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ON THE CAUCHY PROBLEM FOR THE FOKKER-PLANCK-BOLTZMANN EQUATION WITH INFINITE INITIAL ENERGY

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ABSTRACT. We prove a new existence result for the Fokker-Planck-Boltzmann equation with an initial data with infinite energy in the framework of renormalization. We extend the result of DiPerna-Lions by assuming $f_0(|x|^\alpha + |x-v|^2) \in L^1$ instead of $f_0(|x|^2 + |v|^2) \in L^1$.

1. INTRODUCTION

This paper is devoted to study the existence of solution to the Fokker-Planck-Boltzmann equation in the case of initial data with infinite energy. We prove the global existence of renormalized solution for a general class of initial data with finite mass, entropy.

The Fokker-Planck-Boltzmann (FPB) equation, as an approximation to the Boltzmann equation, has been studied by DiPerna & Lions [1]. More precisely, the authors investigated the Cauchy problem

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \Delta_v f + Q(f,f) & \text{in } [0,+\infty) \times R^N \times R^N, \\ f(t,x,v)|_{t=0} = f_0(x,v) & \text{on } R^N \times R^N, \end{cases} \tag{1.1}$$

where Q is a quadratic collision operator defined by

$$Q(f,f) = \iint_{R^N \times S^{N-1}} (f'f'_* - ff_*) B(v - v_*, w)\, dw\, dv_* \tag{1.2}$$

and

$$f_0(1 + |x|^2 + |v|^2 + |\log f_0) \in L^1(R^N \times R^N). \tag{1.3}$$

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As usual, we denote $f = f(t, x, v), f' = f(t, x, v'), f_* = f(t, x, v_*), f'_* = f(t, x, v'_*)$. The post-collisional velocities v' and v'_* are obtained from the pre-collisional velocities v and v_* by

$$\begin{cases} v' = v - (v - v_*, w)w, \\ v'_* = v_* + (v - v_*, w)w. \end{cases} \tag{1.4}$$

Here $w \in S^{N-1}$ indicates the unit angular vector and $(\cdot, \cdot)$ the usual scalar product. For the detailed representation the reader can refer to [2]. The collision kernel B is a given nonnegative function on $R^N \times S^{N-1}$ which is symmetric with respect to v and v_*, that is

$$B \geq 0, \quad B(v - v_*, w) \text{ depends only on } |v - v_*| \text{ and } (v - v_*, w).$$

The main problem in dealing with the Boltzmann equation originates in the lack of estimates of the collision term Q. The renormalized solution was introduced in [1, 3] in order to resolve the difficulty of defining Q. They have proved the stability result and the existence of global solution in time.

It must be mentioned that renormalization has been an effective tool in analysis of large data Cauchy problem for Boltzmann equation and other problems [4–9].

The Fokker-Planck-Boltzmann equation is concerned with the particles system where the diffusive effect can not be neglected. Actually, in many real situations the diffusion exists. So, one should approximate these kinetic problems of the FPB equations with small parameter, but not the Boltzmann equation. Even if the study for FPB equation is easy to handle, their mathematical analysis is not trivial. In paper [10] the first author and Yang study the FPB equation without angular cutoff and extend the result of [1]. The corresponding result about the asymptotics of FPB equation with small parameter to the Boltzmann equation can be found in paper [11]. Here, we will extend the result of [1, 10] to the case of initial data with infinite energy but with angular cutoff assumption.

In order to explain our result, we first recall the definition of renormalized solution of FPB equation (1.1) and the known result.

Definition 1. *We say nonnegative function $f \in C(0, T; L^1(R^N \times R^N))$ is the renormalized solution of* (1.1) *provided that it is initially equal to f_0, and that for all nonlinear function $\beta \in C^2[0, +\infty)$ satisfying*

$$\beta(0) = 0, |\beta'(t)| \leq \frac{C}{1+t}, |\beta''(t)| \leq \frac{C}{1+t^2}, \tag{1.5}$$

and $\beta(f)$ solves

$$(\partial_t + v \cdot \nabla_x - \Delta_v)\beta(f) = \beta'(f)Q(f, f) - \beta''(f)|\nabla_v f|^2 \tag{1.6}$$

in the sense of distributions.

