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PARTIAL INVERSE SPECTRAL PROBLEM FOR THE STURM-LIOUVILLE OPERATOR WITH DELAY

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Dedicated to the memory of our excellent teacher, scientist, and great man academician Fikret Vajzović

ABSTRACT. In this paper we study an inverse spectral boundary value problem for second-order differential equations on a finite interval with delay. For given conditions, we find the expression for the asymptotics of the eigenvalues of the posted problem. Then we examine the inverse problem and by using two sequences of eigenvalues obtained by varying the boundary condition at the right end, we construct the operator and prove its uniqueness.

1. INTRODUCTION

This paper deals with a Sturm-Liouville boundary value problem with delay defined by

$$-y''(x)+q(x)y(x-\tau)=\lambda y(x)=z^2y(x), \tag{1.1}$$

$$y'(0,z)=0, \tag{1.2}$$

$$y'(\pi,z)+H_jy(\pi,z)=0, \quad j=1,2, \tag{1.3}$$

when

$$\tau\in\left[\frac{\pi}{3},\frac{2\pi}{5}\right).$$

The first correct results in this direction were obtained in [6]. Later a similar result was obtained in [15]. In these two papers the simplest uniqueness results were proved in the case when we compare an arbitrary potential with the zero potential. The next step in this theory was made in papers [13], [5] and [17]. In these papers the case $\tau\geq\frac{\pi}{2}$ was considered. In this case the inverse problem becomes linear. In particular in [5] it was shown that this inverse problem is overdetermined, and the correct statement of the inverse problem was suggested. The case $\tau<\frac{\pi}{2}$ is more

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difficult since the inverse problem becomes nonlinear. This case was studied papers [2], [3], [16], [11] and [18]. In particular, in [3] and [18] the case $\tau \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right)$ was considered, and the solution of the inverse problem was obtained. Papers [17] and [14] are devoted to the similar problem with Dirichlet's boundary conditions.

In this paper, we solve the inverse problem for the potential q that satisfies the following conditions

$$\begin{aligned}
&q \in L_2\left[0,\pi\right], \\
&q \equiv 0, x \in \left[0,\tau\right], \\
&q(x) = q\left(x+\frac{\tau}{2}\right), x \in \left[\frac{3\tau}{2}, \pi-\tau\right], \\
&I_1 = \int_{\tau}^{\pi} q(t)dt \neq 0.
\end{aligned}$$

We also consider that $\int\limits_{\pi-\tau}^{2\tau} q(t_1)dt_1, \tau \in \left(\frac{\pi}{3}, \frac{2\pi}{5}\right)$ is known.

2. The Asymptotics of Eigenvalues

It is well known (see [9])) that characteristic functions F_j of the problem (1.1), (1.2) and (1.3) can be expressed in the form

$$\begin{aligned}
F_j(z) = &-z\sin\pi z + H_j\cos\pi z + a_{c^2}(z) + \frac{H_j}{z}a_{sc}(z) + \frac{1}{z}a_{csc}(z) + \\
&+\frac{H_j}{z^2}a_{s^2c(z)}, j=1,2,
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
a_{c^2}(z) &= \int_{\tau}^{\pi} q(t_1)\cos z(\pi-t_1)\sin z(t_1-\tau)dt_1, \\
a_{sc}(z) &= \int_{\tau}^{\pi} q(t_1)\sin z(\pi-t_1)\cos z(t_1-\tau)dt_1, \\
a_{s^2c}(z) &= \int_{2\tau}^{\pi} q(t_1)\int_{\tau}^{t_1-\tau} q(t_2)\sin z(\pi-t_1)\sin z(t_1-\tau-t_2)\cos z(t_2-\tau)dt_2dt_1, \\
a_{csc}(z) &= \int_{2\tau}^{\pi} q(t_1)\int_{\tau}^{t_1-\tau} q(t_2)\cos z(\pi-t_1)\sin z(t_1-\tau-t_2)\cos z(t_2-\tau)dt_2dt_1.
\end{aligned}$$

