (1.1) and (3.9) and the hypotheses we have

$$\begin{aligned} w(x,n) &\le \left|h(x,n)-\bar{h}(x,n)\right| \\ &\quad+\sum_{s=0}^{n-1}\sum_{G}\left|F(x,n,y,s,u(y,s))-F(x,n,y,s,v(y,s))\right| \\ &\quad+\sum_{s=0}^{n-1}\sum_{G}\left|F(x,n,y,s,v(y,s))-\bar{F}(x,n,y,s,v(y,s))\right| \\ &\quad\le \varepsilon+b(x,n)\sum_{s=0}^{n-1}\sum_{G}p(y,s)\,w(y,s)\,. \end{aligned} \tag{3.11}$$

Now a suitable application of Theorem 1 to (3.11) yields

$$\begin{aligned} &|u(x,n)-v(x,n)| \\ &\le \varepsilon\left[1+b(x,n)\sum_{s=0}^{n-1}\sum_{G}p(y,s)\prod_{\sigma=s+1}^{n-1}\left[1+\sum_{G}p(y,\sigma)\,b(y,\sigma)\right]\right], \end{aligned} \tag{3.12}$$

for $(x,n)\in E$. From (3.12) it follows that the solution of equation (1.1) depends continuously on the functions involved therein.

We next consider the following mixed sum-difference equations of Volterra-Fredholm type

$$z(x,n)=g(x,n)+\sum_{s=0}^{n-1}\sum_{G}H(x,n,y,s,z(y,s),\mu), \tag{3.13}$$

$$z(x,n)=g(x,n)+\sum_{s=0}^{n-1}\sum_{G}H(x,n,y,s,z(y,s),\mu_0), \tag{3.14}$$

for $(x,n)\in E$, where $g\in D(E,R)$, $H\in D\left(E^2\times R^2,R\right)$ and μ,μ_0 are parameters. □

Finally, we present the following theorem which shows the dependency of solutions of equations (3.13), (3.14) on parameters.

Theorem 6. *Suppose that the function H in equations* (3.13), (3.14) *satisfy the conditions*

$$|H(x,n,y,s,u,\mu)-H(x,n,y,s,v,\mu)|\le b(x,n)\,p(y,s)\,|u-v|\,, \tag{3.15}$$

$$|H(x,n,y,s,u,\mu)-H(x,n,y,s,u,\mu_0)|\le c(y,s)\,|\mu-\mu_0|\,, \tag{3.16}$$

where $b, p, c \in D(E, R_+)$. *Let* $z_1(x, n)$ *and* $z_2(x, n)$ *be the solutions of equations* (3.13) *and* (3.14) *respectively. Assume that*

$$\sum_{s=0}^{n-1} \sum_{G} c(y, s) \leq M, \tag{3.17}$$

where $M \geq 0$ *is a constant. Then*

$$\begin{aligned} &|z_1(x, n) - z_2(x, n)| \\ &\leq M |\mu - \mu_0| \left[1 + b(x, n) \sum_{s=0}^{n-1} \sum_{G} p(y, s) \prod_{\sigma=s+1}^{n-1} \left[1 + \sum_{G} p(y, \sigma) b(y, \sigma)\right]\right], \end{aligned} \tag{3.18}$$

for $(x, n) \in E$.

Proof. Let $z(x, n) = |z_1(x, n) - z_2(x, n)|, (x, n) \in E$. Using the facts that $z_1(x, n)$ and $z_2(x, n)$ are the solutions of equations (3.13) and (3.14) and hypotheses we have

$$\begin{aligned} z(x, n) &\leq \sum_{s=0}^{n-1} \sum_{G} |H(x, n, y, s, z_1(y, s), \mu) - H(x, n, y, s, z_2(y, s), \mu)| \\ &\quad + \sum_{s=0}^{n-1} \sum_{G} |H(x, n, y, s, z_2(y, s), \mu) - H(x, n, y, s, z_2(y, s), \mu_0)| \\ &\leq b(x, n) \sum_{s=0}^{n-1} \sum_{G} p(y, s) |z_1(y, s) - z_2(y, s)| + \sum_{s=0}^{n-1} \sum_{G} c(y, s) |\mu - \mu_0| \\ &\leq M |\mu - \mu_0| + b(x, n) \sum_{s=0}^{n-1} \sum_{G} p(y, s) z(y, s). \end{aligned} \tag{3.19}$$

Now an application of Theorem 1 to (3.19) yields (3.18), which shows the dependency of solutions of equations (3.13) and (3.14) on parameters. □

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57 Shri Niketan Colony
Near Abhinay Talkies
Aurangabad 431 001 (Marashtra) India
E-mail: bgpachpatte@gmail.com