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TRIGONOMETRIC APPROXIMATION OF FUNCTIONS IN WEIGHTED L^p SPACES

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Abstract. The approximation properties of means of trigonometric Fourier series in weighted L^p spaces $(1 < p < \infty)$ with Muckenhoupt weights are investigated.

1. Introduction and results

A measurable 2π-periodic function $w : [0, 2\pi] \to [0, \infty]$ is said to be a weight function if the set $w^{-1}(\{0, \infty\})$ has the Lebesque measure zero. We denote by $L^p_w = L^p_w [0, 2\pi]$, where $1 \leq p < \infty$ and w a weight function, the weighted Lebesque space of all measurable 2π- periodic functions f, that is, the space of all such functions for which

$$\|f\|_{p,w} = \left(\int_0^{2\pi} |f(x)|^p \, w(x) \, dx \right)^{1/p} < \infty.$$

Let $1 < p < \infty$. A weight function w belongs to the Muckenhoupt class $\mathcal{A}_p$ if

$$\sup_I \left(\frac{1}{|I|} \int_I w(x) \, dx \right) \left(\frac{1}{|I|} \int_I [w(x)]^{-1/p-1} \, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all intervals I with length $|I| \leq 2\pi$.

The weight functions belong to the $\mathcal{A}_p$ class, introduced by Muckenhoupt ([13]), play a very important role in different fields of mathematical analysis.

Let $1 < p < \infty$, $w \in \mathcal{A}_p$ and let $f \in L^p_w$. The modulus of continuity of the function f is defined by

$$\Omega(f, \delta)_{p,w} = \sup_{|h| \leq \delta} \|\Delta_h(f)\|_{p,w}, \quad \delta > 0,$$

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where

$$\Delta_h(f)(x) = \frac{1}{h}\int_0^h |f(x+t) - f(x)|\, dt.$$

The existence of the modulus $\Omega(f,\delta)_{p,w}$ follows from the boundedness of the Hardy-Littlewood maximal function in the space L^p_w (see [13]). The modulus of continuity $\Omega(f,\cdot)_{p,\omega}$, defined by N. X. Ky [10], is nondecreasing, nonnegative, continuous function such that

$$\lim_{\delta\to 0}\Omega(f,\delta)_{p,\omega} = 0, \quad \Omega(f_1+f_2,\cdot)_{p,\omega} \le \Omega(f_1,\cdot)_{p,\omega} + \Omega(f_2,\cdot)_{p,\omega}.$$

The modulus of continuity $\Omega(f,\cdot)_{p,w}$ is defined in this way, since the space L^p_w is noninvariant, in general, under the usual shift $f(x)\to f(x+h)$. Note that, in the case $w\equiv 1$ the modulus $\Omega(f,\cdot)_{p,\omega}$ and the classical integral modulus of continuity $\omega_p(f,\cdot)$ are equivalent (see [10]).

We define the Lipschitz class $Lip(\alpha,p,w)$ for $0<\alpha\le 1$ by

$$Lip(\alpha,p,w) = \left\{ f\in L^p_w : \Omega(f,\delta)_{p,w} = O(\delta^\alpha), \quad \delta>0 \right\}.$$

Let $f\in L^1$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx). \tag{1}$$

Let $S_n(f)(x)$, $(n=0,1,\dots)$ be the nth partial sums of the series (1) at the point x, that is,

$$S_n(f)(x) = \sum_{k=0}^{n} A_k(f)(x),$$

where

$$A_0(f)(x) = \frac{a_0}{2}, \quad A_k(f)(x) = a_k\cos kx + b_k\sin kx, \quad k=1,2,\dots .$$

