STABILITY OF ALMOST CLOSED OPERATORS ON A HILBERT SPACE

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Abstract. We introduce the notion of almost closed linear operators acting in a Hilbert space. This class of operators contains the set of all closed linear operators and is invariant under addition, composition and limits.

1. Introduction

Let us denote by $B(H)$ the algebra of bounded linear operators on a separable Hilbert space $H$ equipped with an inner product $\langle \cdot, \cdot \rangle_H$, $(\|x\|_H = \sqrt{\langle x, x \rangle_H})$. $C(H)$ is the set of all closed linear operators of dense domain in $H$. If $A \in C(H)$, the domain of $A$ is denoted by $D(A)$ and its graph by $G(A) = \{ (x, Ax) : x \in D(A) \}$, in particular $G(A)$ is a closed subspace of $H \oplus H$. $N(A)$ and $R(A)$ denotes respectively the null space and the range of $A$. The adjoint of $A$ is denoted by $A^*$ and $I$ is the identity operator on $H$.

The natural operations sum, product and limits are well defined on $B(H)$. This is thanks to the domain of the bounded operators which is always taken to be the whole Hilbert space $H$.

However, one has to be careful with those manipulations when dealing with unbounded operators, this is essentially due to the domains.

If $A, B \in C(H)$, their sum $A + B$ and product $AB$ are respectively defined on $H$ by

\[
\begin{align*}
(A + B)x &= Ax + Bx, \quad \text{for all } x \in D(A + B) = D(A) \cap D(B) \\
ABx &= A(Bx), \quad \text{for all } x \in B^{-1}(D(A))
\end{align*}
\]

when the operators $A + B$ and $AB$ can just not make any sense.

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The first deficiency is that if \( D(A + B) \) or \( D(AB) \) is trivial, i.e. it reduces to zero even if strong conditions are imposed on \( A \) and \( B \).

Let’s recall that the Fourier transformation \( \mathcal{F} \) defines a unitary operator on \( L^2(\mathbb{R}) \) and \( C_0^\infty(\mathbb{R}) \). This last result is known as the Paley-Wiener theorem (see e.g. [12], [13]), it permits us to construct an example of two unbounded linear operators with trivial sum.

If \( A\varphi(x) = x\varphi(x) \) and \( B\varphi(x) = x^2\varphi(x) \) defined on \( L^2(\mathbb{R}) \) with the following domains:

\[
D(A) = C_0^\infty(\mathbb{R}) \quad \text{and} \quad D(B) = \{ \varphi \in C^\infty(\mathbb{R}); \mathcal{F}\varphi \in C_0^\infty(\mathbb{R}) \}.
\]

Then, \( A \) and \( B \) are essentially selfadjoint and \( D(A + B) = \{0\} \).

For the product, P.R. Chernoff [2] gave a simpler and explicit semibounded operator \( A \) satisfying \( D(A^2) = \{0\} \). His idea was based on the Cayley transform of unbounded operators. \( A \) is the multiplication operator by \( i(F + 1)(F - 1)^{-1} \), where \( F \) is some complex function defined on the Hardy space on the unit circle.

The second deficiency is that if \( A, B \in C(H) \), then \( A + B \) and \( AB \) need not be closed on \( H \).

For instance, if we take \( (e_n)_{n \in \mathbb{N}} \) be a basis of \( H \), \( a_n = e_{2n} \) and \( b_n = e_{2n} + \frac{1}{(n+1)}e_{2n+1} \) for all \( n \in \mathbb{N} \).

Let \( M \) and \( N \) the two linear subspaces of \( H \) respectively spanned by \( (a_n)_{n \in \mathbb{N}} \) and \( (b_n)_{n \in \mathbb{N}} \). Then, \( M \cap N = \{0\} \) and \( M + N \) is not closed in \( H \).

