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THE G–TRANSLATIVITY OF ABEL-TYPE TRANSFORMATIONS

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Abstract. Suppose $0 < t_n < 1$ and $\lim_{n\to\infty} t_n = 1$, then the Abel-type matrix, denoted by $A_{\alpha,t}$, is the matrix defined by

$$a_{nk} = \binom{k+\alpha}{k} t_n^k (1-t_n)^{\alpha+1}, \quad \alpha > -1.$$

Recently the author proved that the Abel-type matrix $A_{\alpha,t}$ is ℓ-translative [2]. In this paper, we investigate the G-translativity of these transformations.

1. Basic notations and definitions

Let $A = (\alpha_{nk})$ be an infinite matrix defining a sequence to a sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} \alpha_{nk} x_k$$

where $(Ax)_n$ denotes the nth term of the image sequence Ax. Let y be a complex number sequence. Throughout this paper, we shall use the following basic notations and definitions.

$$G = \left\{ y : y_k = 0\left(r^k\right), \quad 0 < r < 1 \right\}$$
$$G(A) = \{y : Ay \in G\}$$
$$c(A) = \{y : y \text{ is summable by } A\}.$$

Definition 1.1. *If X and Y are sets of complex number sequences, then the matrix A is called an $X - Y$ matrix if the image Au of u under the transformation A is in Y whenever u is in X.*

Definition 1.2. *The summability matrix A is said to be G-translative for the sequence $u = \{u_0, u_1, u_2, \dots\}$ in $G(A)$ provided that each of the sequences T_u and S_u is in $G(A)$, where $T_u = \{u_1, u_2, \dots\}$ and $S_u = \{0, u_0, u_1, \dots\}$.*

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Definition 1.3. *The sequence x is said to be A_α-summable to L if*

(1) $\sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k x^k$ *is convergent, for* $0 < x < 1$,

(2) $\lim_{x\to 1} (1-x)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k x^k = L$.

Here A_α is the Abel-type power series method discussed by David Borwin in [1].

2. Main results

Proposition 2.1. *If $A_{\alpha,t}$ is a $G-G$ matrix, then $(1-t)^{\alpha+1} \in G$.*

Proof. Suppose $(1-t)^{\alpha+1}$ is not in G. This implies that the first column of the matrix $A_{\alpha,t}$ is not in G because $a_n,\, 0 = (1-t_n)^{\alpha+1}$. Hence $A_{\alpha,t}$ is not a $G-G$ matrix. □

Theorem 2.2. *Every $G-G$ $A_{\alpha,t}$ matrix is G-translative for each sequences $x \in G(A_{\alpha,t})$ for which $\left\{\frac{x_k}{k+\alpha}\right\} \in G(A_{\alpha,t})$, $k = 1, 2, 3, \ldots$.*

Proof. Suppose x is a sequence in $G(A_{\alpha,t})$ for which $\left\{\frac{x_k}{k+\alpha}\right\} \in G(A_{\alpha,t})$. We will show that

(1) $T_x \in G(A_{\alpha,t})$, and
(2) $S_x \in G(A_{\alpha,t})$,

where T_x and S_x are as defined above. Let us first show that (1) holds. Note that

$$
\begin{aligned}
\left|(A_{\alpha,t})_n\right| &= (1-t_n)^{\alpha+1} \left| \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_{k+1} t_n^k \right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n} \left| \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_{k+1} t_n^{k+1} \right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n} \left| \sum_{k=1}^{\infty} \binom{k-1+\alpha}{k-1} x_k t_n^k \right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n} \left| \sum_{k=1}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \frac{k}{k+\alpha} \right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n} \left| \sum_{k=1}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \left(1 - \frac{\alpha}{k+\alpha}\right) \right| \\
&\le A_n + B_n,
\end{aligned}
$$

where

$$A_n = \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\right|$$

and

$$B_n = \frac{|\alpha|\,(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+\alpha}t_n^k\right|.$$