Let

$$\hat{q}(\theta) = \begin{cases} 0, & \theta \in \left[0,\frac{\tau}{2}\right) \cup \left(\pi-\frac{\tau}{2},\pi\right] \\ q(\theta+\frac{\tau}{2}), & \theta \in \left[\frac{\tau}{2}, \pi-\frac{\tau}{2}\right] \end{cases}$$

$$I_1=\int_{\tau}^{\pi} q(t_1)dt_1=\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \hat{q}(\theta)d\theta,$$

$$I_2=\int_{2\tau}^{\pi} q(t_1)\int_{\tau}^{t_1-\tau} q(t_2)dt_2dt_1=\int_{\frac{3\tau}{2}}^{\pi-\frac{\tau}{2}} \hat{q}(\theta)\int_{\frac{\tau}{2}}^{\theta-\frac{\tau}{2}} \hat{q}(\theta_1)d\theta_1 d\theta,$$

$$\hat{a}(z)=\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \hat{q}(\theta)\cos z(\pi-2\theta)d\theta,$$

$$\hat{b}(z)=\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \hat{q}(\theta)\sin z(\pi-2\theta)d\theta.$$

Then

$$\begin{aligned} a_{c^2}(z)&=\frac{1}{2}\hat{a}(z)+\frac{I_1}{2}\cos z(\pi-\tau),\\ a_{sc}(z)&=\frac{1}{2}\hat{b}(z)+\frac{I_1}{2}\sin z(\pi-\tau). \end{aligned} \tag{2.2}$$

Now, let

$$Q^{(2)}(\theta)=\begin{cases} 0, & \theta\in[0,\tau)\cup(\pi-\tau,\pi] \\ q(\theta+\tau)\int_{\tau}^{\theta} q(t_1)dt_1+\int_{\theta+\tau}^{\pi} q(t_1-\theta)q(t_1)dt_1-q(\theta)\int_{\theta+\tau}^{\pi} q(t_1)dt_1, & \\ & \theta\in[\tau,\pi-\tau] \end{cases} \tag{2.3}$$

$$a^{(2)}(z)=\int_{\tau}^{\pi-\tau} Q^{(2)}(\theta)\cos z(\pi-2\theta)d\theta,$$

$$b^{(2)}(z)=\int_{\tau}^{\pi-\tau} Q^{(2)}(\theta)\sin z(\pi-2\theta)d\theta.$$

Then

$$\begin{aligned} a_{s^2c}(z)&=\frac{1}{4}a^{(2)}(z)-\frac{I_2}{4}\cos z(\pi-2\tau),\\ a_{csc}(z)&=-\frac{1}{4}b^{(2)}(z)+\frac{I_2}{4}\sin z(\pi-2\tau). \end{aligned} \tag{2.4}$$

Using relations (2.2) and (2.4) functions (2.1) can be written as

$$\begin{aligned} F_j(z)=&-z\sin\pi z+H_j\cos\pi z+\frac{1}{2}\hat{a}(z)+\frac{I_1}{2}\cos z(\pi-\tau)+\\ &+\frac{H_j}{2z}\hat{b}(z)+\frac{H_jI_1}{2z}\sin z(\pi-\tau)-\frac{1}{4z}b^{(2)}(z)+\\ &+\frac{I_2}{4z}\sin z(\pi-2\tau)+\frac{H_j}{4z^2}a^{(2)}(z)-\frac{H_jI_2}{4z^2}\cos z(\pi-2\tau). \end{aligned} \tag{2.5}$$