Let \( A \) and \( B \) be the projection operators respectively on \( M \) and \( N \). Then \( A \) and \( B \) are closed operators on \( H \), but \( A + B \) is only the restriction of the identity to \( M + N \). \( A + B \) could not be closed operator since \( M + N \) is not closed in \( H \).

An example with \( A \) bounded in \( H \) and \( B \) closed, but \( AB \) not closed is given in [10].

In [9] the example of an unbounded selfadjoint operators \( A = -i \frac{d}{dx} \) and \( B \) the multiplication operator by \( |x| \) on their respective domains \( D(A) = H^1(\mathbb{R}) \) the Sobolev space \( \{ \varphi \in L^2(\mathbb{R}) ; \varphi' \in L^2(\mathbb{R}) \} \) (\( \varphi' \) is the derivative of \( \varphi \) in the sense of distributions) and \( D(B) = \{ \varphi \in L^2(\mathbb{R}) ; |x| \varphi \in L^2(\mathbb{R}) \} \) is given. Then \( D(AB) = \{ \varphi \in L^2(\mathbb{R}) ; |x| \varphi, -i(|x| \varphi)' \in L^2(\mathbb{R}) \} \) is dense in \( L^2(\mathbb{R}) \) since it contains \( C_0^\infty(\mathbb{R}) \), but \( AB \) is certainly not closed.

In particular, the sum \( A + B \) and the product \( AB \) of two closed operators \( A \) and \( B \) is closed if some standard necessary conditions are imposed on \( A \) and \( B \) (see e.g. [8], [11]).

Different definitions of a product of closed operators were given which have properties not shared by the usual product.

J. Dixmier [4], gave a new definition of a product \( \times \) such that \( A \times B \in C(H) \) if \( A, B \in C(H) \). We say that \( x \in D(A \times B) \) and \( y = (A \times B)x \) if
there exists two sequences, \((x_n)_n\) in \(D(B)\) and \((y_n)_n\) in \(R(A)\), \(x_n \longrightarrow x\) and \(y_n \longrightarrow y\), such that \(A^{-1}y_n - Bx_n \longrightarrow 0\) (for some well choosen \(A^{-1}y_n\) and \(Bx_n\)).

The MM-product proposed by B. Messirdi and M.H. Mortad in [10] is based upon the bissecting \(F(A)\) of an operator \(A\) in \(C(H)\), \(F(A) = AS_A(I + S_A)^{-1}\) if \(S_A = \sqrt{R_A}\) and \(R_A = (I + A^*A)^{-1}\); \(R_A, S_A, F(A) \in B(H)\). If \(A\) and \(B\) are in \(C(H)\), this product is defined by \(A \bullet B = F^{-1}(F(A)F(B))\).

The product \(\bullet\) is an internal law on \(C(H)\) but not commutative. It is, however, associative but it does not have an identity element. Nevertheless, this law has a fundamental property about adjoints that is not shared by the usual product and the product of Dixmier in the unbounded case. We remark here that the MM-product can be also adapted to the sum of unbounded operators on \(H\).

Now if we talk about adjoints, the results are not better because the adjoint of the sum and the product are generally not equal to the sum and the product of adjoints. The following relations hold for closed linear operators on \(H\) (see e.g. [8],[11]):

1) \(A^* + B^* \subset (A + B)^*\)
2) \((A + B)^* = A^* + B^*\) if \(A \in B(H)\)
3) \(B^*A^* \subset (AB)^*\) if \(D(AB)\) is dense in \(H\)
4) \((AB)^* = B^*A^*\) if \(A \in B(H)\)
5) \((AB)^* = B^*A^*\) if \(D(AB)\) is dense in \(H\) and \(B^{-1} \in B(H)\)
6) \((AB)^* = B^* \times A^*\) and \((A \times B)^* = B^*A^*\)
7) \((A \bullet B)^* = B^* \bullet A^*\).

