SARAJEVO JOURNAL OF MATHEMATICS
Vol.5 (17) (2009), 73–87

EXISTENCE RESULTS FOR NONOSCILLATORY SOLUTIONS OF THIRD ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS

E. THANDAPANI, R. KARUNAKARAN AND I. M. AROCKIASAMY

Abstract. In this paper the authors consider the third order neutral difference equation

$$\Delta^3 (x_n + p_n x_{n-k}) + q_n f(x_{n-\ell}) = h_n$$

where $\{p_n\}, \{q_n\}, \{h_n\}$ are real sequences. They use Krasnoselskii's fixed point theorem to establish the existence of nonoscillatory solutions. The results are illustrated with examples.

1. Introduction

In this paper, we consider the third order neutral difference equation

$$\Delta^3 (x_n + p_n x_{n-k}) + q_n f(x_{n-\ell}) = h_n \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, k and ℓ are positive integers, $\{p_n\}, \{q_n\}$ and$\{h_n\}$ are real sequences defined for all $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, n_0+2, \dots\}$, and n_0 a nonnegative integer, and f is a continuous real valued function. Here we allow $\{q_n\}$ and $\{h_n\}$ to be oscillatory.

Let $\theta = \max\{k, \ell\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq \mathbb{N}(n_0 - \theta)$ and that satisfies equation (1.1). A solution $\{x_n\}$ of equation (1.1) is nonoscillatory if it is either eventually positive or eventually negative and oscillatory otherwise.

The oscillatory and nonoscillatory behavior of solutions of difference equations has been considered in [1-8] and conditions for the existence of nonoscillatory solutions using either Schauder fixed point theorem or Banach contraction principle are obtained. The aim of this paper is to obtain a sufficient

2000 *Mathematics Subject Classification.* 39A10.
Key words and phrases. Nonoscillation, third order, nonlinear, difference equations.

condition for the existence of nonoscillatory solutions of equation (1.1) without using nondecreasing conditions and any sign conditions on the sequences $\{q_n\}$ and $\{h_n\}$ via Krasnoselskii's fixed point theorem.

In Section 2, we establish condition for the existence of nonoscillatory solutions of equation (1.1).

2. Existence results for nonoscillatory solutions

In this section we establish sufficient condition for the existence of bounded nonoscillatory solution of equation (1.1).

Lemma 2.1. [7], [Krasnoselskii's Fixed Point Theorem]
Let X be a Banach space let Ω be a bounded closed convex subset of X and let S_1, S_2 be maps of Ω into X such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contraction and S_2 is completely continuous, then the equation $S_1x + S_2y = x$ has a solution in Ω.

Lemma 2.2. [7], [Schauder's Fixed Point Theorem]
Let Ω be a closed, convex and nonempty subset of a Banach space X. Let $S : \Omega \to \Omega$ be a continuous mapping such that $S\Omega$ is relatively compact subset of X. Then S has at least one fixed point in Ω. That is, there exists an $x \in \Omega$ such that $Sx = x$.

We need the following factorial function in our main results.

Definition 2.3. *The factorial function $t^{(r)}$ is defined as follows according to the value of r :*

(a) *if* $r = 1, 2, 3, \ldots,$ *then* $t^{(r)} = t\,(t-1)\,(t-2)\cdots(t-r+1)\,,$
(b) *if* $r = 0$, *then* $t^{(r)} = 1$.

Whenever $t^{(r)}$ is defined we have $\Delta t^{(r)} = rt^{(r-1)}$, and $\sum t^{(r)} = \frac{t^{(r+1)}}{r+1} + c$.

Theorem 2.4. *Assume that $-1 < c_1 \leq p_n \leq 0$ and that*

$$\sum_{n=n_0}^{\infty} n^{(2)}\,|q_n| < \infty, \tag{2.1}$$

and

$$\sum_{n=n_0}^{\infty} n^{(2)}\,|h_n| < \infty. \tag{2.2}$$

Then equation (1.1) *has a bounded nonoscillatory solution.*

Proof. By (2.1) and (2.2), we choose a $N \in \mathbb{N}\,(n_0)$ sufficiently large such that

$$\sum_{n=N}^{\infty} n^{(2)}\left(|q_n|\,M_1 + |h_n|\right) \leq \frac{2}{3}\,(1+c_1)\,,$$

where $M_1 = \max_{\frac{2(1+c_1)}{3} \le x \le \frac{4}{3}} \{|f(x)|\}$. Let $\mathcal{B}_{n_0}$ be the set of all real sequences with the norm $\|x\| = \sup_{n \geqslant n_0} |x_n| < \infty$. Then $\mathcal{B}_{n_0}$ is a Banach space. We define a closed, bounded and convex subset Ω of $\mathcal{B}_{n_0}$ as follows.

$$\Omega = \left\{ x = \{x_n\} \in \mathcal{B}_{n_0} : \frac{2}{3}(1+c_1) \le x_n \le \frac{4}{3}, n \in \mathbb{N}(n_0) \right\}.$$

Define two maps S_1 and $S_2 : \Omega \to \mathcal{B}_{n_0}$ as follows.

