THE $G$–TRANSLATIVITY OF ABEL-TYPE TRANSFORMATIONS

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Abstract. Suppose $0 < t_n < 1$ and $\lim_{n \to \infty} t_n = 1$, then the Abel-type matrix, denoted by $A_{\alpha, t}$, is the matrix defined by

$$a_{nk} = \left(\frac{k + \alpha}{k}\right) t_n^k (1 - t_n)^{\alpha + 1}, \quad \alpha > -1.$$ 

Recently the author proved that the Abel-type matrix $A_{\alpha, t}$ is $\ell$-translative [2]. In this paper, we investigate the $G$-translativity of these transformations.

1. Basic notations and definitions

Let $A = (\alpha_{nk})$ be an infinite matrix defining a sequence to a sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} \alpha_{nk}x_k$$

where $(Ax)_n$ denotes the $n$th term of the image sequence $Ax$. Let $y$ be a complex number sequence. Throughout this paper, we shall use the following basic notations and definitions.

$$G = \left\{ y : y_k = 0 \left( r^k \right), \quad 0 < r < 1 \right\}$$

$$G(A) = \left\{ y : Ay \in G \right\}$$

$$c(A) = \left\{ y : y \text{ is summable by } A \right\}.$$

Definition 1.1. If $X$ and $Y$ are sets of complex number sequences, then the matrix $A$ is called an $X–Y$ matrix if the image $Au$ of $u$ under the transformation $A$ is in $Y$ whenever $u$ is in $X$. 

Definition 1.2. The summability matrix $A$ is said to be $G$-translative for the sequence $u = \{u_0, u_1, u_2, \ldots\}$ in $G(A)$ provided that each of the sequences $T_u$ and $S_u$ is in $G(A)$, where $T_u = \{u_1, u_2, \ldots\}$ and $S_u = \{0, u_0, u_1, \ldots\}$.

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Definition 1.3. The sequence \( x \) is said to be \( A_\alpha \)-summable to \( L \) if

1. \( \sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k x^k \) is convergent, for \( 0 < x < 1 \),
2. \( \lim_{x \to 1} (1 - x)^{\alpha + 1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k x^k = L \).

Here \( A_\alpha \) is the Abel-type power series method discussed by David Borwin in [1].

2. Main results

Proposition 2.1. If \( A_{\alpha,t} \) is a \( G - G \) matrix, then \( (1 - t)^{\alpha + 1} \in G \).

Proof. Suppose \( (1 - t)^{\alpha + 1} \) is not in \( G \). This implies that the first column of the matrix \( A_{\alpha,t} \) is not in \( G \) because \( a_n, 0 = (1 - t_n)^{\alpha + 1} \). Hence \( A_{\alpha,t} \) is not a \( G - G \) matrix. \( \square \)

Theorem 2.2. Every \( G - G \) \( A_{\alpha,t} \) matrix is \( G \)-translative for each sequences \( x \in G (A_{\alpha,t}) \) for which \( \left\{ \frac{x_k}{k+\alpha} \right\} \in G (A_{\alpha,t}) \), \( k = 1, 2, 3, \ldots \).

Proof. Suppose \( x \) is a sequence in \( G (A_{\alpha,t}) \) for which \( \left\{ \frac{x_k}{k+\alpha} \right\} \in G (A_{\alpha,t}) \). We will show that

1. \( T_x \in G (A_{\alpha,t}) \), and
2. \( S_x \in G (A_{\alpha,t}) \),

where \( T_x \) and \( S_x \) are as defined above. Let us first show that (1) holds. Note that

\[
|\langle A_{\alpha,t} \rangle_n| = \left| (1 - t_n)^{\alpha + 1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_{k+1} t_n^k \right| \\
= \frac{(1 - t_n)^{\alpha + 1}}{t_n} \left| \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_{k+1} t_n^{k+1} \right| \\
= \frac{(1 - t_n)^{\alpha + 1}}{t_n} \left| \sum_{k=1}^{\infty} \binom{k-1+\alpha}{k-1} x_k t_n^k \right| \\
= \frac{(1 - t_n)^{\alpha + 1}}{t_n} \left| \sum_{k=1}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \left( 1 - \frac{\alpha}{k+\alpha} \right) \right| \\
\leq A_n + B_n,
\]
where
\[ A_n = \frac{(1 - t_n)^{\alpha+1}}{t_n} \left| \sum_{k=1}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \right| \]
and
\[ B_n = \frac{|\alpha| (1 - t_n)^{\alpha+1}}{t_n} \left| \sum_{k=1}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \right|. \]