Now if we show that both $\{A_n\}$ and $\{B_n\}$ are in G, then (1) holds. But the conditions that $\{A_n\} \in G$ and $\{B_n\} \in G$ follow easily from the assumptions that $x \in G\,(A_{\alpha,t})$ and $\left\{\frac{x_k}{k+\alpha}\right\} \in G\,(A_{\alpha,t})$ respectively. Next, we will show that (2) holds as follows. We have

$$\begin{aligned}\left|(A_{\alpha,t}S_x)_n\right| &= (1-t_n)^{\alpha+1}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_{k-1}t_n^k\right|\\ &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha+1}{k+1}x_k t_n^{k+1}\right|\\ &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k+1}\left(\frac{k+\alpha+1}{k+1}\right)\right|\\ &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k+1}\left(1+\frac{\alpha}{k+1}\right)\right|\\ &\le C_n + D_n,\end{aligned}$$

where

$$C_n = (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\right|$$

and

$$D_n = (1-t_n)^{\alpha+1}\,|\alpha|\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+1}t_n^{k+1}\right|.$$

If we show that $\{C_n\}$ and $\{D_n\}$ are in G, then (2) holds. But the assumptions that $x \in G\,(A_{\alpha,t})$ and $\left\{\frac{x_k}{k+\alpha}\right\} \in G\,(A_{\alpha,t})$ imply that both $\{C_n\}$ and $\{D_n\}$ are in G respectively hence the theorem holds. □

Corollary 2.3. *Every $G-G$ $A_{\alpha,t}$ matrix is G-translative for each sequence $x \in G\,(A_{\alpha,t})$ for which $\left\{\frac{x_k}{k+\alpha}\right\} \in G$, $k = 1, 2, 3, \ldots$.*

Theorem 2.4. *Suppose $-1 < \alpha \leq 0$, then every $G - G$ $A_{\alpha,t}$ matrix is G-translative for each A_α-summable* [1] *sequence x in $G(A_{\alpha,t})$.*

Proof. Since the assumption holds for $\alpha = 0$, we will only consider the case $-1 < \alpha < 0$.

Let $x \in (c(A_\alpha) \cap G(A_{\alpha,t}))$. We will show that:

(1) $T_x \in G(A_{\alpha,t})$ and
(2) $S_x \in G(A_{\alpha,t})$

Let us first show that (1) holds.

$$\begin{aligned}
\left|(A_{\alpha,t}T_x)_n\right| &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^k\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^{k+1}\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k-1+\alpha}{k-1}x_k t_n^k\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\frac{k}{k+\alpha}\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\left(1-\frac{\alpha}{k+\alpha}\right)\right| \\
&\leq E_n + F_n,
\end{aligned}$$

where

$$E_n = (1-t_n)^{\alpha+1}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\right|$$

and

$$F_n = -\alpha(1-t_n)^{\alpha+1}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}t_n^k\frac{x_k}{k+\alpha}\right|.$$

Now if we show that both $\{E_n\}$ and $\{F_n\}$ are in G then (1) holds. From the conditions that $\{E_n\} \in G$ follows the assumption that $x \in G(A_{\alpha,t})$ and $\{F_n\} \in G$ will be shown as follows. We have

$$F_n < (1-t_n)^{\alpha+1}|x_1| + (1-t_n)^{\alpha+1}\left|\sum_{k=2}^{\infty}\binom{k+\alpha}{k}t_n^k\frac{x_k}{k+\alpha}\right| = P_n + Q_n,$$

where

$$P_n = |x_1|\,(1-t_n)^{\alpha+1}$$

and

$$Q_n = (1-t_n)^{\alpha+1}\left|\sum_{k=2}^{\infty}\binom{k+\alpha}{k} t_n^k \frac{x_k}{k+\alpha}\right|.$$

By Proposition 2.1, and the hypothesis that $A_{\alpha,t}\ G-G$ we have that $\{P_n\}\in G$, hence there remains only to show that $\{Q_n\}\in G$ as $\{F_n\}\in G$. Observe that

$$\begin{aligned} Q_n &= (1-t_n)^{\alpha+1}\left|\sum_{k=2}^{\infty}\binom{k+\alpha}{k} x_k \left(\int_0^{t_n} t_n^{k+\alpha-1}dt\right)\right| \\ &= \frac{(1-t_n)^{\alpha+1}}{t_n^{\alpha+1}}\left|\int_0^{t_n} dt\left(\sum_{k=2}^{\infty}\binom{k+\alpha}{k} x_k t^{k+\alpha-1}\right)\right|. \end{aligned}$$