Let $(p_n)_0^\infty$ be a sequence of positive numbers. We consider two means of the series (1) defined by

$$N_n(f)(x) = \frac{1}{P_n}\sum_{m=0}^{n} p_{n-m}S_m(f)(x)$$

and

$$R_n(f)(x) = \frac{1}{P_n}\sum_{m=0}^{n} p_m S_m(f)(x),$$

where $P_n = \sum_{m=0}^{n} p_m$, $p_{-1} = P_{-1} := 0$. In the case $p_n = 1, n \geq 0$, both of $N_n(f)(x)$ and $R_n(f)(x)$ are equal to the Cesàro mean

$$\sigma_n(f)(x) = \frac{1}{n+1} \sum_{m=0}^{n} S_m(f)(x).$$

The approximation properties of the means σ_n in Lipschitz classes $Lip(\alpha, p)$, $1 \leq p < \infty$, $0 < \alpha \leq 1$ were investigated by Quade in [14]. The generalizations of Quade's results were studied by Mohapatra and Russell [12], Chandra ([1], [2], [3], [4]) and Leindler [11]. In [1], Chandra obtained estimates for $\|f - N_n(f)\|_p$, where $1 < p < \infty$. Chandra also gave estimates for the difference $\|f - R_n(f)\|_p$, where $f \in Lip(\alpha, p)$, $1 < p < \infty$, $0 < \alpha \leq 1$ (see [2]). In the paper [4], Chandra gave some conditions on the sequence $(p_n)_0^\infty$ and obtained very satisfactory results about approximation by the means $N_n(f)$ and $R_n(f)$ in $Lip(\alpha, p)$, $1 \leq p < \infty$, $0 < \alpha \leq 1$.

In the present paper, we give the weighted versions of the results obtained by Chandra in [4] in the case $1 < p < \infty$. Our main results are the following.

Theorem 1. *Let* $1 < p < \infty$, $w \in \mathcal{A}_p$, $0 < \alpha \leq 1$, *and let* $(p_n)_0^\infty$ *be a monotonic sequence of positive real numbers such that*

$$(n+1) p_n = O(P_n). \tag{2}$$

Then, for every $f \in Lip(\alpha, p, w)$ *the estimate*

$$\|f - N_n(f)\|_{p,w} = O\left(n^{-\alpha}\right), \quad n = 1, 2, \ldots$$

holds.

Theorem 2. *Let* $1 < p < \infty$, $w \in \mathcal{A}_p$, $0 < \alpha \leq 1$, *and let* (p_n) *be a sequence of positive real numbers satisfying the relation*

$$\sum_{m=0}^{n-1} \left| \frac{P_m}{m+1} - \frac{P_{m+1}}{m+2} \right| = O\left(\frac{P_n}{n+1}\right). \tag{3}$$

Then, for $f \in Lip(\alpha, p, w)$ *the estimate*

$$\|f - R_n(f)\|_{p,w} = O\left(n^{-\alpha}\right), \quad n = 1, 2, \ldots$$

satisfied.

If we take $p_n = A_n^{\beta-1}$ $(\beta > 0)$, where

$$A_0^\beta = 1, \quad A_k^\beta = \frac{\beta(\beta+1)\ldots(\beta+k)}{k!}, \quad k \geq 1,$$

we get

$$N_n(f)(x) = \sigma_n^\beta(f)(x) = \frac{1}{A_n^\beta} \sum_{m=0}^{n} A_{n-m}^{\beta-1} S_m(f)(x).$$

Hence we can estimate the deviation of $f \in Lip(\alpha, p, w)$ from the Cesàro means $\sigma_n^\beta(f)$:

Corollary 3. *Let* $1 < p < \infty$, $w \in \mathcal{A}_p$, $0 < \alpha \leq 1$ *and* $\beta > 0$. *Then, for* $f \in Lip(\alpha, p, w)$,

$$\left\| f - \sigma_n^\beta(f) \right\|_{p,w} = O\left(n^{-\alpha}\right), \quad n = 1, 2, \ldots .$$

The trigonometric approximation problems in weighted L^p spaces with Muckenhoupt weights where $1 < p < \infty$ were studied by several authors. Gadjieva [5] obtained the direct and inverse theorems of trigonometric approximation in the spaces L_w^p. Later, Ky investigated the same problems and obtained similar results by using a different modulus of continuity, which in special case coincides with the modulus $\Omega(f, \cdot)_{p,\omega}$ ([9], [10]). The improvement of the inverse theorem of Gadjieva was obtained in [6]. Later, in the more general spaces, namely weighted Orlicz spaces, the direct and inverse theorems of trigonometric approximation and the complete characterization of the generalized Lipschitz classes were obtained [8].