Now if we show that both \( \{A_n\} \) and \( \{B_n\} \) are in \( G \), then (1) holds. But the conditions that \( \{A_n\} \in G \) and \( \{B_n\} \in G \) follow easily from the assumptions that \( x \in G(A_{\alpha,t}) \) and \( \left\{ \frac{x_k}{k+\alpha} \right\} \in G(A_{\alpha,t}) \) respectively. Next, we will show that (2) holds as follows. We have
\[
\left| (A_{\alpha,t}Sx)_n \right| = (1 - t_n)^{\alpha+1} \left| \sum_{k=1}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \right|
\]
\[
= (1 - t_n)^{\alpha+1} \left| \sum_{k=0}^{\infty} \binom{k+\alpha+1}{k+1} x_k t_n^{k+1} \right|
\]
\[
= (1 - t_n)^{\alpha+1} \left| \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_k t_n^{k+1} \left( 1 + \frac{\alpha}{k+1} \right) \right|
\]
\[
\leq C_n + D_n,
\]
where
\[ C_n = (1 - t_n)^{\alpha+1} \left| \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \right| \]
and
\[ D_n = (1 - t_n)^{\alpha+1} |\alpha| \left| \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_k t_n^{k+1} \right|. \]

If we show that \( \{C_n\} \) and \( \{D_n\} \) are in \( G \), then (2) holds. But the assumptions that \( x \in G(A_{\alpha,t}) \) and \( \left\{ \frac{x_k}{k+\alpha} \right\} \in G(A_{\alpha,t}) \) imply that both \( \{C_n\} \) and \( \{D_n\} \) are in \( G \) respectively hence the theorem holds. \( \square \)

**Corollary 2.3.** Every \( G - G \) \( A_{\alpha,t} \) matrix is \( G \)-translative for each sequence \( x \in G(A_{\alpha,t}) \) for which \( \left\{ \frac{x_k}{k+\alpha} \right\} \in G, \ k = 1, 2, 3, \ldots \).
Theorem 2.4. Suppose \(-1 < \alpha \leq 0\), then every \(G - G A_{\alpha,t}\) matrix is \(G\)-translative for each \(A_{\alpha}\)-summable \([1]\) sequence \(x\) in \(G (A_{\alpha,t})\).

Proof. Since the assumption holds for \(\alpha = 0\), we will only consider the case \(-1 < \alpha < 0\).

Let \(x \in (c(A_{\alpha}) \cap G (A_{\alpha,t}))\). We will show that:

1. \(T_x \in G (A_{\alpha,t})\) and \(S_x \in G (A_{\alpha,t})\)

Let us first show that (1) holds.

\[
|(A_{\alpha,t}T_x)_n| = (1 - t_n)^{\alpha+1} \left| \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_{k+1} t_n^k \right|
\]

\[
= \frac{(1 - t_n)^{\alpha+1}}{t_n} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_{k+1} t_n^{k+1}
\]

\[
= \frac{(1 - t_n)^{\alpha+1}}{t_n} \sum_{k=1}^{\infty} \binom{k+\alpha}{k-1} x_k t_n^k
\]

\[
= \frac{(1 - t_n)^{\alpha+1}}{t_n} \sum_{k=1}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \frac{k}{k+\alpha}
\]

\[
= \frac{(1 - t_n)^{\alpha+1}}{t_n} \sum_{k=1}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \left(1 - \frac{\alpha}{k+\alpha}\right)
\]

\[
\leq E_n + F_n,
\]

where

\[
E_n = (1 - t_n)^{\alpha+1} \left| \sum_{k=1}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \right|
\]

and

\[
F_n = -\alpha (1 - t_n)^{\alpha+1} \left| \sum_{k=1}^{\infty} \binom{k+\alpha}{k} t_n^k \frac{x_k}{k+\alpha} \right|.
\]