It’s known that evolution problems will, in general, lead to not closable operators with “bad” spectral properties (see for further examples [7]). To avoid the problems with closures altogether and to be able to treat for example linear evolution equations for all linear operators appearing in applications, some authors have tried to weaken the closedness of operators ([1],[4],[10]). We introduce in this paper the notion of almost closed operators on \(H\). This class of operators contain \(C(H)\) and is invariant under addition, composition and limits.

Almost closed operators satisfy the usual properties of the adjoint of linear operators.

We will here assume that the basic space \(H\) to be a Hilbert space, this is done mostly for convenience, the almost closed operators can be considered also on Banach spaces.

2. Almost closed linear operators

It is interesting to recall in the beginning the well-known procedure of making a closed linear operator \(A\) bounded on \(H\) by renorming its domain
with the graph norm \( \|x\|_A = (\|x\|^2_H + \|Ax\|^2_H)^{1/2} \) defined by the graph inner product \( \langle x, y \rangle_A = \langle x, y \rangle_H + \langle Ax, Ay \rangle_H \) for all \( x, y \in D(A) \).

**Proposition 1.** Let \( A \) be an unbounded linear operator on \( H \) with domain \( D(A) \) dense in \( H \). Then, \( A \in C(H) \) if and only if \( (D(A), \langle ., . \rangle_A) \) is a Hilbert space.

Proposition 1 implies that the existence of such a procedure permits us to define a new class of weakly closed linear operators as follows:

**Definition 2.** A linear operator \( A \) with domain \( D(A) \) is called almost closed on a Hilbert space \( H \) if there exists an inner product \( \langle ., . \rangle_A \) on \( D(A) \) such that \( H/A = (D(A), \langle ., . \rangle_A) \) is complete, \( H/A \hookrightarrow H \) and \( A \) is continuous from \( H/A \) onto \( H \) (i.e. \( A \in B(H_A, H) \)).

If \( A \) is almost closed operator then \( D(A) \) is paracomplete or operator range and we can observe that \( A \) can always be decomposed in a certain special ways (see [3],[5],[6]). This procedure of factorization suggests to us to use the notion of almost closable operators. These interesting questions will be developed in another paper.

Obviously, if \( H_A \) is a Hilbert space, then \( A \) is almost closed if and only if the graph \( G(A) \) of \( A \) is closed in \( H_A \oplus H \), thus if \((x_n), n \in \mathbb{N} \subset D(A) \) converges to \( x \) in \( H_A \) and \((Ax_n), n \in \mathbb{N} \) converges to \( y \) in \( H \), then \( x \in D(A) \) and \( y = Ax \).

Before to study this class of operators and to show that the property of being almost closed is algebraically stable, we remark that an important class of examples of almost closed operators are sums and products of operators of \( C(H) \).

Let \( A, B \in C(H) \), such that \( D(A) \cap D(B) \) and \( D(BA) \) are not trivial. Then, \( A + B \) and \( BA \) are almost closed on \( H \).

Indeed, we choose as \( H_{A+B} \) and \( H_{BA} \) the Hilbert space \( H_A \) which is \( D(A) \) equipped with the graph inner product \( \langle ., . \rangle_A \). If \( (x_n), n \in \mathbb{N} \subset D(A) \cap D(B) \) converges to \( x \) in \( H_A \) and \( ((A + B)x_n), n \in \mathbb{N} \) converges to \( y \) in \( H \), then,

\[
\begin{align*}
x_n &\rightarrow x \text{ in } H, \ x \in D(A) \\
Ax_n &\rightarrow Ax \text{ in } H \\
Bx_n &\rightarrow y - Ax \text{ in } H.
\end{align*}
\]

Since \( B \) is closed, we deduce that \( x \in D(B) \) and \( Bx = y - Ax \). Then, it follows that \( A + B \) is closed from \( H_A \) onto \( H \). The proof is analogue for the product \( BA \).