$$(S_1 x)_n = \begin{cases} 1 + c_1 - p_n x_{n-k}, & n \geqslant N, \\ (S_1 x)_N, & n_0 \le n \le N \end{cases}$$

$$(S_2 x)_n = \begin{cases} \frac{1}{2} \sum\limits_{s=n}^{\infty} (s-n+2)^{(2)} \left(q_s f(x_{s-\ell}) - h_s \right), & n \geqslant N, \\ (S_2 x)_N, & n_0 \le n \le N. \end{cases}$$

(i) We shall show that for any $x, y \in \Omega, (S_1 x)_n + (S_2 y)_n \in \Omega$. In fact for every $x, y \in \Omega$ and $n \geqslant N$, we get

$$\begin{aligned} &(S_1 x)_n + (S_2 y)_n \\ &\le 1 + c_1 - p_n x_{n-k} + \frac{1}{2} \sum_{s=n}^{\infty} (s-n+2)^{(2)} \left(|q_s| \left| f(y_{s-\ell}) \right| + |h_s| \right) \\ &\le 1 + c_1 - \frac{4}{3} c_1 + \frac{1}{2} \sum_{s=n}^{\infty} s^{(2)} \left(|q_s| M_1 + |h_s| \right) \\ &\le 1 + c_1 - \frac{4}{3} c_1 + \frac{1+c_1}{3} = \frac{4}{3}. \end{aligned}$$

Furthermore we have,

$$\begin{aligned} &(S_1 x)_n + (S_2 y)_n \\ &\geqslant 1 + c_1 - p_n x_{n-k} - \frac{1}{2} \sum_{s=n}^{\infty} (s-n+2)^{(2)} \left(|q_s| \left| f(y_{s-\ell}) \right| + |h_s| \right) \\ &\geqslant 1 + c_1 - \frac{1}{2} \sum_{s=n}^{\infty} s^{(2)} \left(|q_s| M_1 + |h_s| \right) \\ &\geqslant 1 + c_1 - \frac{1+c_1}{3} = \frac{2(1+c_1)}{3}. \end{aligned}$$

Hence

$$\frac{2(1+c_1)}{3} \le (S_1 x)_n + (S_2 y)_n \le \frac{4}{3} \text{ for } n \geqslant N_0.$$

Thus we have proved that $(S_1 x)_n + (S_2 y)_n \in \Omega$ for any $x, y \in \Omega$.

(ii) We shall show that S_1 is a contraction mapping on Ω. In fact for $x, y \in \Omega$ and $n \geqslant N$ we have

$$\left|(S_1x)_n - (S_1y)_n\right| \leq -p_n \left|x_{n-k} - y_{n-k}\right| \leq -c_1 \left\|x - y\right\|.$$

Since $0 < -c_1 < 1$, we conclude that S_1 is a contraction mapping on Ω.

(iii) Next we show that S_2 is uniformly Cauchy. First we shall show that S_2 is continous. Let $\left\{x^{(i)}\right\}$ be a sequence in Ω such that $x^{(i)} \longrightarrow x = \{x_n\}$ as $i \longrightarrow \infty$. Since, Ω is closed $x \in \Omega$. Furthermore, for $n \geqslant N$ we have,

$$\begin{aligned}\left|\left(S_2x^{(i)}\right)_n - (S_2x)_n\right| &\leq \frac{1}{2}\sum_{s=n}^{\infty} s^{(2)} \left|q_s\right| \left|f\left(x^{(i)}_{s-\ell}\right) - f\left(x_{s-\ell}\right)\right| \\ &\leq \frac{1}{2}\sum_{s=N}^{\infty} s^{(2)} \left|q_s\right| \left|f\left(x^{(i)}_{s-\ell}\right) - f\left(x_{s-\ell}\right)\right|.\end{aligned}$$

Since $\left|f\left(x^{(i)}_{s-\ell}\right) - f\left(x_{s-\ell}\right)\right| \longrightarrow 0$ as $i \longrightarrow \infty$ by applying the Lebesque dominated convergence theorem, we conclude that

$$\lim_{i\longrightarrow\infty} \left\|\left(S_2x^{(i)}\right)_n - (S_2x)_n\right\| = 0.$$

This means that S_2 is continuous. Finally we prove that S_2 is uniformly Cauchy. By (2.1), for any $\varepsilon > 0$, choose $N_1 > N$ large enough so that

$$\frac{1}{2}\sum_{n=N_1}^{\infty} n^{(2)} \left(M_1 \left|q_n\right| + \left|h_n\right|\right) < \frac{\varepsilon}{2}$$

Then for $x \in \Omega, n_2 > n_1 > N_1$.