Now if we show that both \(\{E_n\}\) and \(\{F_n\}\) are in \(G\) then (1) holds. From the conditions that \(\{E_n\} \in G\) follows the assumption that \(x \in G (A_{\alpha,t})\) and \(\{F_n\} \in G\) will be shown as follows. We have

\[
F_n < (1 - t_n)^{\alpha+1} |x_1| + (1 - t_n)^{\alpha+1} \left| \sum_{k=2}^{\infty} \binom{k+\alpha}{k} t_n^k \frac{x_k}{k+\alpha} \right| = P_n + Q_n,
\]

where

\[
P_n = (1 - t_n)^{\alpha+1} |x_1|\] and \(Q_n = (1 - t_n)^{\alpha+1} \left| \sum_{k=2}^{\infty} \binom{k+\alpha}{k} t_n^k \frac{x_k}{k+\alpha} \right|.
\]
where
\[ P_n = |x_1| (1 - t_n)^{\alpha + 1} \]
and
\[ Q_n = (1 - t_n)^{\alpha + 1} \left| \sum_{k=2}^{\infty} \left( \frac{k+\alpha}{k} \right) t_n^k \frac{x_k}{k + \alpha} \right|. \]

By Proposition 2.1, and the hypothesis that \( A_{\alpha,t} \in G \) we have that \( \{P_n\} \in G \), hence there remains only to show that \( \{Q_n\} \in G \) as \( \{F_n\} \in G \).

Observe that
\[ Q_n = (1 - t_n)^{\alpha + 1} \left| \sum_{k=2}^{\infty} \left( \frac{k+\alpha}{k} \right) t_n^k \int_0^{t_n} x_k t^{k+\alpha-1} dt \right|. \]

The interchanging of the integral and the summation is legitimate as the radius of convergence of the power series
\[ \sum_{k=2}^{\infty} \left( \frac{k+\alpha}{k} \right) x_k t^{k+\alpha-1} \]
is at least 1 and hence the power series converges absolutely and uniformly for \( 0 \leq t \leq t_n \). Now we let
\[ G(t) = \sum_{k=2}^{\infty} \left( \frac{k+\alpha}{k} \right) x_k t^{k+\alpha-1}. \]
Then, we have
\[ G(t) (1 - t)^{\alpha + 1} = (1 - t)^{\alpha + 1} \sum_{k=0}^{\infty} \left( \frac{k+\alpha}{k} \right) x_k t^{k+\alpha-1} \]
and the hypothesis that \( x \in c (A_{\alpha}) \) implies that
\[ \lim_{t \to t} G(t) (1 - t)^{\alpha + 1} = A \text{ (finite), for } 0 < t < 1. \] (i)

We also have
\[ \lim_{t \to 0} G(t) (1 - t)^{\alpha + 1} = 0. \] (ii)

Now (i) and (ii) yield that
\[ \left| G(t) (1 - t)^{\alpha + 1} \right| \leq M_1, \text{ for some } M_1 > 0, \]
and hence
\[ |G(t)| \leq M_1 (1 - t)^{-(\alpha + 1)}. \]
So, we have

\[ Q_n = \frac{(1 - t_n)^{\alpha + 1}}{t^{\alpha}} \left| \int_0^{t_n} G(t) \, dt \right| \]

\[ \Rightarrow Q_n \leq M_2 (1 - t_n)^{\alpha + 1} \int_0^{t_n} |G(t)| \, dt \] for some \( M_2 > 0 \)

\[ \leq M_1 M_2 (1 - t_n)^{\alpha + 1} \int_0^{t_n} (1 - t_n)^{-(\alpha + 1)} \, dt \]

\[ = \frac{M_1 M_2}{\alpha} (1 - t_n) \frac{1}{\alpha} (1 - t_n)^{\alpha + 1} \]

\[ \leq -2M_1 M_2 (1 - t_n)^{\alpha + 1}. \]

By Proposition 2.1 and the assumption that \( A_{\alpha, t} G - G \) we have that \((1 - t)^{\alpha + 1} \in G\), and hence \(\{Q_n\} \in G\). Next we show that (2) holds. We have

\[ \left| (A_{\alpha, t} S X)_n \right| = (1 - t_n)^{\alpha + 1} \sum_{k=1}^{\infty} \binom{k + \alpha}{k} x_{k-1} t_n^k \]

\[ = (1 - t_n)^{\alpha + 1} \sum_{k=0}^{\infty} \binom{k + \alpha + 1}{k + 1} x_k t_n^{k+1} \]

\[ = (1 - t_n)^{\alpha + 1} \sum_{k=0}^{\infty} \binom{k + \alpha}{k} x_k t_n^{k+1} \left( \frac{k + \alpha + 1}{k + 1} \right) \]