The class of almost closed operators contains \( C(H) \) (see proposition 1), but there exists almost closed operators \( A \) which are not closable if for example the graph \( G(A) \) of \( A \) is dense in \( H \oplus H \).
We can also construct closable linear operators which are not almost closed. Consider the identity operator $I$ on the Hilbert space $l^2$ of sequences $f = (f_n)_{n \in \mathbb{N}}$ of real or complex numbers such that $\sum_{n=0}^{\infty} |f_n|^2 < +\infty$, with domain $l^2_0$. $l^2_0$ is the subset of $l^2$ consisting of all sequences $(f_n)_{n \in \mathbb{N}}$ such that $f_n = 0$ for all $n \geq n_0$.

Clearly, $I$ is closable but not almost closed. Since if we suppose $I$ almost closed, then there exists a Hilbert space $H \hookrightarrow l^2$ such that the graph $G(I) = l^2_0 \oplus l^2_0$ is closed in $H \oplus l^2$. Thus, $G(I)$ is a complete metric space. However $G(I)$ is also the union of countably many finite dimensional subspaces and is thus of first category. But, by Baire’s theorem, complete metric spaces are of the second category, which is a contradiction.

We establish now some fundamental properties of almost closedness.

**Theorem 3.** Let $A$ be almost closed operator on $H$ with domain $D(A)$ and associated Hilbert space $H_A = (D(A), [\cdot, \cdot]_A)$. Then,

1) $N(A)$ is closed linear subspace of $H_A$,
2) If $D(A) = H$, then $A \in \mathcal{B}(H)$,
3) If $A$ is invertible, then $A^{-1}$ is also almost closed on $H$. In particular, if $R(A) = H$ and $A$ is invertible then $A \in \mathcal{B}(H)$.

**Proof.** 1) Follows directly from Definition 2.

2) The identity operator in $H$ is bijective and bicontinuous by the open mapping theorem. Consequently, the topologies induced on $H$ by $\|\cdot\|_H$ and $\|\cdot\|_{H_A}$ are equivalent. Then, $A \in \mathcal{B}(H)$ since $A \in \mathcal{B}(H_A, H)$.

3) $D(A^{-1}) = R(A)$ is a Hilbert space, denoted by $H_{A^{-1}}$, with respect to the metric generated by the inner product:

$$[y, z]_{A^{-1}} = (y, z)_H + [A^{-1}y, A^{-1}z]_A,$$

for all $y, z \in R(A)$.

Furthermore,

$$\|A^{-1}y\|_{H_A} \leq \|y\|_{H_{A^{-1}}}, \quad \text{for all } y \in R(A)$$

where $\|y\|_{H_{A^{-1}}} = ([y, y]_{A^{-1}})^{1/2}$.

Since $H_A \hookrightarrow H$, we obtain that $A^{-1} \in \mathcal{B}(H_{A^{-1}}, H)$ and thus $A^{-1}$ is almost closed on $H$.

**Theorem 4.** Let $A$ be almost closed operator on $H$. Then, $A$ has a densely defined almost closed extension.

**Proof.** Suppose $D(A)$ is not dense in $H$, then its closure $\overline{D(A)}$ has a complement $\hat{D}$ in $H$. Let $P$ the orthogonal projection of $H$ on $\overline{D(A)}$ along $\hat{D}$. $P$ is closed operator and $AP$ is an almost closed extension of $A$.

We need to verify that sums, products and limits of almost closed operators are also almost closed operators.
Theorem 5. If $A$ and $B$ are almost closed operators on $H$, then $A + B$ and $AB$ are also almost closed when $D(A) \cap D(B) \neq \{0\}$ and $D(AB) \neq \{0\}$.