$$\begin{aligned}&\left|(S_2x)_{n_2} - (S_2x)_{n_1}\right| \\ &\leq \frac{1}{2}\sum_{s=n_2}^{\infty} s^{(2)} \left(\left|q_s\right| \left|f\left(x_{s-\ell}\right)\right| + \left|h_s\right|\right) + \frac{1}{2}\sum_{s=n_1}^{\infty} s^{(2)} \left(\left|q_s\right| \left|f\left(x_{s-\ell}\right)\right| + \left|h_s\right|\right) \\ &\leq \frac{1}{2}\sum_{s=n_2}^{\infty} s^{(2)} \left(\left|q_s\right| M_1 + \left|h_s\right|\right) + \frac{1}{2}\sum_{s=n_2}^{\infty} s^{(2)} \left(\left|q_s\right| \left|f\left(x_{s-\ell}\right)\right| + \left|h_s\right|\right) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Therefore $(S_2x)_n$ is uniformly Cauchy. By Lemma 2.2, there is an $x^* \in \Omega$ such that $S_1x^* + S_2x^* = x^*$. It is easy to see that $x^* = \{x^*\}$ is a nonoscillatory solution of the equation (1.1). This completes the proof of Theorem 2.4. $\square$

Example 2.5. Consider the difference equation

$$\Delta^3\left(x_n - \frac{1}{2}x_{n-1}\right) + \frac{1}{n\,(n+1)\,(n+2)\,(n+3)}x_{n-1} = \frac{15-2n}{(n-1)\,n\,(n+1)\,(n+2)\,(n+3)}, n \geq 2. \qquad (2.3)$$

Here $p_n = -\frac{1}{2}, q_n = \frac{1}{n(n+1)(n+2)(n+3)}$ and $h_n = \frac{15-2n}{(n-1)n(n+1)(n+2)(n+3)}$. It is easy to see that all conditions of Theorem 2.4 are satisfied and hence equation (2.3) has a bounded nonoscillatory solution. In fact $\{x_n\} = \left\{1 + \frac{1}{n}\right\}$ is one such solution of equation (2.3).

Theorem 2.6. *Assume that* $-\infty < p_n \equiv c_2 < -1$ *and that* (2.1) *and* (2.2) *hold. Then* (1.1) *has a bounded nonoscillatory solution.*

Proof. By (2.1) and (2.2) we choose $N \in \mathbb{N}\,(n_0)$ sufficiently large such that

$$-\frac{1}{c_2}\sum\_s = n + k^{\infty} s^{(2)}\left(|q_s|\,M_2 + |h_s|\right) \leq -\frac{(c_2+1)}{2}$$

where $M_2 = \max_{-\frac{(c_2+1)}{2} \leq x \leq -2c_2} \{|f(x)|\}$.

Let $\mathcal{B}_{n_0}$ be the space defined as in the proof of Theorem 2.4. We define a closed bounded and convex subset Ω of $\mathcal{B}_{n_0}$ as follows:

$$\Omega = \left\{x = \{x_n\} \in \mathcal{B}_{n_0} : -\frac{(c_2+1)}{2} \leq x \leq -2c_2, n \geqslant n_0\right\}.$$

Define two maps S_1 and $S_2 : \Omega \longrightarrow \mathcal{B}_{n_0}$ as follows.

$$(S_1x)_n = \begin{cases} -c_2 - 1 - \frac{1}{p_n}x_{n+k}, & n \geqslant N \\ (S_1x)_N\,, & n_0 \leq n \leq N, \end{cases}$$

$$(S_2x)_n = \begin{cases} \frac{1}{2p_n}\sum\limits_{s=n+k}^{\infty}(s-n-k)^{(2)}\left(q_s f\left(x_{s-\ell}\right) - h_s\right), & n \geqslant N, \\ (S_2x)_N\,, & n_0 \leq n \leq N. \end{cases}$$

We shall show that for any $x, y \in \Omega$.In fact for any $x, y \in \Omega$, we get

$$\begin{aligned}
&(S_1x)_n + (S_2y)_n \\
&\leq -c_2 - 1 - \frac{1}{p_n}x_{n+k} - \frac{1}{2p_n}\sum_{s=n+k}^{\infty}(s-n-k+2)^{(2)}\left(|q_s|\left|f\left(y_{s-\ell}\right)\right| + |h_s|\right) \\
&\leq -c_2 - 1 + 2 - \frac{1}{2c_2}\sum_{s=N+k}^{\infty} s^{(2)}\left(|q_s|\,M_2 + |h_s|\right) \\
&\leq -c_2 + 1 - \frac{(c_2+1)}{2} \leq -2c_2.
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
&(S_1x)_n + (S_2y)_n \\
&\geqslant -c_2 - 1 - \frac{1}{p_n}x_{n+k} + \frac{1}{2p_n}\sum_{s=n+k}^{\infty}(s-n+2)^{(2)}\left(|q_s|\left|f\left(y_{s-\ell}\right)\right| + |h_s|\right) \\
&\geqslant -c_2 - 1 + \frac{1}{2c_2}\sum_{s=N}^{\infty}s^{(2)}\left(|q_s|\,M_2 + |h_s|\right) \\
&\geqslant (-c_2 - 1) + \frac{(c_2+1)}{2} = \frac{-(c_2+1)}{2}.
\end{aligned}$$

Hence $\frac{-(c_2+1)}{2} \leq (S_1x)_n + (S_2y)_n \leq -2c_2$, for $n \geqslant n_0$.

