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STABILITY OF ALMOST CLOSED OPERATORS ON A HILBERT SPACE

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Abstract. We introduce the notion of almost closed linear operators acting in a Hilbert space. This class of operators contains the set of all closed linear operators and is invariant under addition, composition and limits.

1. Introduction

Let us denote by $\mathcal{B}(H)$ the algebra of bounded linear operators on a separable Hilbert space H equipped with an inner product $\langle .,. \rangle_H$,$(\|x\|_H = [\langle x,x\rangle_H]^{1/2})$. $C(H)$ is the set of all closed linear operators of dense domain in H. If $A \in C(H)$, the domain of A is denoted by $D(A)$ and its graph by $G(A) = \{(x, Ax)\ ;\ x \in D(A)\}$, in particular $G(A)$ is a closed subspace of $H \oplus H$. $N(A)$ and $R(A)$ denotes respectively the null space and the range of A. The adjoint of A is denoted by A^* and I is the identity operator on H.

The natural operations sum, product and limits are well defined on $\mathcal{B}(H)$. This is thanks to the domain of the bounded operators which is always taken to be the whole Hilbert space H.

However, one has to be careful with those manipulations when dealing with unbounded operators, this is essentially due to the domains.

If $A, B \in C(H)$, their sum $A+B$ and product AB are respectively defined on H by

$$\begin{cases} (A+B)x = Ax + Bx, & \text{for all } x \in D(A+B) = D(A) \cap D(B) \\ ABx = A(Bx), & \text{for all } x \in B^{-1}(D(A)) \end{cases}$$

when the operators $A + B$ and AB can just not make any sense.

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The first deficiency is that if $D(A+B)$ or $D(AB)$ is trivial, i.e it reduces to zero even if strong conditions are imposed on A and B.

Let's recall that the Fourier transformation $\mathcal{F}$ defines a unitary operator on $L^2(\mathbb{R})$ and $C_0^\infty\mathbb{R}\cap\mathcal{F}C_0^\infty(\mathbb{R})=\{0\}$. This last result is known as the Paley-Wiener theorem (see e.g.[12],[13]), it permits us to construct an example of two unbounded linear operators with trivial sum.

If $A\varphi(x)=x\varphi(x)$ and $B\varphi(x)=x^2\varphi(x)$ defined on$L^2(\mathbb{R})$ with the following domains:

$$D(A)=C_0^\infty(\mathbb{R}) \text{ and } D(B)=\left\{\varphi\in C^\infty(\mathbb{R});\, \mathcal{F}\varphi\in C_0^\infty(\mathbb{R})\right\}.$$

Then, A and B are essentially selfadjoint and $D(A+B)=\{0\}$.

For the product, P.R. Chernoff [2] gave a simpler and explicit semi-bounded operator A satisfying $D(A^2)=\{0\}$. His idea was based on the Cayley transform of unbounded operators. A is the multiplication operator by $i(F+1)(F-1)^{-1}$, where F is some complex function defined on the Hardy space on the unit circle.

The second deficiency is that if $A,B\in C(H)$, then $A+B$ and AB need not be closed on H.

For instance, if we take $(e_n)_{n\in\mathbb{N}}$ be a basis of H, $a_n=e_{2n}$ and $b_n=e_{2n}+\frac{1}{(n+1)}e_{2n+1}$ for all $n\in\mathbb{N}$.

Let M and N the two linear subspaces of H respectively spaned by $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$. Then, $M\cap N=\{0\}$ and $M+N$ is not closed in H.

Let A and B be the projection operators respectively on M and N. Then A and B are closed operators on H, but $A+B$ is only the restriction of the identity to $M+N$. $A+B$ could not be closed operator since $M+N$ is not closed in H.

An example with A bounded in H and B closed, but AB not closed is given in [10].

In [9] the example of an unbounded selfadjoint operators $A=-i\frac{d}{dx}$ and B the multiplication operator by $|x|$ on their respective domains $D(A)=H^1(\mathbb{R})$ the Sobolev space $\{\varphi\in L^2(\mathbb{R})\ ;\ \varphi'\in L^2(\mathbb{R})\}$ (φ' is the derivative of φ in the sense of distributions) and $D(B)=\{\varphi\in L^2(\mathbb{R})\ ;\ |x|\,\varphi\in L^2(\mathbb{R})\}$ is given. Then $D(AB)=\{\varphi\in L^2(\mathbb{R})\ ;\ |x|\,\varphi,\ -i(|x|\,\varphi)'\in L^2(\mathbb{R})\}$ is dense in $L^2(\mathbb{R})$ since it contains $C_0^\infty(\mathbb{R})$, but AB is certainly not closed.