Proof. We take $H_{A+B} = (D(A) \cap D(B), [, ,]_{A+B})$ and $H_{AB} = (D(AB), [, ,]_{AB})$, where :

$$\begin{cases} [x, y]_{A+B} = [x, y]_A + [x, y]_B + \langle Ax, Ay \rangle_H, \text{ for all } x, y \in D(A) \cap D(B) \\
[x, y]_{AB} = [x, y]_B + [Bx, By]_A, \text{ for all } x, y \in D(AB). \end{cases}$$

Consequently, $H_{A+B}$ and $H_{AB}$ are Hilbert spaces, $H_{A+B} \hookrightarrow H$ and $H_{AB} \hookrightarrow H$.

i) Let $(x_n)_{n \in \mathbb{N}}$ a sequence of vectors of $H_{A+B}$ that converges to $x$ in $H_{A+B}$. Then, $(x_n)_{n \in \mathbb{N}}$ converges to $x$ respectively in $H_A$ and $H_B$ and $(Ax_n)_{n \in \mathbb{N}}$ converges to $Ax$ in $H$. The boundedness of $B$ from $H_B$ onto $H$ implies that $(Bx_n)_{n \in \mathbb{N}}$ converges to $Bx$ in $H$ and then $((A + B)x_n)_{n \in \mathbb{N}}$ converges to $(A + B)x$ in $H$. Hence, $(A + B) \in B(H_{A+B}, H)$.

ii) If $(x_n)_{n \in \mathbb{N}}$ converges to $x$ in $H_{AB}$, then $(x_n)_{n \in \mathbb{N}}$ converges to $x$ in $H_B$ and $(Bx_n)_{n \in \mathbb{N}}$ converges to $Bx$ in $H_A$. According to the boundedness of $A$ from $H_A$ onto $H$, we deduce that $(ABx_n)_{n \in \mathbb{N}}$ converges to $ABx$ in $H$. Consequently, $AB \in B(H_{AB}, H)$.

□

Theorem 6. For all $\varepsilon > 0$, let $A_\varepsilon$ be almost closed operator on $H$ with the associated Hilbert space $H_\varepsilon = (D(A_\varepsilon), [, ,]_{A_\varepsilon})$. Assume that there exists an Hilbert space $L$ such that $L \hookrightarrow H_\varepsilon$ for all $\varepsilon > 0$ and $\sup_{\varepsilon > 0} \|A_\varepsilon x\|_H < +\infty$, for all $x \in L$.

Then, $Ax = \lim_{\varepsilon \to 0} A_\varepsilon x$ with domain

$$D(A) = \left\{ x \in \bigcap_{\varepsilon > 0} D(A_\varepsilon) \cap L; \lim_{\varepsilon \to 0} A_\varepsilon x \text{ exists in } H \right\}$$

is almost closed operator on $H$.

Proof. We take $H_A = (D(A), [, ,]_A)$, where

$$[x, y]_A = \langle x, y \rangle_L + \lim_{\varepsilon \to 0} \langle A_\varepsilon x, A_\varepsilon y \rangle_H = \langle x, y \rangle_L + \langle Ax, Ay \rangle_H, \text{ for all } x, y \in D(A).$$

$[ , ,]_A$ is clearly an inner product on $D(A)$, we show that $H_A$ is complete. Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $H_A$. Then, $(x_n)_{n \in \mathbb{N}}$ is Cauchy in $L$ and thus in $H_\varepsilon$ for all $\varepsilon > 0$. Thus, $(x_n)_{n \in \mathbb{N}}$ converges to $x$ in $L$, $H$ and $H_\varepsilon$ for all $\varepsilon > 0$, then $x \in \bigcap_{\varepsilon > 0} D(A_\varepsilon) \cap L$. It follows from the almost closedness of the operators $H_\varepsilon$ that $(A_\varepsilon x_n)_{n \in \mathbb{N}}$ converges to $A_\varepsilon x$ in $H$ for all $\varepsilon > 0$.

Since $(x_n)_{n \in \mathbb{N}}$ is Cauchy in $H_A$ there exists $C > 0$ such that

$$\|x_n\|_{H_A} = [x_n, x_n]_A^{1/2} \leq C, \text{ for all } n \in \mathbb{N}$$
and
\[ \|x\|_{H_A}^2 = \lim_{n \to +\infty} \|x_n\|_L^2 + \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \|A_\varepsilon x_n\|_H^2 \leq 2C^2. \]

We obtain that \( x \in H_A \).