Thus we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$. We shall show that S_1 is a contraction mapping on Ω. In fact for $x, y \in \Omega$and $n \geqslant N$ we have

$$\left|(S_1x)_n + (S_1y)_n\right| \leq \frac{-1}{p_n}\left|x_{n+k} - y_{n+k}\right| \leq -\frac{1}{c_2}\left\|x-y\right\|.$$

Since $0 < -\frac{1}{c_2} < 1$, we conclude that S_1 is a contraction mapping on Ω. Proceeding similarly as in the proof of Theorem 2.4 we obtain S_2 is uniformly Cauchy. By Lemma 2.1 there is an $x^* \in \Omega$ such that $S_1x^* + S_2x^* = x^*$. Clearly $x^* = \{x_n^*\}$ is a bounded nonoscillatory solution of equation (1.1). This completes the proof of Theorem 2.6. □

Example 2.7. Consider the difference equation

$$\Delta^3\left(x_n - 2x_{n-1}\right) + \frac{1}{2^n}x_{n-1} = \frac{11\left(2^n\right)+16}{8(2^{2n})},\ \ n \geq 2. \tag{2.4}$$

Here $p_n = -2, q_n = \frac{1}{2^n}$ and $h_n = \frac{11(2^n)+16}{8(2^{2n})}$. It is easy to see that all conditions of Theorem 2.6 are satisfied and hence equation (2.4) has a bounded nonoscillatory solution. In fact $\{x_n\} = \left\{1 + \frac{1}{2^n}\right\}$ is one such solution of equation (2.4).

Theorem 2.8. *Assume that* $0 \leq p_n \leq c_3 < 1$ *and that* (2.1) *and* (2.2) *hold. Then* (1.1) *has a bounded nonoscillatory solution.*

Proof. By (2.1) and (2.2) we choose $N \in \mathbb{N}\left(n_0\right)$ sufficiently large such that

$$\frac{1}{2}\sum_{s=N}^{\infty}s^{(2)}\left(|q_s|\,M_3 + |h_s|\right) \leq 1 - c_3,$$

where $M_3 = \max_{2(1-c_3)\leq x\leq 4}\left\{f(x)\right\}$.

Let $\mathcal{B}_{n_0}$ be the space as defined in the proof of Theorem 2.4. We define a closed, bounded and convex subset Ω of $\mathcal{B}_{n_0}$as follows:

$$\Omega = \left\{x = \{x_n\} \in \mathcal{B}_{n_0} : 2(1-c_3) \leq x_n \leq 4\right\}.$$

We define two maps S_1 and $S_2 \,:\Omega \longrightarrow \mathcal{B}_{n_0}$ as follows.

$$S_1x = \begin{cases} 3 + c_3 - p_n x_{n-k}, & n \ge N; \\ (S_1x)_N\,, & n_0 \le n \le N. \end{cases}$$

$$S_2x = \begin{cases} \frac{1}{2}\sum\limits_{s=n}^{\infty} (s-n+2)^{(2)} \left(q_s f\left(x_{s-\ell}\right) - h_s\right), & n \ge N; \\ (S_2x)_N\,, & n_0 \le n \le N. \end{cases}$$

We shall show that for any $x, y \in \Omega$, $S_1x + S_2y \in \Omega$. In fact for any $x, y \in \Omega$ and $n \ge N$, we obtain

$$\begin{aligned} &(S_1x)_n + (S_2y)_n \\ &\le 3 + c_3 - p_n x_{n-k} + \frac{1}{2}\sum_{s=n}^{\infty} (s-n+2)^{(2)} \left(|q_s|\left|f\left(y_{s-\ell}\right)\right| + |h_s|\right) \\ &\le 3 + c_3 + \frac{1}{2}\sum_{s=N}^{\infty} s^{(2)} \left(|q_s|\, M_3 + |h_s|\right) \\ &\le 3 + c_3 + 1 - c_3 = 4. \end{aligned}$$

Furthermore we have

$$\begin{aligned} &(S_1x)_n + (S_2y)_n \\ &\ge 3 + c_3 - p_n x_{n-k} - \frac{1}{2}\sum_{s=N}^{\infty} (s-n+2)^{(2)} \left(|q_s|\left|f\left(x_{s-\ell}\right)\right| + |h_s|\right) \\ &\ge 3 + c_3 - 4c_3 - \frac{1}{2}\sum_{s=N}^{\infty} s^{(2)} \left(|q_s|\, M_3 + |h_s|\right) \\ &\ge 3 + c_3 - 4c_3 - (1 - c_3) = 2(1 - c_3). \end{aligned}$$

Hence,

$$2(1 - c_3) \le (S_1x)_n + (S_2y)_n \le 4, \text{ for } n \ge n_0.$$

Thus we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$. Proceeding similarly as in the proof of Theorem 2.4 we obtain the mapping S_1 is a contraction mapping on Ω and the mapping S_2 is uniformly Cauchy. By Lemma 2.1, there is an $x^* \in \Omega$ such that $S_1x^* + S_2x^* = x^*$. Clearly $x^* = x_n^*$ is a bounded nonoscillatory solution of the equation (1.1). This completes the proof of Theorem 2.8. □