In particular, the sum $A+B$ and the product AB of two closed operators A and B is closed if some standard necessary conditions are imposed on A and B (see e.g. [8], [11]).

Different definitions of a product of closed operators were given which have properties not shared by the usual product.

J. Dixmier [4], gave a new definition of a product $\times$ such that $A\times B\in C(H)$ if $A,B\in C(H)$. We say that $x\in D(A\times B)$ and $y=(A\times B)x$ if there exists two sequences, $(x_n)_n$ in $D(B)$ and $(y_n)_n$ in $R(A)$, $x_n \longrightarrow x$ and $y_n \longrightarrow y$, such that $A^{-1}y_n - Bx_n \longrightarrow 0$ (for some well choosen $A^{-1}y_n$ and Bx_n).

The MM-product proposed by B. Messirdi and M.H. Mortad in [10] is based upon the bissecting $F(A)$ of an operator A in $C(H)$, $F(A) = AS_A(I+S_A)^{-1}$ if $S_A = \sqrt{R_A}$ and $R_A = (I+ A^*A)^{-1}$; R_A,$S_A, F(A) \in \mathcal{B}(H)$. If A and B are in $C(H)$, this product is defined by $A \bullet B = F^{-1}(F(A)F(B))$.

The product $\bullet$ is an internal law on $C(H)$ but not commutative. It is, however, associative but it does not have an identity element. Neverthless, this law has a fundamental property about adjoints that is not shared by the usual product and the product of Dixmier in the unbounded case. We remark here that the MM-product can be also adapted to the sum of unbounded operators on H.

Now if we talk about adjoints, the results are not better because the adjoint of the sum and the product are generally not equal to the sum and the product of adjoints. The following relations hold for closed linear operators on H (see e.g. [8],[11]):

1) $A^* + B^* \subset (A + B)^*$
2) $(A + B)^* = A^* + B^*$ if $A \in \mathcal{B}(H)$
3) $B^*A^* \subset (AB)^*$ if $D(AB)$ is dense in H
4) $(AB)^* = B^*A^*$ if $A \in \mathcal{B}(H)$
5) $(AB)^* = B^*A^*$ if $D(AB)$ is dense in H and $B^{-1} \in \mathcal{B}(H)$
6) $(AB)^* = B^* \times A^*$ and $(A \times B)^* = B^*A^*$
7) $(A \bullet B)^* = B^* \bullet A^*$.

It's known that evolution problems will, in general, lead to not closable operators with "bad" spectral properties (see for further examples [7]). To avoid the problems with closures altogether and to be able to treat for example linear evolution equations for all linear operators appearing in applications, some authors have tried to weaken the closedness of operators ([1],[4],[10]). We introduce in this paper the notion of almost closed operators on H. This class of operators contain $C(H)$ and is invariant under addition, composition and limits.

Almost closed operators satisfy the usual properties of the adjoint of linear operators.

We will here assume that the basic space H to be a Hilbert space, this is done mostly for convenience, the almost closed operators can be considered also on Banach spaces.

2. Almost closed linear operators

It is interesting to recall in the beginning the well-known procedure of making a closed linear operator A bounded on H by renorming its domain with the graph norm $\|x\|_A = (\|x\|_H^2 + \|Ax\|_H^2)^{1/2}$ defined by the graph inner product $\langle x, y\rangle_A = \langle x, y\rangle_H + \langle Ax, Ay\rangle_H$ for all $x, y \in D(A)$.

Proposition 1. *Let A be an unbounded linear operator on H with domain $D(A)$ dense in H. Then, $A \in C(H)$ if and only if $(D(A), \langle ., .\rangle_A)$ is a Hilbert space.*

Proposition 1, implies that the existence of such a procedure permits us to define a new class of weakly closed linear operators as follows :

Definition 2. *A linear operator A with domain $D(A)$ is called almost closed on a Hilbert space H if there exists an inner product $[., .]_A$ on $D(A)$ such that $H_A = (D(A), [., .]_A)$ is complete, $H_A \hookrightarrow H$ and A is continuous from H_A onto H (ie. $A \in \mathcal{B}(H_A, H)$).*

If A is almost closed operator then $D(A)$ is paracomplete or operator range and we can observe that A can always be decomposed in a certain special ways (see [3],[5],[6]). This procedure of factorization suggests to us to use the notion of almost closable operators. These interesting questions will be developed in another paper.