Remark. In fact, it can be checked that we just have to choose β as a given function on $[0, +\infty)$, for instance, $\beta(t) = \log(1 + t)$.

Following [1], suppose that $B \in L^1_{loc}(R^N \times S^{N-1})$. Let

$$A(z) = \int_{S^{N-1}} B(z, w)dw \text{ for } z \in R^N,$$

$$\int_{B_R} \frac{A(v - z)}{(1 + |v|^2)} dz \to 0 \text{ as } |z| \to \infty \text{ for all } R < +\infty. \quad (1.7)$$

In 1988, DiPerna and Lions [1] proved the following result.

Theorem A. *Given an initial data f_0 satisfying* (1.3) *and B satisfies* (1.7). *Then, there exist a renormaliezed solution f to FPB equation* (1.1) *satisfying $f|_{t=0} = f_0$.*

In this paper, we do not assume bounded energy anymore. Instead, due to dispersion effects(see [12]) we assume

$$\int_{R^N \times R^N} f_0(x, v)(1 + |x|^\alpha + |x - v|^2 + |\log f_0|)\, dx\, dv < +\infty \quad (1.8)$$

for some $\alpha > 0$. Since the assumption (1.8) on f_0 leads to velocity moment bounds, we can derive the following estimate

$$f(1 + |x|^\alpha + |v|^\alpha + |\log f|) \in L^1((0, T) \times R^N \times R^N) \quad (1.9)$$

which give us enough a priori estimates to deduce the stability result of renormalized solutions like [1].

It seems strange to allow infinite energy, while we always think of the particles system as finite (mass, energy, etc.) from the physical point of view. It is known that the kinetic energy is not conserved for the FPB equation. However, one can remark that $\iint_{R^N \times R^N} f(t, x, v)|x-(1+t)v|^2 dxdv$ is bounded for any $t \in [0, T]$(see below Prop. 1). It is sufficient for us to assume that $f_0|x - v|^2 \in L^1$ instead of $f_0|v|^2 \in L^1$, in other word, replacing the finiteness of kinetic energy by position potential energy. By the way, Mischler and Perthame studied the Boltzmann equation with the initial values of infinite energy [13]. Our result can be considered as the extension of [13], while our proof does not need the assumption that f has compact support.

Now let us explain our result. Suppose that

$$\int_{B_R(v)} \frac{A(z)dz}{(1 + |v|^\alpha)} \to 0 \text{ as } |v| \to \infty \text{ for every } R < +\infty \quad (1.10)$$

here A is defined as the above. Then we have

Theorem 1. *Assume B satisfies* (1.10) *and f_0 satisfies* (1.8)*. Then, there exists a global renormalized solution f to the FPB equation* (1.1) *such that $f|_{t=0} = f_0$.*

Our paper is organized as following: In the second section we shall establish some new estimates for a classical solution to FPB equation. The idea is to choose appropriate multiplicators as in [13]. In the third section we prove the theorem of existence of renormalized solution following the line of [1] in the case of infinite energy. Like [1] we employ the theory of hypoelloptic operator [14] and the contraction mapping principle to obtain the existence and uniqueness.

2. Some elementary estimates

In this section we present some new estimates for the solution to the Fokker-Planck-Boltzmann equation. Assume that f is the classical solution of FPB equation which decays to zero at infinite.

For FPB equation (1.1), we can derive easily the conservations of mass and momentum, i.e.

$$\frac{d}{dt}\int_{R^N\times R^N} f\begin{pmatrix}1\\ v\end{pmatrix} dx\,dv = 0 \tag{2.1}$$

and the entropy equation

$$\frac{d}{dt}\int_{R^N\times R^N} f\log f\,dx\,dv = -4\int_{R^N\times R^N}|\nabla_v\sqrt{f}|^2 dx\,dv - \frac{1}{4}D(f) \tag{2.2}$$

where

$$D(f) = \int_{R^N\times R^N} dv\,dv_*\int_{S^{N-1}} dw\,B(v-v_*,w)(f'f'_* - ff_*)\log\frac{f'f'_*}{ff_*}. \tag{2.3}$$

Note that $D(f)\geq 0$, one deduces that $\int_{R^N\times R^N} f\log f\,dx\,dv$ is a nonincreasing function of time variable t.