Example 2.9. Consider the difference equation

$$\Delta^3\left(x_n+\frac{1}{n+1}x_{n-1}\right)+\frac{24(n+2)^3}{(n+1)^3(n+3)(n+4)(n+5)(n+6)}x_{n-1}^3$$
$$=\frac{24(n+1)}{(n+2)(n+3)(n+4)(n+5)(n+6)},\quad n\geq 2. \quad (2.5)$$

Here

$$p_n=\frac{1}{n+1},\quad q_n=\frac{24(n+2)^3}{(n+1)^3(n+3)(n+4)(n+5)(n+6)}$$

and

$$h_n=\frac{24(n+1)}{(n+2)(n+3)(n+4)(n+5)(n+6)}.$$

It is easy to see that all conditions of Theorem 2.8 are satisfied and hence equation (2.5) has a bounded nonoscillatory solution. In fact $\{x_n\}=\left\{\frac{n+2}{n+3}\right\}$ is one such solution of equation (2.5).

Theorem 2.10. *Assume that* $1<c_4\equiv p_n<\infty$ *and that* (2.1) *and* (2.2) *hold. Then* (1.1) *has a bounded nonoscillatory solution.*

Proof. By (2.1) and (2.2) we choose $N\in\mathbb{N}(n_0)$ sufficiently large so that

$$\frac{1}{2c_4}\sum_{s=n+k}^{\infty}s^{(2)}\left(|q_s|\,M_4+|h_s|\right)<c_4-1,$$

where $M_4=\max_{2(c_4-1)\leq x\leq 4c_4}\{f(x)\}$. Let $\mathcal{B}_{n_0}$ be the space as in the proof of Theorem 2.4. We define a closed bounded and convex subset Ω of B_{n_0} as follows:

$$\Omega=\left\{x=\{x_n\}\in\mathcal{B}_{n_0}:2(c_4-1)\leq x_n\leq 4c_4,n\geq n_0\right\}.$$

Define two maps S_1 and $S_2:\Omega\longrightarrow\mathcal{B}_{n_0}$ as follows:

$$(S_1x)_n=\begin{cases}3c_4+1-\frac{1}{p_n}x_{n+k}, & n\geq N;\\ (S_1x)_N, & n_0\leq n\leq N.\end{cases}$$

$$(S_2x)_n=\begin{cases}\frac{1}{2p_n}\sum\limits_{s=n+k}^{\infty}(s-n-k+2)^{(2)}\left(q_sf\left(x_{s-\ell}\right)-h_s\right), & n\geq N;\\ (S_2x)_N, & n_0\leq n\leq N.\end{cases}$$

We shall show that for any $x, y \in \Omega$, $S_1x+S_2y \in \Omega$. In fact for every $x, y \in \Omega$ and $n \geq N$, we get

$$\begin{aligned}
&(S_1x)_n + (S_2y)_n \\
&\leq 3c_4 + 1 - \frac{1}{p_n}x_{n+k} + \frac{1}{2p_n}\sum_{s=n+k}^{\infty}(s-n-k+2)^{(2)}\left(|q_s|\left|f\left(y_{s-\ell}\right)\right| + |h_s|\right) \\
&\leq 3c_4 + 1 + \frac{1}{2c_4}\sum_{s=N+k}^{\infty}s^{(2)}\left(|q_s|\,M_4 + |h_s|\right) \\
&\leq 3c_4 + 1 + (c_4 - 1) = 4c_4.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&(S_1x)_n + (S_2y)_n \\
&\geq 3c_4 + 1 - \frac{1}{p_n}x_{n+k} - \frac{1}{2p_n}\sum_{s=n+k}^{\infty}(s-n+2)^{(2)}\left(|q_s|\left|f\left(y_{s-\ell}\right)\right| + |h_s|\right) \\
&\geq 3c_4 + 1 - 4 - \frac{1}{2c_4}\sum_{s=N}^{\infty}s^{(2)}\left(|q_s|\,M_4 + |h_s|\right) \\
&\geq 3c_4 - 3 - (c_4 - 1) = 2\left(c_4 - 1\right).
\end{aligned}$$

Hence,

$$2\left(c_4 - 1\right) \leq (S_1x)_n + (S_2y)_n \leq 4c_4 \quad \text{for } n \geq n_0.$$

Thus we have proved that $S_1x + S_2y \in \Omega$ for every $x, y \in \Omega$. Proceeding similarly as in the proof of Theorem 2.4 we obtain that the mapping S_1 is a contraction mapping on Ω and the mapping S_2 is uniformly Cauchy. By Lemma 2.1, there is an $x^* \in \Omega$ such that $S_1x^* + S_2x^* = x^*$. Clearly $x^* = x_n^*$ is a bounded nonoscillatory solution of the equation (1.1). This completes the proof of Theorem 2.10. □

Example 2.11. Consider the difference equation

$$\Delta^3\left(x_n + 2x_{n-1}\right) + \frac{1}{3^n}x_{n-1} = \frac{81 - 29\left(3^n\right)}{27(3^{2n})}, \ \ n \geq 2. \tag{2.6}$$

Here $p_n = 2, q_n = \frac{1}{3^n}$ and $h_n = \frac{81-29(3^n)}{27(3^{2n})}$. It is easy to see that all conditions of Theorem 2.10 are satisfied and hence equation (2.6) has a bounded nonoscillatory solution. In fact $\{x_n\} = \left\{1 + \frac{1}{3^n}\right\}$ is one such solution of equation (2.6).