The interchanging of the integral and the summation is legitimate as the radius of convergence of the power series 1

$$\sum_{k=2}^{\infty}\binom{k+\alpha}{k} x_k t^{k+\alpha-1}$$

is at least 1 and hence the power series converges absolutely and uniformly for $0\le t\le t_n$. Now we let

$$G(t)=\sum_{k=2}^{\infty}\binom{k+\alpha}{k} x_k t^{k+\alpha-1}.$$

Then, we have

$$G(t)\,(1-t)^{\alpha+1} = (1-t)^{\alpha+1}\sum_{k=0}^{\infty}\binom{k+\alpha}{k} x_k t^{k+\alpha-1}$$

and the hypothesis that $x\in c\,(A_\alpha)$ implies that

$$\lim_{t\to\bar{t}} G\,(t)\,(1-t)^{\alpha+1} = A \ \text{(finite)}, \ \text{for } 0<t<1. \tag{i}$$

We also have

$$\lim_{t\to 0} G\,(t)\,(1-t)^{\alpha+1} = 0. \tag{ii}$$

Now (i) and (ii) yield that

$$\left|G\,(t)\,(1-t)^{\alpha+1}\right| \le M_1, \ \text{for some } M_1>0,$$

and hence

$$|G\,(t)| \le M_1\,(1-t)^{-(a+1)}.$$

So, we have

$$
\begin{aligned}
& Q_n = \frac{(1-t_n)^{\alpha+1}}{t^{\alpha}} \left| \int_0^{t_n} G(t)\ dt \right| \\
\Rightarrow\ & Q_n \le M_2 (1-t_n)^{\alpha+1} \int_0^{t_n} |G(t)|\ dt \text{ for some } M_2 > 0 \\
& \le M_1 M_2 (1-t_n)^{\alpha+1} \int_0^{t_n} (1-t_n)^{-(\alpha+1)}\, dt \\
& = \frac{M_1 M_2}{\alpha}(1-t_n) - \frac{M_1 M_2}{\alpha}(1-t_n)^{\alpha+1} \\
& \le \frac{-2M_1 M_2}{\alpha}(1-t_n)^{\alpha+1}.
\end{aligned}
$$

By Proposition 2.1 and the assumption that $A_{\alpha,t}\ G - G$ we have that $(1-t)^{(\alpha+1)} \in G$, and hence $\{Q_n\} \in G$. Next we show that (2) holds. We have

$$
\begin{aligned}
\left|(A_{\alpha,t} S_x)_n\right| &= (1-t_n)^{\alpha+1} \left| \sum_{k=1}^{\infty} \binom{k+\alpha}{k} x_{k-1} t_n^k \right| \\
&= (1-t_n)^{\alpha+1} \left| \sum_{k=0}^{\infty} \binom{k+\alpha+1}{k+1} x_k t_n^{k+1} \right| \\
&= (1-t_n)^{\alpha+1} \left| \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_k t_n^{k+1} \left(\frac{k+\alpha+1}{k+1} \right) \right| \\
&= (1-t_n)^{\alpha+1} \left| \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_k t_n^{k+1} \left(1 + \frac{\alpha}{k+1} \right) \right| \\
&= R_n + S_n,
\end{aligned}
$$

where

$$
R_n = (1-t_n)^{\alpha+1} \left| \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \right|
$$

and

$$
S_n = -(1-t_n)^{\alpha+1}\, \alpha \left| \sum_{k=0}^{\infty} \binom{k+\alpha}{k} \frac{x_k}{k+1} t_n^{k+1} \right|.
$$

Now if we show that both $\{R_n\}$ and $\{S_n\}$ are in G, then (2) follows. But the assumption that $x \in G(A_{\alpha,t})$ implies that $\{R_n\} \in G$, and $\{S_n\} \in G$ follows using the same argument that we used in showing $\{Q_n\} \in G$ bebore. □

Theorem 2.5. *Suppose $0 < \alpha$ and $1 - t \in G$, then every $G - G$ $A_{\alpha,t}$ matrix is G-translative for each A_α-summable sequence x in $G(A_{\alpha,t})$.*

Proof. Let $x \in (c(A_\alpha) \cap G(A_{\alpha,t}))$. We will show that:

(1) $T_x \in G(A_{\alpha,t})$ and
(2) $S_x \in G(A_{\alpha,t})$.