Obviously, if H_A is a Hilbert space, then A is almost closed if and only if the graph $G(A)$ of A is closed in $H_A \oplus H$, thus if $(x_n)_n$ converges to x in H_A and $(Ax_n)_n$ converges to y in H, then $x \in D(A)$ and $y = Ax$.

Before to study this class of operators and to show that the property of being almost closed is algebraically stable, we remark that an important class of examples of almost closed operators are sums and products of operators of $C(H)$.

Let $A, B \in C(H)$, such that $D(A) \cap D(B)$ and $D(BA)$ are not trivial. Then, $A + B$ and BA are almost closed on H.

Indeed, we choose as H_{A+B} and H_{BA} the Hilbert space H_A which is $D(A)$ equipped with the graph inner product $\langle ., .\rangle_A$. $H_A \hookrightarrow H$. If $(x_n)_{n\in\mathbb{N}} \subset D(A) \cap D(B)$ converges to x in H_A and $((A + B)x_n)_{n\in\mathbb{N}}$ converges to y in H, then,

$$\begin{cases} x_n \longrightarrow x \text{ in } H, \ x \in D(A) \\ Ax_n \longrightarrow Ax \text{ in } H \\ Bx_n \longrightarrow y - Ax \text{ in } H. \end{cases}$$

Since B is closed, we deduce that $x \in D(B)$ and $Bx = y - Ax$. Then, it follows that $A + B$ is closed from H_A onto H. The proof is analogue for the product BA.

The class of almost closed operators contains $C(H)$ (see proposition 1), but there exists almost closed operators A which are not closable if for example the graph $G(A)$ of A is dense in $H \oplus H$.

We can also construct closable linear operators which are not almost closed. Consider the identity operator I on the Hilbert space l^2 of sequences $f = (f_n)_{n\in\mathbb{N}}$ of real or complex numbers such that $\sum_{n=0}^{\infty} |f_n|^2 < +\infty$, with domain l_0^2. l_0^2 is the subset of l^2 consisting of all sequences $(f_n)_{n\in\mathbb{N}}$ such that $f_n = 0$ for all $n \geq n_0$.

Clearly, I is closable but not almost closed. Since if we suppose I almost closed, then there exists a Hilbert space $H_I \hookrightarrow l^2$ such that the graph $G(I) = l_0^2 \oplus l_0^2$ is closed in $H_I \oplus l^2$. Thus, $G(I)$ is a complete metric space. However $G(I)$ is also the union of countably many finite dimensional subspaces and is thus of first category. But, by Baire's theorem, complete metric spaces are of the second category, which is a contradiction.

We establish now some fundamental properties of almost closedness.

Theorem 3. *Let A be almost closed operator on H with domain $D(A)$ and associated Hilbert space $H_A = (D(A), [.,.]_A)$. Then,*

1) *$N(A)$ is closed linear subspace of H_A,*
2) *If $D(A) = H$, then $A \in \mathcal{B}(H)$,*
3) *If A is invertible, then A^{-1} is also almost closed on H. In particular, if $R(A) = H$ and A is invertible then $A \in \mathcal{B}(H)$.*

Proof. 1) Follows directly from Definition 2.

2) The identity operator in H is bijective and bicontinuous by the open mapping theorem. Consequently, the topologies induced on H by $\|-\|_H$ and $\|-\|_{H_A}$ are equivalent. Then, $A \in \mathcal{B}(H)$ since $A \in \mathcal{B}(H_A, H)$.

3) $D(A^{-1}) = R(A)$ is a Hilbert space, denoted by $H_{A^{-1}}$, with respect to the metric generated by the inner product:

$$[y, z]_{A^{-1}} = \langle y, z\rangle_H + [A^{-1}y, A^{-1}z]_A, \ \text{ for all } y, z \in R(A).$$

Furthermore,

$$\left\|A^{-1}y\right\|_{H_A} \leq \|y\|_{H_{A^{-1}}}, \ \text{ for all } y \in R(A)$$

where $\|y\|_{H_{A^{-1}}} = ([y, y]_{A^{-1}})^{1/2}$.