Theorem 2.12. *Assume that* $p_n \equiv 1$ *and that* (2.1) *and* (2.2) *hold. Then* (1.1) *has a bounded nonoscillatory solution.*

Proof. By (2.1) and (2.2) we choose $N > n_0$ sufficiently large so that

$$\frac{1}{2}\sum_{s=N+k}^{\infty} s^{(2)}\left(|q_s|\, M_5 + |h_s|\right) \leq 1,$$

where $M_5 = \max_{2\leq x\leq 4}\{f(x)\}$. We define a closed bounded and convex subset Ω of $\mathcal{B}_{n_0}$ as follows.

$$\Omega = \{x = \{x_n\} \in \mathcal{B}_{n_0} : 2 \leq x_n \leq 4,\ \ n \geq n_0\}.$$

Define a map $S : \Omega \longrightarrow \mathcal{B}_{n_0}$ as follows.

$$(Sx)_n = \begin{cases} 3 + \frac{1}{2}\sum\limits_{j=1}^{\infty}\sum\limits_{s=n+(2j-1)k}^{n+2jk}(s-n+2)^{(2)}\left(q_s f\left(x_{s-\ell}\right) - h_s\right), & n \geq N;\\ (S_1 x)_N, & n_0 \leq n \leq N.\end{cases}$$

We shall show that $S\Omega \subset \Omega$ for every $x \in \Omega$ and $n \geq N$, we get

$$\begin{aligned}(Sx)_n &\leq 3 + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{s=n+(2j-1)k}^{n+2jk}(s-n+2)^{(2)}\left(|q_s||f\left(x_{s-\ell}\right)| + |h_s|\right)\\ &\leq 3 + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{s=n+(2j-1)k}^{n+2jk} s^{(2)}\left(|q_s|M_5 + |h_s|\right) \leq 4.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}(Sx)_n &\geq 3 - \frac{1}{2}\sum_{j=1}^{\infty}\sum_{s=n+(2j-1)k}^{n+2jk}(s-n+2)^{(2)}\left(|q_s||f\left(x_{s-\ell}\right)| + |h_s|\right)\\ &\geq 3 - \frac{1}{2}\sum_{j=1}^{\infty}\sum_{s=n+(2j-1)k}^{n+2jk} s^{(2)}\left(|q_s|M_5 + |h_s|\right) \geq 2.\end{aligned}$$

Hence, $S\Omega \subset \Omega$.

Proceeding similarly as in the proof of Theorem 2.4 we obtain that the mapping S is uniformly Cauchy. By Lemma 2.2, there is an $x^* \in \Omega$ such that $Sx^* = x^*$, that is

$$x_n^* = \begin{cases} 3 + \frac{1}{2}\sum\limits_{j=1}^{\infty}\sum\limits_{s=n+(2j-1)k}^{n+2jk}(s-n+2)^{(2)}\left(q_s f\left(x_{s-\ell}\right) - h_s\right), & n \geq N;\\ x_N^*, & n_0 \leq n \leq N.\end{cases}$$

It follows that

$$x_n + x_{n-k} = 6 + \frac{1}{2}\sum_{s=n}^{\infty}(s-n+2)^{(2)}\left(q_s f\left(x_{s-\ell}\right) - h_s\right).$$

Clearly $x^* = \{x_n^*\}$ is a bounded nonoscillatory solution of the equation (1.1). This completes the proof of Theorem 2.12. $\square$

Example 2.13. Consider the difference equation

$$\Delta^3 (x_n + x_{n-1}) + \frac{12}{n(n+1)(n+2)(n+3)} x_{n-1} = \frac{-12}{(n-1)n(n+1)(n+2)(n+3)}, \quad n \geq 2. \tag{2.7}$$

Here

$$p_n = 1, \quad q_n = \frac{12}{n(n+1)(n+2)(n+3)}$$

and

$$h_n = \frac{-12}{(n-1)n(n+1)(n+2)(n+3)}.$$

It is easy to see that all conditions of Theorem 2.12 are satisfied and hence equation (2.7) has a bounded nonoscillatory solution. In fact $\{x_n\} = \left\{1 + \frac{1}{n}\right\}$ is one such solution of equation (2.7).

Theorem 2.14. *Assume that $p_n \equiv -1$ and that*

$$\sum_{s=n_0}^{\infty} s^{(3)} |q_s| < \infty, \tag{2.8}$$

and

$$\sum_{s=n_0}^{\infty} s^{(3)} |h_s| < \infty. \tag{2.9}$$

Then (1.1) *has a bounded nonoscillatory solution.*

Proof. First note that the assumptions (2.8) and (2.9) are equivalent to

$$\sum_{j=0}^{\infty} \sum_{s=n_0+jk}^{\infty} s^{(2)} |q_s| < \infty \tag{2.10}$$

and

$$\sum_{j=0}^{\infty} \sum_{s=n_0+jk}^{\infty} s^{(2)} |h_s| < \infty \tag{2.11}$$

respectively. We choose a sufficiently large $N \in \mathbb{N}(n_0)$ such that

$$\frac{1}{2} \sum_{j=1}^{\infty} \sum_{s=N+jk}^{\infty} s^{(2)} \left(|q_s| M_6 + |h_s| \right) \leq 1,$$

where $M_6 = \max_{0 \leq x \leq 1} \{f(x)\}$. We define a closed, bounded, and convex subset Ω on $\mathcal{B}_{n_0}$ as follows.