\[ = (1 - t_n)^{\alpha + 1} \sum_{k=0}^{\infty} \binom{k + \alpha}{k} x_k t_n^{k+1} \left( 1 + \frac{\alpha}{k + 1} \right) \]

\[ = R_n + S_n, \]

where

\[ R_n = (1 - t_n)^{\alpha + 1} \sum_{k=0}^{\infty} \binom{k + \alpha}{k} x_k t_n^k \]

and

\[ S_n = - (1 - t_n)^{\alpha + 1} \frac{\alpha}{\alpha} \sum_{k=0}^{\infty} \binom{k + \alpha}{k} \frac{x_k}{k + 1} t_n^{k+1}. \]

Now if we show that both \(\{R_n\}\) and \(\{S_n\}\) are in \(G\), then (2) follows. But the assumption that \(x \in G (A_{\alpha, t})\) implies that \(\{R_n\} \in G\), and \(\{S_n\} \in G\) follows using the same argument that we used in showing \(\{Q_n\} \in G\) before.

\[ \square \]
Theorem 2.5. Suppose $0 < \alpha$ and $1 - t \in G$, then every $G - G$ $A_{\alpha,t}$ matrix is $G$-translative for each $A_{\alpha}$-summable sequence $x$ in $G(A_{\alpha,t})$.

Proof. Let $x \in (c(A_{\alpha}) \cap G(A_{\alpha,t}))$. We will show that:

1. $T_x \in G(A_{\alpha,t})$ and
2. $S_x \in G(A_{\alpha,t})$.

Let us first show that (1) holds.

\[
\left| (A_{\alpha,t} T_x)_n \right| = (1 - t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_k t_n^k
\]

\[
= (1 - t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha+1}{k+1} x_{k+1} t_n^{k+1}
\]

\[
= (1 - t_n)^{\alpha+1} \sum_{k=1}^{\infty} \binom{k-1+\alpha}{k-1} x_k t_n^k
\]

\[
= (1 - t_n)^{\alpha+1} \sum_{k=1}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \left( 1 - \frac{\alpha}{k+\alpha} \right)
\]

\[
\leq H_n + L_n
\]

where

\[
H_n = (1 - t_n)^{\alpha+1} t_n \sum_{k=1}^{\infty} \binom{k+\alpha}{k} x_k t_n^k
\]

and

\[
L_n = -\alpha (1 - t_n)^{\alpha+1} t_n \sum_{k=1}^{\infty} \binom{k+\alpha}{k} x_k t_n^k x_k \frac{1}{k+\alpha}.
\]

Now if we show that both $\{H_n\}$ and $\{L_n\}$ are in $G$ hence (1) holds. But the conditions that $\{H_n\} \in G$ follows from the assumption that $x \in G(A_{\alpha,t})$ and $\{L_n\} \in G$ will be shown as follows. Note that

\[
L_n < (1 - t_n)^{\alpha+1} |x_1| + (1 - t_n)^{\alpha+1} \sum_{k=2}^{\infty} \binom{k+\alpha}{k} t_n^k x_k \frac{1}{k+\alpha} = Y_n + Z_n,
\]

where

\[
Y_n = |x_1| (1 - t_n)^{\alpha+1}
\]
and

\[ Z_n = (1 - t_n)^{\alpha+1} \left| \sum_{k=2}^{\infty} \binom{k+\alpha}{k} t_n^k x_k \right|. \]

By Proposition 2.1, the assumption that \( A_{\alpha,t} \) \( G \)-translative for the sequence \( x \) such that \( \sum_{k=1}^{\infty} x_k \) has bounded partial sum.

Proposition 2.6. Suppose \(-1 < \alpha \leq 0\), then every \( G \)-translative for the sequence \( x \) such that \( \sum_{k=1}^{\infty} x_k \) has bounded partial sum.

Proposition 2.7. Every \( G \)-translative for the unbounded sequence \( x \) given by

\[ x_k = (-1)^k (k + 1). \]

Proposition 2.8. Every \( G \)-translative for each sequence \( x \) \in \( G \).

Proposition 2.9. Suppose \(-1 < \alpha \leq 0\), then every \( G \)-translative for the sequence \( x \) such that \( \sum_{k=1}^{\infty} x_k \) is conditionally convergent.
**References**


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