Proposition 1. *Let f be the classical solution of FPB equation* (1.1)*, then*

$$\sup_{t\in[0,T]}\int_{R^N\times R^N} f|x-(1+t)v|^2 dx\,dv \leq \int_{R^N\times R^N} f_0|x-v|^2 dx\,dv + \frac{2N}{3}(1+T)^3\int_{R^N\times R^N} f_0 dx\,dv \tag{2.4}$$

Proof. Multiply the FPB equation (1.1) by $|x-(1+t)v|^2$ and integrate in the velocity and space variables. Note that $|x-(1+t)v|^2$ is a collisional invariant, one gets

$$\frac{d}{dt}\int_{R^N\times R^N} f|x-(1+t)v|^2 dx\, dv =$$

$$\int_{R^N\times R^N} f(\partial_t + v\cdot\nabla_x)|x-(1+t)v|^2 dx\, dv + \int_{R^N\times R^N} f\Delta_v|x-(1+t)v|^2 dx\, dv$$

$$= 2N(1+t)^2\int_{R^N\times R^N} f\, dx\, dv.$$

By the conservation of mass we obtain the result. □

Proposition 2. *Let f be the classical solution of* (1.1) *and α be a given real number in* $(0,2)$. *Then for every $t\geq 0$*

$$\int_{R^N\times R^N} f(1+|x|^2)^{\frac{\alpha}{2}} dx\, dv \leq e^{\alpha t}\int_{R^N\times R^N} f_0\left((1+|x|^2)^{\frac{\alpha}{2}} + |x-v|^2\right) dx\, dv. \tag{2.5}$$

Proof. We multiply the equation (1.1) by $(1+|x|^2)^{\frac{\alpha}{2}}$ and integrate over x and v to obtain

$$\frac{d}{dt}\int_{R^N\times R^N} f(1+|x|^2)^{\frac{\alpha}{2}} dx\, dv = \alpha\int_{R^N\times R^N} f(1+|x|^2)^{\frac{\alpha}{2}-1} x\cdot v\, dx\, dv$$

since $(1+|x|^2)^{\frac{\alpha}{2}}$ is a collisional invariant. Note that

$$\begin{aligned} 2|x\cdot v| &\leq |x|^2 + |v|^2 \\ &\leq \left(1+\frac{1}{(1+t)^2}\right)|x|^2 + \frac{1}{(1+t)^2}|x-(1+t)v|^2 \end{aligned}$$

and then by the Gronwall type inequality we obtain the result (see the proof of Lemma 2 of [13] for the details). □

Proposition 3. *Let f be the solution of* (1.1) *and α be a given real number in* $(0,2)$. *Then there exists a constant C_T dependent on T,α,f_0 such that*

$$\sup_{t\in[0,T]}\int_{R^N\times R^N} f(1+|x|^\alpha + |v|^\alpha + |\log f|) dx\, dv \leq C_T, \tag{2.6}$$

$$\int_0^T dt\int_{R^N\times R^N} D(f) dx\, dv \leq C_T. \tag{2.7}$$

Furthermore,

$$\int_0^T dt\int_{R^N\times R^N} dx\, dv(|\nabla_v\sqrt{f}|^2 + D(f)) \leq C_T. \tag{2.8}$$

Proof. Firstly, we have

$$\int_{R^N\times R^N} f|x|^\alpha dx\, dv \le \int_{R^N\times R^N} f(1+|x|^2)^{\frac{\alpha}{2}} dx\, dv. \tag{2.9}$$

Secondly, we consider the velocity moment of α order. Note that $|(1+t)v| \le |x| + |x-(1+t)v|$, hence

$$\int_{R^N\times R^N} f|v|^\alpha dx\, dv \le \int_{R^N\times R^N} f\frac{1}{(1+t)^\alpha}(|x|+|x-(1+t)v|)^\alpha dx\, dv. \tag{2.10}$$

By virtue of the binomial formula, the right hand side is bounded by

$$C\int_{R^N\times R^N} f\left((1+|x|^2)^{\frac{\alpha}{2}} + |x-(1+t)v|^2\right) dx\, dv$$

C is a nonnegative constant only dependent on α. Apply preceding Prop.1, 2 and together with (2.1) and (1.8) we derive

$$\sup_{t\in[0,T]}\int_{R^N\times R^N} f(1+|x|^\alpha+|v|^\alpha)dx\, dv \le C_T \tag{2.11}$$

and C_T depends on T, α and f_0.