$$\Omega = \{x = \{x_n\} \in \mathcal{B}_{n_0} : 2 \leq x_n \leq 4, \ n \geq n_0\}.$$

Define a map $S : \Omega \longrightarrow \mathcal{B}_{n_0}$ as follows:

$$(Sx)_n = \begin{cases} 3 - \frac{1}{2}\sum\limits_{j=1}^{\infty}\sum\limits_{s=n+jk}^{\infty}(s-n+2)^{(2)}\left(q_s f\left(x_{s-\ell}\right) - h_s\right), & n \geq N; \\ (Sx)_N, & n_0 \leq n \leq N. \end{cases}$$

We shall show that $S\Omega \subset \Omega$ .In fact for every $x \in \Omega$ and $n \geq N$, we get

$$\begin{aligned}(Sx)_n &\leq 3 + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{s=n+jk}^{\infty}(s-n+2)^{(2)}\left(|q_s||f\left(x_{s-\ell}\right)| + |h_s|\right) \\ &\leq 3 + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{s=n+jk}^{\infty}s^{(2)}\left(|q_s|M_6 + |h_s|\right) \leq 4.\end{aligned}$$

Furthermore we have

$$\begin{aligned}(Sx)_n &\geq 3 - \frac{1}{2}\sum_{j=1}^{\infty}\sum_{s=n+jk}^{\infty}(s-n+2)^{(2)}\left(|q_s|\left|f\left(x_{s-\ell}\right)\right| + h_s\right) \\ &\geq 3 - \frac{1}{2}\sum_{j=1}^{\infty}\sum_{s=n+jk}^{\infty}s^{(2)}\left(|q_s|M_6 + |h_s|\right) \geq 2.\end{aligned}$$

Hence, $S\Omega \subset \Omega$. We now show that S is continuous.

Let $\left\{x^{(i)}\right\}$ be a sequence in Ω such that $x^{(i)} \rightarrow x = \{x_n\}$ as $i \rightarrow \infty$. Since, Ω is closed $x \in \Omega$. Furthermore, for $n \geqslant N$ we have,

$$\left|\left(Sx^{(i)}\right)_n - (Sx)_n\right| \leq \frac{1}{2}\sum_{j=1}^{\infty}\sum_{s=n+jk}^{\infty}s^{(2)}\left(|q_s|\left|f\left(x_{s-\ell}^{(i)}\right) - f\left(x_{s-\ell}\right)\right|\right).$$

Since $\left|f\left(x_{s-\ell}^{(i)}\right) - f\left(x_{s-\ell}\right)\right| \rightarrow 0$ as $i \rightarrow \infty$ by applying the Lebesque dominated convergence theorem, we conclude that

$$\lim_{i \longrightarrow \infty}\left\|\left(Sx^{(i)}\right)_n - (Sx)_n\right\| = 0.$$

This means that S is continuous. In the following we show that S is uniformly Cauchy. By (2.10) and (2.11), for any $\varepsilon > 0$, choose $N_1 > N$ large enough so that

$$\frac{1}{2}\sum_{j=1}^{\infty}\sum_{s=N_1+jk}^{\infty}s^{(2)}\left(M_6\left|q_s\right| + |h_s|\right) < \frac{\varepsilon}{2}.$$

Then for $x \in \Omega, n_2 > n_1 \geq N$,

$$\begin{aligned}
&\left|(Sx)_{n_2} - (S_2x)_{n_1}\right| \\
&\leq \frac{1}{2}\sum_{j=1}^{\infty}\sum_{s=n_2+jk}^{\infty} s^{(2)}\left(|q_s|\left|f\left(x_{s-\ell}\right)\right| + |h_s|\right) \\
&\qquad\qquad + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{s=n_1+jk}^{\infty} s^{(2)}\left(|q_s|\left|f\left(x_{s-\ell}\right)\right| + |h_s|\right) \\
&\leq \frac{1}{2}\sum_{j=1}^{\infty}\sum_{s=n_2+jk}^{\infty} s^{(2)}\left(|q_s| M_6 + |h_s|\right) + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{s=n_1+jk}^{\infty} s^{(2)}\left(|q_s| M_6 + |h_s|\right) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore Sx is uniformly Cauchy. By Lemma 2.2, there is an $x^* \in \Omega$ such that $Sx^* = x^*$. That is

$$x_n^* = \begin{cases} 3 - \frac{1}{2}\sum\limits_{j=1}^{\infty}\sum\limits_{s=n+jk}^{\infty} (s-n+2)^{(2)}\left(q_s f\left(x_{s-\ell}^*\right) - h_s\right), & n \geq N; \\ x_N^*, & n_0 \leq n \leq N. \end{cases}$$