Theorem 2.1. *The zeros z_{nj} of characteristic functions F_j have the following asymptotic expansions*

$$z_{nj} = n + \frac{C_{1j}(n)}{n} + \frac{C_{2j}(n)}{n^2} + o\left(\frac{C_{2j}(n)}{n^2}\right), \tag{2.6}$$

where

$$C_{1j}(n) = \zeta_{0j} + \zeta_1 \cos n\tau + \frac{1}{4}\zeta_{2n}, \quad C_{2j}(n) = \eta_{1j} \sin n\tau + \eta_2 \sin 2n\tau$$

$$\zeta_{0j} = \frac{H_j}{\pi}, \quad \zeta_1 = \frac{I_1}{2\pi}, \quad \zeta_{2n} = \hat{a}_{2n} \in l_2, \quad \eta_2 = \frac{(\pi-\tau)I_j^2}{4\pi^2} - \frac{I_2}{4\pi}, \quad \eta_{1j} = -\frac{\tau I_j H_j}{2\pi^2}.$$

Proof. See [9]. □

Corollary 2.1. *The asymptotic behavior of eigenvalues is given by*

$$\lambda_{nj} = n^2 + 2\zeta_{oj} + 2\zeta_1 \cos n\tau + \frac{1}{2}\zeta_{2n} + \frac{1}{n}\Big(\eta_{1j} \sin n\tau + \eta_2 \sin 2n\tau\Big) + o\left(\frac{1}{n}\right). \tag{2.7}$$

In the following the set $\prod = \{\tau, H, q\}$ is called the set of parameters of operator D^2 and the set $\bigwedge = \{\lambda_{nj}, n \in N_0, j = 1,2\}$ is called the spectral characteristics of the operator D^2. In the next parts of the paper we will will construct the set $\prod$ with some additional conditions on the potential q, provided the set $\bigwedge$ known.

3. Recovering Numbers τ, H_j, and a_0

We suppose that

$$\hat{a}_0 = \frac{2}{\pi}\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \hat{q}(\theta)d\theta = \frac{2}{\pi}I_1 \neq 0,$$

i.e. sequences λ_{nj} oscillate as $n \to \infty$.

Let

$$\mu_{nj} = \frac{\lambda_{n+2,j} - \lambda_{n-2,j} - (n+2)^2 + (n-2)^2}{\lambda_{n+1,j} - \lambda_{n-1,j} - (n+1)^2 - (n-1)^2}.$$

From (2.7) it follows

$$\tau = \arccos \frac{1}{2}\lim_{n\to\infty} \mu_{nj}. \tag{3.1}$$

Furthermore, we have

$$H_2 - H_1 = \pi \lim_{n\to\infty}(\lambda_{n2} - \lambda_{n1}) = \pi(\zeta_{02} - \zeta_{01}).$$

Let us choose subsequences $\{n_k^{(l)}\}$ of the sequence $\{n\}, l = 1,2$ for which $\cos n_k^{(1)}\tau \cdot \cos n_k^{(2)}\tau \neq 0$ and $|\cos n_k^{(2)}\tau - \cos n_k^{(1)}\tau| \geq \delta > 0, \quad (\forall k)$.

Then we have the following

$$2\zeta_{0j} = \lim_{k\to\infty} \frac{\left(\lambda_{n_k^{(2)}j} - \left(n_k^{(2)}\right)^2\right)\cos n_k^{(1)}\tau - \left(\lambda_{n_k^{(1)}j} - \left(n_k^{(1)}\right)^2\right)\cos n_k^{(2)}\tau}{\cos n_k^{(2)}\tau - \cos_k^{(1)}\tau},$$

i.e.

$$H_j = \pi\zeta_{oj}, \quad j = 1, 2. \tag{3.2}$$

We also get

$$2\zeta_1 = \lim_{k\to\infty} \frac{\lambda_{n_k^{(l)}j} - \left(n_k^{(l)}\right)^2 - 2\zeta_{0j}}{\cos n_k^{(l)}\tau}, \quad l = 1, 2, \quad j = 1, 2,$$

i.e.