Remark. Theorem 1, Theorem 2 and Corollary 3 also hold in reflexive weighted Orlicz spaces L_w^M.

The general information on weighted Orlicz spaces and approximation results in these spaces can be found in [8].

2. Some auxiliary results

Lemma 4. *Let* $1 < p < \infty$, $w \in \mathcal{A}_p$ *and* $0 < \alpha \leq 1$. *Then, the estimate*

$$\| f - S_n(f) \|_{p,w} = O\left(n^{-\alpha}\right) \tag{4}$$

holds for every $f \in Lip(\alpha, p, w)$ *and* $n = 1, 2, \ldots$.

Proof. Let t_n^* $(n = 0, 1, \ldots)$ be the trigonometric polynomial of best approximation to f, that is,

$$\| f - t_n^* \|_{p,w} = \inf \| f - t_n \|_{p,w},$$

where the infimum is taken over all trigonometric polynomials t_n of degree at most n. From Theorem 2 of [10], we have

$$\| f - t_n^* \|_{p,w} = O\left(\Omega(f, 1/n)_{p,w}\right)$$

and hence

$$\| f - t_n^* \|_{p,w} = O\left(n^{-\alpha}\right).$$

By the uniform boundedness of the partial sums $S_n(f)$ in the space L_w^p (see [7]), we get

$$\| f - S_n(f) \|_{p,w} \leq \| f - t_n^* \|_{p,w} + \| t_n^* - S_n(f) \|_{p,w}$$

$$= \|f - t_n^*\|_{p,w} + \|S_n(t_n^* - f)\|_{p,w}$$
$$= O\left(\|f - t_n^*\|_{p,w}\right)$$
$$= O\left(n^{-\alpha}\right).$$

□

Lemma 5. *Let $1 < p < \infty$ and $w \in \mathcal{A}_p$. Then, for $f \in Lip(1, p, w)$ the estimate*

$$\|S_n(f) - \sigma_n(f)\|_{p,w} = O\left(n^{-1}\right), \quad n = 1, 2, ... \tag{5}$$

holds.

Proof. If $f \in Lip(1, p, w)$, from Theorem 3 of [10] it can be deduced that f is absolutely continuous and $f' \in L_w^p$. If f has the Fourier series

$$f(x) \sim \sum_{k=0}^{\infty} A_k(f)(x),$$

then the Fourier series of the conjugate function $\widetilde{f'}$ is

$$\widetilde{f'}(x) \sim \sum_{k=1}^{\infty} kA_k(f)(x).$$

On the other hand,

$$S_n(f)(x) - \sigma_n(f)(x) = \sum_{k=1}^{n} \frac{k}{n+1} A_k(f)(x)$$
$$= \frac{1}{n+1} S_n\left(\widetilde{f'}\right)(x).$$

Hence, by considering the uniform boundedness of the partial sums and the conjugation operator in the space L_w^p (see [7]), we obtain

$$\|S_n(f) - \sigma_n(f)\|_{p,w} = O\left(n^{-1}\right)$$

for $n = 1, 2, \ldots$. □

Lemma 6. ([4]). *Let (p_n) be a non-increasing sequence of positive numbers. Then,*

$$\sum_{m=1}^{n} m^{-\alpha} p_{n-m} = O\left(n^{-\alpha} P_n\right)$$

for $0 < \alpha < 1$.

3. Proof of the new results

Proof of Theorem 1. Let $0<\alpha<1$. Since

$$f(x)=\frac{1}{P_n}\sum_{m=0}^{n}p_{n-m}f(x),$$

we have

$$f(x)-N_n(f)(x)=\frac{1}{P_n}\sum_{m=0}^{n}p_{n-m}\left\{f(x)-S_m(f)(x)\right\}.$$

By Lemma 4, Lemma 6 and condition (2) we obtain

$$\begin{aligned}\|f-N_n(f)\|_{p,w}&\le\frac{1}{P_n}\sum_{m=0}^{n}p_{n-m}\|f-S_m(f)\|_{p,w}\\&=\frac{1}{P_n}\sum_{m=1}^{n}p_{n-m}O\left(m^{-\alpha}\right)+\frac{p_n}{P_n}\|f-S_0(f)\|_{p,w}\\&=\frac{1}{P_n}O\left(n^{-\alpha}P_n\right)+O\left(\frac{1}{n+1}\right)\\&=O\left(n^{-\alpha}\right).\end{aligned}$$