Since $H_A \hookrightarrow H$, we obtain that $A^{-1} \in \mathcal{B}(H_{A^{-1}}, H)$ and thus A^{-1} is almost closed on H. □

Theorem 4. *Let A be almost closed operator on H. Then, A has a densely defined almost closed extension.*

Proof. Suppose $D(A)$ is not dense in H, then its closure $\overline{D(A)}$ has a complement $\widehat{D}$ in H. Let P the orthogonal projection of H on $\overline{D(A)}$ along $\widehat{D}$. P is closed operator and AP is an almost closed extension of A. □

We need to verify that sums, products and limits of almost closed operators are also almost closed operators.

Theorem 5. *If A and B are almost closed operators on H, then $A+B$ and AB are also almost closed when $D(A)\cap D(B)\neq\{0\}$ and $D(AB)\neq\{0\}$.*

Proof. We take $H_{A+B}=(D(A)\cap D(B),[.,.]_{A+B})$ and $H_{AB}=(D(AB),[.,.]_{AB})$, where :

$$\begin{cases}[x,y]_{A+B}=[x,y]_A+[x,y]_B+\langle Ax,Ay\rangle_H\,, & \text{for all } x,y\in D(A)\cap D(B)\\ [x,y]_{AB}=[x,y]_B+[Bx,By]_A, & \text{for all } x,y\in D(AB).\end{cases}$$

Clearly, H_{A+B} and H_{AB} are Hilbert spaces, $H_{A+B}\hookrightarrow H$ and $H_{AB}\hookrightarrow H$.

i) Let $(x_n)_{n\in\mathbb{N}}$ a sequence of vectors of H_{A+B} that converges to x in H_{A+B}. Then, $(x_n)_{n\in\mathbb{N}}$ converges to x respectively in H_A and H_B and $(Ax_n)_{n\in\mathbb{N}}$ converges to Ax in H. The boundedness of B from H_B onto H implies that $(Bx_n)_{n\in\mathbb{N}}$ converges to Bx in H and then $((A+B)x_n)_{n\in\mathbb{N}}$ converges to $(A+B)x$ in H. Hence, $(A+B)\in\mathcal{B}(H_{A+B},H)$.

ii) If $(x_n)_{n\in\mathbb{N}}$ converges to x in H_{AB}, then $(x_n)_{n\in\mathbb{N}}$ converges to x in H_B and $(Bx_n)_{n\in\mathbb{N}}$ converges to Bx in H_A. According to the boundedness of A from H_A onto H, we deduce that $(ABx_n)_{n\in\mathbb{N}}$ converges to ABx in H. Consequently, $AB\in\mathcal{B}(H_{AB},H)$. □

Theorem 6. *For all $\varepsilon>0$, let A_ε be almost closed operator on H with the associated Hilbert space $H_\varepsilon=(D(A_\varepsilon),[.,.]_{A_\varepsilon})$. Assume that there exists an Hilbert space L such that $L\hookrightarrow H_\varepsilon$ for all $\varepsilon>0$ and $\sup_{\varepsilon>0}\|A_\varepsilon x\|_H<+\infty$, for all $x\in L$.*

Then, $Ax=\lim\limits_{\varepsilon\longrightarrow 0}A_\varepsilon x$ with domain

$$D(A)=\Big\{x\in\bigcap_{\varepsilon>0}D(A_\varepsilon)\cap L;\ \lim_{\varepsilon\to 0}A_\varepsilon x \text{ exists in } H\Big\}$$

is almost closed operator on H.

Proof. We take $H_A=(D(A),[.,.]_A)$, where

$$\begin{aligned}[x,y]_A&=\langle x,y\rangle_L+\lim_{\varepsilon\to 0}\langle A_\varepsilon x,A_\varepsilon y\rangle_H\\ &=\langle x,y\rangle_L+\ \langle Ax,Ay\rangle_H\,,\ \text{ for all } x,y\in D(A).\end{aligned}$$

$[.,.]_A$ is clearly an inner product on $D(A)$, we show that H_A is complete. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in H_A. Then, $(x_n)_{n\in\mathbb{N}}$ is Cauchy in L and thus in H_ε for all $\varepsilon>0$. Thus, $(x_n)_{n\in\mathbb{N}}$ converges to x in L, H and H_ε for all $\varepsilon>0$, then $x\in\bigcap_{\varepsilon>0}D(A_\varepsilon)\cap L$. It follows from the almost closedness of the operators H_ε that $(A_\varepsilon x_n)_{n\in\mathbb{N}}$ converges to $A_\varepsilon x$ in H for all $\varepsilon>0$.

Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy in H_A there exists $C>0$ such that

$$\|x_n\|_{H_A}=[x_n,x_n]_A^{1/2}\le C\ ,\ \text{ for all } n\in\mathbb{N}$$

and

$$\|x\|_{H_A}^2 = \lim_{n\to+\infty} \|x_n\|_L^2 + \lim_{\varepsilon\to 0}\lim_{n\to+\infty} \|A_\varepsilon x_n\|_H^2 \le 2C^2.$$

We obtain that $x \in H_A$.

Let $\eta > 0$. Then, by vertue of the assumption $\sup_{\varepsilon>0} \|A_\varepsilon x\|_H < +\infty$ on L and the uniform boundedness principle, there exists $i_0 \in \mathbb{N}$ such that for all $n, m \ge i_0$ and $\varepsilon > 0$,

$$\|x_n - x_m\|_{H_A} \le \frac{\eta}{2}$$
$$\|A_\varepsilon x_n - A_\varepsilon x_m\|_H \le C\frac{\eta}{2}.$$

Moreover, we have

$$\begin{aligned}\|x_n - x\|_{H_A} &= \Big[\lim_{n\to+\infty} \|x_n - x_m\|_L^2 + \lim_{\varepsilon\to 0}\lim_{n\to+\infty} \|A_\varepsilon x_n - A_\varepsilon x_m\|_H^2\Big]^{1/2} \\ &\le \frac{\eta}{2}(1+C^2)^{1/2}, \text{ for all } n \ge i_0.\end{aligned}$$

Consequently, H_A is a Hilbert space, $H_A \hookrightarrow H$ and A is bounded from H_A onto H. □

We know that almost closedness is stable under an almost closed perturbation. We now try to extend this stability to a relatively bounded perturbation.

Theorem 7. *Let A and B be unbounded linear operators on H, and let B be A-bounded with A-bound smaller than 1. Then $A + B$ is almost closed if and only if A is almost closed.*

Proof. $D(A) \subset D(B)$ and for all $x \in D(A)$ we have

$$\|Bx\|_H \le a\,\|Ax\|_H + b\,\|x\|_H$$

where a, b are nonnegative constants, $a < 1$.

Hence, we obtain for all $x \in D(A)$

$$\begin{cases}\|Ax\|_H \le \frac{1}{1-a}\,\|(A+B)x\|_H + \frac{b}{1-a}\,\|x\|_H \\ \|(A+B)x\|_H \le (1+a)\,\|Ax\|_H + b\,\|x\|_H\,.\end{cases}$$

If A is almost closed there exists a Hilbert space H_A such that $H_A \hookrightarrow H$ and $A \in \mathcal{B}(H_A, H)$. $D(A+B) = D(A)$, let $H_{A+B} = H_A$. Then, there exists $K > 0$ such that for all $x \in D(A)$

$$\begin{aligned}\|(A+B)x\|_H &\le K[\|Ax\|_H + \|x\|_{H_A}] \\ &\le K(1 + \|A\|_{\mathcal{B}(H_A,H)})\,\|x\|_{H_A}\,.\end{aligned}$$

Thus, $(A + B)$ is almost closed on H.

Conversely, if $(A+B)$ is almost closed on a Hilbert space H_{A+B} , $H_{A+B} \hookrightarrow H$ and $(A+B) \in \mathcal{B}(H_{A+B}\ , H)$, let again $H_A = H_{A+B}$. Then, there exists $K' > 0$ such that for all $x \in D(A)$

$$\begin{aligned} \|Ax\|_H &\le K'[\|(A+B)x\|_H + \|x\|_{H_A}] \\ &\le K'(1 + \|A+B\|_{\mathcal{B}(H_A,H)})\, \|x\|_{H_A}. \end{aligned}$$

This shows that A is almost closed on H. □

Remark 8. If $A \in \mathcal{B}(H)$ and B is almost closed operator on H, then $(A+B)^* = A^* + B^*$ and $(AB)^* = B^* A^*$.

This follows directly from the almost closedness of B and the properties of adjoint in the continuous case.

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