It follows that

$$x_n - x_{n-k} = -\frac{1}{2}\sum_{s=n}^{\infty}(s-n+2)^{(2)}\left(q_s f\left(x_{s-\ell}\right) - h_s\right), n \geq N.$$

Clearly $x^* = \{x_n^*\}$ is a bounded nonoscillatory solution of the equation (1.1). This completes the proof of Theorem 2.14. $\square$

Example 2.15. Consider the difference equation

$$\begin{aligned}
\Delta^3\left(x_n - x_{n-1}\right) &+ \frac{24}{(n-1)n\,(n+1)\,(n+2)\,(n+3)}x_{n-1} \\
&= \frac{-24}{(n-1)^2\,n\,(n+1)\,(n+2)\,(n+3)}, \quad n \geq 2. \qquad (2.12)
\end{aligned}$$

Here

$$p_n = -1, q_n = \frac{24}{(n-1)n\,(n+1)\,(n+2)\,(n+3)}$$

and

$$h_n = \frac{-24}{(n-1)^2\,n\,(n+1)\,(n+2)\,(n+3)}.$$

It is easy to see that all conditions of Theorem 2.12 are satisfied and hence equation (2.12) has a bounded nonoscillatory solution. One can easily check that $\{x_n\} = \left\{1 + \frac{1}{n}\right\}$ is one such solution of equation (2.12).

Remark 2.16. *Simple modifications are necessary in the proofs to discuss the existence of nonoscillatory solutions of neutral functional difference equations of the form*

$$\Delta^3 (x_n + p_n x_{n-k}) + F(n, x_{n-\ell}) = h_n, \ \ n \geq n_0$$

where $F(n, x_{n-\ell}) : \mathbb{N}(n_0) \times \mathbb{R} \to \mathbb{R}$ *is continuous and bounded.*

Acknowledgment. The authors wish to thank the anonymous referee for his/her suggestions which improved the content of the paper.

References

[1] R. P. Agarwal,*Difference Equations and Inequalities* , 2^{nd} Edition, Marcel Dekker, New York, 2000.
[2] R. P. Agarwal, M. Bohner, S. R. Grace and D. O Regan, *Discrete Oscillation Theory*, Hindawi publishing corporation, New York, 2005.
[3] R. P. Agarwal and S. R. Grace, *The oscillation of higher - order nonlinear difference equations of neutral type*, Appl. Math. Lett., 12 (1999), 77–83.
[4] R. P. Agarwal and P. J. Y. Wong, *Advanced Topics in Difference Equations,* Kluwer Publications, Dordrecht, 1997.
[5] R. P. Agarwal and S. R. Grace, *Oscillation of certain third order difference equations*, Comput.Math.Applic.,42 (2001), 379–384.
[6] Y. Bolat and Ö. Akin, *Oscillatory behaviour of a higher-order nonlinear neutral type functional difference equation with oscillating coefficients*, Appl. Math. Lett., 17 (2004), 1073–1078.
[7] L. H. Erbe, Q. K. Kong, and B. G. Zang, *Oscillation Theory for Functional Differential Equations,* Marcel Dekker, New York, 1995.
[8] J. R. Graef and E. Thandapani, *Oscillatory and asymptotic behaviour of solutions of third order delay difference equations*, Funk.Ekv., 42 (1999), 355–369.
[9] V. Lakshmikantham and D. Trigiante, *Theory of Difference Equations, Numerical Methods and Applications,* Academic Press, New York, 1998.
[10] W. T. Li, *Classifications and existence of positive solutions of higher-order nonlinear neutral difference equations*, Appl. Math. Comp., 152 (2004), 351–366.
[11] M. Migda, E. Schmeidel and A. Drozdowicz, *Nonoscillation results for some third order nonlinear difference equations*, Colloquium on Differential and Difference Equations, Brno (2002), 185–192.
[12] S. H. Saker, *Oscillation of third order difference equations*, Portugal. Math., 61 (2004), 249–257.
[13] B. Smith, *Oscillatory and asymptotic behaviour in certain third order difference equations*, Rocky Mountain J.Math., 17 (1987), 597–606.
[14] B. Smith and W. E. Taylor, *Nonlinear third order difference equation: Oscillatory and asymptotic behaviour*, Tamk. J. Math., 19 (1988), 91–95.
[15] B. Smith and W. E. Taylor, *Asymptotic behaviour of third order difference equations*, Portugal. Math., 44 (1987), 113–117.
[16] B. Smith and W. E. Taylor, *Oscillation and nonoscillation in nonlinear third order difference equations*, Int. J. Math. Math. Sci., 13 (1990), 281–286.

[17] S. H. Saker, *Oscillation and asymptotic behavior of third order nonlinear neutral delay difference equations*, Dynamic Systems and Applications, 15 (2006), 549–568.

(Received: February 16, 2008)
(Revised: January 27, 2009)

E. Thandapani and I. M. Arockiasamy
Ramanujan Institute for Advanced
Study in Mathematics
University of Madras
Chepauk, Chennai-600 005, India
E–mail: ethandapani@yahoo.co.in

R. Karunakaran
Department of Mathematics
Periyar University
Salem-636 011, Tamilnadu, India