Let \( \eta > 0 \). Then, by virtue of the assumption \( \sup_{\varepsilon > 0} \|A_\varepsilon x\|_H < +\infty \) on \( L \) and the uniform boundedness principle, there exists \( i_0 \in \mathbb{N} \) such that for all \( n, m \geq i_0 \) and \( \varepsilon > 0 \),
\[ \|x_n - x_m\|_{H_A} \leq \frac{\eta}{2}, \quad \|A_\varepsilon x_n - A_\varepsilon x_m\|_H \leq C \frac{\eta}{2}. \]
Moreover, we have
\[ \|x_n - x\|_{H_A} = \left( \lim_{n \to +\infty} \|x_n - x_m\|_L^2 + \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \|A_\varepsilon x_n - A_\varepsilon x_m\|_H^2 \right)^{1/2} \leq \frac{\eta}{2} \left(1 + C^2\right)^{1/2}, \text{ for all } n \geq i_0. \]

Consequently, \( H_A \) is a Hilbert space, \( H_A \hookrightarrow H \) and \( A \) is bounded from \( H_A \) onto \( H \). \( \square \)

We know that almost closedness is stable under an almost closed perturbation. We now try to extend this stability to a relatively bounded perturbation.

**Theorem 7.** Let \( A \) and \( B \) be unbounded linear operators on \( H \), and let \( B \) be \( A \)-bounded with \( A \)-bound smaller than 1. Then \( A + B \) is almost closed if and only if \( A \) is almost closed.

**Proof.** \( D(A) \subset D(B) \) and for all \( x \in D(A) \) we have
\[ \|Bx\|_H \leq a \|Ax\|_H + b \|x\|_H \]
where \( a, b \) are nonnegative constants, \( a < 1 \).

Hence, we obtain for all \( x \in D(A) \)
\[ \begin{cases} \|Ax\|_H \leq \frac{1}{1-a} \|(A + B)x\|_H + \frac{b}{1-a} \|x\|_H, \\ \|(A + B)x\|_H \leq (1 + a) \|Ax\|_H + b \|x\|_H. \end{cases} \]

If \( A \) is almost closed there exists a Hilbert space \( H_A \) such that \( H_A \hookrightarrow H \) and \( A \in \mathcal{B}(H_A, H) \). \( D(A + B) = D(A) \), let \( H_{A+B} = H_A \). Then, there exists \( K > 0 \) such that for all \( x \in D(A) \)
\[ \|(A + B)x\|_H \leq K \|Ax\|_H + \|x\|_{H_A} \]
\[ \leq K \left(1 + \|A\|_{\mathcal{B}(H_A, H)}\right) \|x\|_{H_A}. \]

Thus, \( (A + B) \) is almost closed on \( H \).
Conversely, if \((A+B)\) is almost closed on a Hilbert space \(H_{A+B}, H_{A+B} \hookrightarrow H\) and \((A+B) \in \mathcal{B}(H_{A+B}, H)\), let again \(H_A = H_{A+B}\). Then, there exists \(K' > 0\) such that for all \(x \in D(A)\)

\[
\|Ax\|_H \leq K'[\|(A+B)x\|_H + \|x\|_{H_A}]
\leq K'(1 + \|A+B\|_{\mathcal{B}(H_A,H)}) \|x\|_{H_A}.
\]

This shows that \(A\) is almost closed on \(H\). \(\square\)

**Remark 8.** If \(A \in \mathcal{B}(H)\) and \(B\) is almost closed operator on \(H\), then \((A+B)^* = A^* + B^*\) and \((AB)^* = B^*A^*\).

This follows directly from the almost closedness of \(B\) and the properties of adjoint in the continuous case.

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**References**


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