Now let $\alpha=1$. It is clear that

$$N_n(f)(x)=\frac{1}{P_n}\sum_{m=0}^{n}P_{n-m}A_m(f)(x).$$

By Abel transform,

$$\begin{aligned}S_n(f)(x)-N_n(f)(x)&=\frac{1}{P_n}\sum_{m=1}^{n}\left(P_n-P_{n-m}\right)A_m(f)(x)\\&=\frac{1}{P_n}\sum_{m=1}^{n}\left(\frac{P_n-P_{n-m}}{m}-\frac{P_n-P_{n-(m+1)}}{m+1}\right)\left(\sum_{k=1}^{m}kA_k(f)(x)\right)\\&\qquad+\frac{1}{n+1}\sum_{k=1}^{n}kA_k(f)(x),\end{aligned}$$

and hence

$$\begin{aligned}\|S_n(f)-N_n(f)\|_{p,w}&\le\frac{1}{P_n}\sum_{m=1}^{n}\left|\frac{P_n-P_{n-m}}{m}-\frac{P_n-P_{n-(m+1)}}{m+1}\right|\\&\qquad\times\left\|\sum_{k=1}^{m}kA_k(f)\right\|_{p,w}+\frac{1}{n+1}\left\|\sum_{k=1}^{n}kA_k(f)\right\|_{p,w}.\end{aligned}$$

Since

$$S_n(f)(x) - \sigma_n(f)(x) = \frac{1}{n+1}\sum_{k=1}^{n} kA_k(f)(x),$$

by Lemma 5 we get

$$\left\|\sum_{k=1}^{n} kA_k(f)\right\|_{p,w} = (n+1)\left\|S_n(f) - \sigma_n(f)\right\|_{p,w} = O(1).$$

Hence,

$$\begin{aligned}
&\left\|S_n(f) - N_n(f)\right\|_{p,w} \\
&\quad\leq \frac{1}{P_n}\sum_{m=1}^{n}\left|\frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1}\right| O(1) + O\left(n^{-1}\right) \\
&\quad= O\left(\frac{1}{P_n}\right)\sum_{m=1}^{n}\left|\frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1}\right| + O\left(n^{-1}\right). \quad (6)
\end{aligned}$$

By a simple computation, one can see that

$$\frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1} = \frac{1}{m(m+1)}\left(\sum_{k=n-m+1}^{n} p_k - mp_{n-m}\right),$$

which shows that

$$\left(\frac{P_n - P_{n-m}}{m}\right)_{m=1}^{n+1}$$

is non-increasing whenever (p_n) is non-decreasing and non-decreasing whenever (p_n) is non-increasing. This implies that

$$\sum_{m=1}^{n}\left|\frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1}\right| = \left|p_n - \frac{P_n}{n+1}\right| = \frac{1}{n+1}O(P_n).$$

This and the inequality (6) yields

$$\left\|S_n(f) - N_n(f)\right\|_{p,w} = O\left(n^{-1}\right).$$

Combining the last estimate with (4) we obtain

$$\left\|f - N_n(f)\right\|_{p,w} = O\left(n^{-1}\right).$$

□

Proof of Theorem 2. Let $0 < \alpha < 1$. By definition of $R_n(f)(x)$,

$$f(x) - R_n(f)(x) = \frac{1}{P_n}\sum_{m=0}^{n} p_m \left\{f(x) - S_m(f)(x)\right\}.$$

From Lemma 4, we get

$$\begin{aligned}\|f-R_n(f)\|_{p,w} &\leq \frac{1}{P_n}\sum_{m=0}^{n} p_m \|f-S_m(f)\|_{p,w} \\ &= O\left(\frac{1}{P_n}\right)\sum_{m=1}^{n} p_m m^{-\alpha} + \frac{p_0}{P_n}\|f-S_0(f)\|_{p,w} \\ &= O\left(\frac{1}{P_n}\right)\sum_{m=1}^{n} p_m m^{-\alpha}. \qquad (7)\end{aligned}$$