Next, let us estimate $\int_{R^N\times R^N} f|\log f|dx\, dv$ employing the classical scheme of [1]:

$$\int_{R^N\times R^N} f|\log f|dx\, dv \le \int_{R^N\times R^N} f\log f dx\, dv + 2\int_{R^N\times R^N} f\log\left(\frac{1}{f}\right)\mathbf{1}_{(f<1)}dx\, dv. \tag{2.12}$$

The second term of the last equation can be split into two parts

$$\begin{aligned}\int_{R^N\times R^N} f\log\left(\frac{1}{f}\right)\mathbf{1}_{(f<1)}dx\, dv &\le \int_{R^N\times R^N} f\log\left(\frac{1}{f}\right)\mathbf{1}_{(f\le e^{-(|x|^\alpha+|v|^\alpha)})}dx\, dv \\ &\quad + \int_{R^N\times R^N} f\log\left(\frac{1}{f}\right)\mathbf{1}_{(e^{-(|x|^\alpha+|v|^\alpha)}\le f<1)}dx\, dv \\ &\le \int_{R^N\times R^N} f(|x|^\alpha+|v|^\alpha)dx\, dv + \int_{R^N\times R^N} f\log\left(\frac{1}{f}\right)\mathbf{1}_{(f\le e^{-(|x|^\alpha+|v|^\alpha)})}dx\, dv.\end{aligned} \tag{2.13}$$

Since $t \log \frac{1}{t} \leq C\sqrt{t}$ for $t \in (0,1)$, $C \geq 0$ is a constant and we note that the integral $\int_{R^N} e^{-\frac{1}{2}|x|^\alpha} dx$ is convergent. Indeed,

$$\begin{aligned}\int_{R^N} e^{-\frac{1}{2}|x|^\alpha} dx &= \int_{S_1(0)} \int_0^\infty e^{-\frac{1}{2}r^\alpha} r^{N-1} dr\, d\sigma \\ &= \omega_N \int_0^\infty e^{-\frac{1}{2}r^\alpha} r^{N-1} dr \\ &= \frac{\omega_N 2^{\frac{n}{\alpha}}}{\alpha} \Gamma\left(\frac{n}{\alpha}\right)\end{aligned}$$

where ω_N denotes the surface area of unit sphere. Thus, combining (2.12), (2.13) with (2.2) and (2.11) we deduce

$$\sup_{t\in[0,T]} \int_{R^N\times R^N} f|\log f| dx\, dv \leq C_T. \tag{2.14}$$

The proof of (2.6) is completed.

The bound 2.8 (and obviously (2.7)) follows from (2.2) and we obtain the result. □

Remark. From (2.8) one can obtain that $f \in L^2((0,T) \times R_x^N; H^1(R_v^N))$.

3. Proof of theorem of existence

This section is mainly devoted the proof of Theorem 1. Let $L = \partial_t + v \cdot \nabla_x - \Delta_v$. From [1], L is a hypoelliptic operator (see also [14]). Let $G(t,x,v)$ be a continuous fundamental solution of L.

We establish the stability theorem in the following:

Theorem 2. *Assume that f^n is a sequence of classical solution of FPB equation* (1.1) *such that $f^n \in W^{2,\infty}((0,\infty) \times R^N \times R^N)$ and*

$$\sup_{t\in(0,T)} \int_{R^N\times R^N} f^n(1 + |x|^\alpha + |v|^\alpha + |\log f^n|) dx\, dv \leq C_T, \tag{3.1}$$

$$\int_0^T dt \int_{R^N\times R^N} dx\, dv \{|\nabla_v \sqrt{f^n}|^2 + D(f^n)\} \leq C_T, \tag{3.2}$$

C_T is a constant independent of n. In addition, assume that B satisfies (1.10)*. Then, $\forall p, T < \infty$ the sequence f^n converges in $L^p(0,T; L^1(R^N \times R^N))$ to a renormalized solution f which satisfies* (3.1) *for a.e. $t \in (0,T)$ and* (3.2)*.*

Proof. The proof of this theorem is completely similar to that of Theorem 1 of [1]. □

Proof of Theorem 1. Firstly we approximate the FPB equation (1.1) by truncating the collision section B and regularizing the initial data f_0.