Let us first show that (1) holds.

$$\begin{aligned}
\left|(A_{\alpha,t}T_x)_n\right| &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^k\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=0}^{\infty}\binom{k+\alpha+1}{k}x_{k+1}t_n^{k+1}\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k-1+\alpha}{k-1}x_k t_n^k\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\frac{k}{k+\alpha}\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\left(1-\frac{\alpha}{k+\alpha}\right)\right| \\
&\leq H_n + L_n
\end{aligned}$$

where

$$H_n = (1-t_n)^{\alpha+1}t_n\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\right|$$

and

$$L_n = -\alpha(1-t_n)^{\alpha+1}t_n\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\frac{x_k}{k+\alpha}\right|.$$

Now if we show that both $\{H_n\}$ and $\{L_n\}$ are in G hence (1) holds. But the conditions that $\{H_n\} \in G$ follows from the assumption that $x \in G(A_{\alpha,t})$ and $\{L_n\} \in G$ will be shown as follows. Note that

$$L_n < (1-t_n)^{\alpha+1}|x_1| + (1-t_n)^{\alpha+1}\left|\sum_{k=2}^{\infty}\binom{k+\alpha}{k}t_n^k\frac{x_k}{k+\alpha}\right| = Y_n + Z_n,$$

where

$$Y_n = |x_1|(1-t_n)^{\alpha+1}$$

and

$$Z_n = (1-t_n)^{\alpha+1} \left| \sum_{k=2}^{\infty} \binom{k+\alpha}{k} t_n^k \frac{x_k}{k+\alpha} \right|.$$

By Proposition 2.1, the assumption that $A_{\alpha,t}$ $G{-}G$ implies that $\{Y_n\} \in G$, hence remains only to show $\{Z_n\} \in G$ to show that $\{L_n\} \in G$. Observe that

$$\begin{aligned} Z_n &= (1-t_n)^{\alpha+1} \left| \sum_{k=2}^{\infty} \binom{k+\alpha}{k} x_k \left(\int_0^{t_n} t_n^{k+\alpha-1} dt \right) \right| \\ &= \frac{(1-t_n)^{\alpha+1}}{t_n^{\alpha}} \left| \int_0^{t_n} dt \left(\sum_{k=2}^{\infty} \binom{k+\alpha}{k} x_k t^{k+\alpha} \right) \right|. \end{aligned}$$

Now following the same argument we used for Q_n above in the proof of Theorem 2.4 it follows that

$$Z_n \leq \frac{M_1 M_2}{\alpha} (1-t_n) - \frac{M_1 M_2}{\alpha} (1-t_n)^{\alpha+1}.$$

By Proposition 2.1, the assumptions that $A_{\alpha,t}$ is $G-G$ implies that $(1-t)^{(\alpha+1)} \in G$. Now $(1-t)^{(\alpha+1)} \in G$ and the assumption that $(1-t) \in G$ imply that $Z \in G$. Next we show that (2) holds by showing $S_x \in G(A_{\alpha,t})$. But this can be easily shown using the same argument used in showing $S_x \in G(A_{\alpha,t})$ in Theorem 2.4. Hence the theorem holds. □

The following basic facts can be easily proved. We state them here as propositions with out proofs.

Proposition 2.6. *Suppose $-1 < \alpha \leq 0$, then every $G-G$ $A_{\alpha,t}$ matrix is G-translative for the sequence x such that $\sum_{k=1}^{\infty} x_k$ has bounded partial sum.*

Proposition 2.7. *Every $G-G$ $A_{\alpha,t}$ matrix is G-translative for the unbounded sequence x given by*

$$x_k = (-1)^k (k+1).$$

Proposition 2.8. *Every $G-G$ $A_{\alpha,t}$ matrix is G-translative for each sequence $x \in G$.*

Proposition 2.9. *Suppose $-1 < \alpha \leq 0$, then every $G-G$ $A_{\alpha,t}$ matrix is G-translative for the sequence x such that $\sum_{k=1}^{\infty}$ is conditionally convergent.*

References

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