By Abel transform,

$$\begin{aligned}\sum_{m=1}^{n} p_m m^{-\alpha} &= \sum_{m=1}^{n-1} P_m \left\{m^{-\alpha}-(m+1)^{-\alpha}\right\} + n^{-\alpha}P_n \\ &\leq \sum_{m=1}^{n-1} m^{-\alpha}\frac{P_m}{m+1} + n^{-\alpha}P_n,\end{aligned}$$

and

$$\begin{aligned}\sum_{m=1}^{n-1} m^{-\alpha}\frac{P_m}{m+1} &= \sum_{m=1}^{n-1}\left(\frac{P_m}{m+1}-\frac{P_{m+1}}{m+2}\right)\left(\sum_{k=1}^{m} k^{-\alpha}\right) + \frac{P_n}{n+1}\sum_{m=1}^{n-1} m^{-\alpha} \\ &= O\left(n^{-\alpha}P_n\right)\end{aligned}$$

by condition (3). This yields

$$\sum_{m=1}^{n} p_m m^{-\alpha} = O\left(n^{-\alpha}P_n\right)$$

and from this and (7) we get

$$\|f-R_n(f)\|_{p,w} = O\left(n^{-\alpha}\right).$$

Let's consider the case $\alpha = 1$. By Abel transform,

$$\begin{aligned}R_n(f)(x) &= \frac{1}{P_n}\sum_{m=0}^{n-1}\left\{P_m\left(S_m(f)(x)-S_{m+1}(f)(x)\right)+P_nS_n(f)(x)\right\} \\ &= \frac{1}{P_n}\sum_{m=0}^{n-1} P_m\left(-A_{m+1}(f)(x)\right) + S_n(f)(x),\end{aligned}$$

and hence

$$R_n(f)(x) - S_n(f)(x) = -\frac{1}{P_n}\sum_{m=0}^{n-1} P_m A_{m+1}(f)(x).$$

Using Abel transform again yields

$$\begin{aligned}\sum_{m=0}^{n-1} P_m A_{m+1}(f)(x) &= \sum_{m=0}^{n-1} \frac{P_m}{m+1}(m+1) A_{m+1}(f)(x) \\ &= \sum_{m=0}^{n-1} \left(\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right)\left(\sum_{k=0}^{m}(k+1) A_{k+1}(f)(x)\right) \\ &\quad + \frac{P_n}{n+1}\sum_{k=0}^{n-1}(k+1) A_{k+1}(f)(x).\end{aligned}$$

Thus, by considering (5) and (3) we obtain

$$\begin{aligned}\left\|\sum_{m=0}^{n-1} P_m A_{m+1}(f)\right\|_{p,w} &\leq \sum_{m=0}^{n-1}\left|\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right|\left\|\sum_{k=0}^{m}(k+1) A_{k+1}(f)\right\|_{p,w} \\ &\quad + \frac{P_n}{n+1}\left\|\sum_{k=0}^{n-1}(k+1) A_{k+1}(f)\right\|_{p,w} \\ &= \sum_{m=0}^{n-1}\left|\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right|(m+2)\left\|S_{m+1}(f) - \sigma_{m+1}(f)\right\|_{p,w} \\ &\quad + P_n\left\|S_n(f) - \sigma_n(f)\right\|_{p,w} \\ &= O(1)\sum_{m=0}^{n-1}\left|\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right| + O\left(\frac{P_n}{n}\right).\end{aligned}$$

This gives

$$\begin{aligned}\left\|R_n(f) - S_n(f)\right\|_{p,w} &= \frac{1}{P_n}\left\|\sum_{m=0}^{n-1} P_m A_{m+1}(f)\right\|_{p,w} \\ &= \frac{1}{P_n} O\left(\frac{P_n}{n}\right) = O\left(\frac{1}{n}\right).\end{aligned}$$

Combining this estimate with (4) yields

$$\left\|f - R_n(f)\right\|_{p,w} = O\left(n^{-1}\right).$$

□

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