Let us define

$$B_n(z,w) = B(z,w)\mathbf{1}_{(|z|\le n)} \wedge \frac{n}{\omega_N}$$

where ω_N is the surface area of unit sphere. So the approximation collision operator is written as

$$Q_n(f,f) = \iint_{R^N\times S^{N-1}} (f'f'_* - ff_*)B_n(v-v_*,w)dw\,dv_*.$$

Let $\rho(x,v)$ be the standard mollifier with respective to x and v, i.e. ρ be a non-negative enen C^∞ function such that

$$\mathbf{supp}\rho(x,v) = \{x\in R^N, v\in R^N; |x|\le 1, |v|\le 1\} \text{ and } \int_{R^N\times R^N}\rho(x,v)dx\,dv = 1.$$

For each n, let

$$f_0^n(x,v) = f_0 * \rho_{\frac{1}{n}}(x,v), \quad \text{where} \quad \rho_{\frac{1}{n}}(x,v) = n^{2N}\rho(nx,nv).$$

Then, we obtain the approximation equation

$$\begin{cases} Lf^n = (1+\frac{1}{n}\int_{R^N} f^n dv)^{-1}Q_n(f^n,f^n), \\ f^n|_{t=0} = f_0^n. \end{cases} \tag{3.3}$$

Next we solve the equation (3.3) by contraction mapping principle. Make use of the fundamental solution of L,

$$f^n(x,v) = \mathcal{Q}_n(f^n,f^n) * G(t,x,v) + f_0^n * G(t,x,v) \tag{3.4}$$

where

$$\mathcal{Q}_n(f^n,f^n) = (1+\frac{1}{n}\int_{R^N} f^n dv)^{-1}Q_n(f^n,f^n).$$

Let the mapping $\mathcal{A}: f^n \to g^n$ defined by

$$g^n(t,x,v) = \int_0^t\int_{R^N}\int_{R^N} \mathcal{Q}_n(f^n,f^n)(s,y,\xi)G(t,x,v;s,y,\xi)ds\,dy\,d\xi + \int_{R^N}\int_{R^N} f_0^n(y,\xi)G(t,x,v;y,\xi)dy\,d\xi. \tag{3.5}$$

Note that

$$\|\mathcal{Q}_n(f^n,f^n)\|_{L^1} \le C_n\|f^n\|_{L^1},$$
$$\|\mathcal{Q}_n(f^n,f^n)\|_{L^\infty} \le C_n\|f^n\|_{L^\infty}.$$

Where C_n only depends on n. This implies that $\mathcal{A}$ maps $L^1 \cap L^\infty((0,T) \times R^N \times R^N)$ into itself for every $T < \infty$. Additionally,

$$\|\mathcal{Q}_n(\phi,\phi) - \mathcal{Q}_n(\psi,\psi)\|_{L^1 \cap L^\infty} \leq C_n \|\phi - \psi\|_{L^1 \cap L^\infty}.$$

Thus, $\mathcal{Q}_n$ is Lipschitz continuous in $L^1 \cap L^\infty((0,T) \times R^N \times R^N)$. Furthermore, by choosing T appropriately we deduce that the mapping $\mathcal{A}$ is a contraction mapping on $L^1 \cap L^\infty((0,T) \times R^N \times R^N)$. The contractive mapping principle then implies that there is a unique solution to (3.3).

By the boot-strap technique we have that for any $T < \infty$ there is a unique solution to (3.3) in $L^1 \cap L^\infty((0,T) \times R^N \times R^N)$.

Finally, we check easily that $f^n \in W^{2,\infty}((0,T) \times R^N \times R^N)$ and (3.1), (3.2) hold uniformly in n. Passing to the limit, and using the stability result we finish the proof of Theorem 1. □

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