$$\hat{a}_0 = \frac{2}{\pi} I_1 = \frac{4}{\pi}\zeta_1 \tag{3.3}$$

With this we have proved the following

Theorem 3.1. *If $\hat{a}_0 \neq 0$, then the set $\bigwedge$ uniquely determines the numbers τ, H_1, H_2 and the 0-th Fourier coefficient of potential q.*

4. Construction of the Integral Equation for the Function q

In the following we assume $\tau \in \left[\frac{\pi}{3}, \frac{2\pi}{5}\right)$. By the Hadamard's factorization theorem it follows

$$F_j(z) = \frac{A_j}{\pi\lambda_{0j}} \prod_{n=1}^{\infty} \frac{n^2}{\lambda_{nj}} \left(-z + \frac{\lambda_{0j}}{z}\right) \sin\pi z \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_{nj} - n^2}{n^2 - z^2}\right), \quad j = 1, 2. \tag{4.1}$$

From (2.5) we also have

$$\begin{aligned} F_j(z) \equiv & -z\sin\pi z + H_j\cos\pi z + \frac{1}{2}\hat{a}(z) + \frac{I_1}{2}\cos z(\pi-\tau) + \frac{H_j}{2z}\hat{b}(z) + \\ & + \frac{H_j I_1}{2z}\sin z(\pi-\tau) - \frac{1}{4z} b^{(2)}(z) + \frac{I_2}{4z}\sin z(\pi - 2\tau) + \\ & + \frac{H_j}{4z^2} a^{(2)}(z) - \frac{H_j I_2}{4z^2}\cos z(\pi - 2\tau), \quad j = 1, 2, \end{aligned} \tag{4.2}$$

where F_j are the products of (4.1), and τ, H_1, H_2 and I_1 the numbers given by (3.1), (3.2) and (3.3).

Let us define functions

$$C(z) = \frac{2}{H_2 - H_1}\left[H_2 F_1(z) - H_1 F_2(z)\right] + 2z\sin\pi z - I_1\cos z(\pi - \tau), \tag{4.3}$$

$$S(z) = \frac{2z}{H_2 - H_1}\left[F_2(z) - F_1(z)\right] - 2z\cos\pi z - I_1\sin z(\pi - \tau). \tag{4.4}$$

It is easily verifiable that by using (4.2) we can write (4.3) and (4.4) as

$$C(z)=\hat{a}(z)-\frac{b^{(2)}(z)}{2z}+\frac{I_2}{2z}\sin z(\pi-2\tau) \tag{4.5}$$

$$S(z)=\hat{b}(z)+\frac{a^{(2)}(z)}{2z}-\frac{I_2}{2z}\cos z(\pi-2\tau) \tag{4.6}$$

Applying integration by parts to fractions $\frac{b^{(2)}(z)}{2z}$, $\frac{a^{(2)}(z)}{2z}$ we get

$$-\frac{b^{(2)}(z)}{2z}+\frac{I_2}{2z}\sin z(\pi-2\tau)=\frac{I_2}{z}\sin z(\pi-2\tau)-a^{IQ^{(2)}}(z), \tag{4.7}$$

$$\frac{a^{(2)}(z)}{2z}-\frac{I_2}{2z}\cos z(\pi-2\tau)=-b^{IQ^{(2)}}(z), \tag{4.8}$$

where

$$a^{IQ^{(2)}}(z)=\int_{\tau}^{\pi-\tau}\left(\int_{\tau}^{\theta}Q^{(2)}(\theta_1)d\theta_1\right)\cos z(\pi-2\theta)d\theta, \tag{4.9}$$

$$b^{IQ^{(2)}}(z)=\int_{\tau}^{\pi-\tau}\left(\int_{\tau}^{\theta}Q^{(2)}(\theta_1)d\theta_1\right)\sin z(\pi-2\theta)d\theta. \tag{4.10}$$

Based on (4.7)–(4.10) relations (4.5) and (4.6) can be written in the form

$$C(z)=\hat{a}(z)-a^{IQ^{(2)}}(z)+\frac{I_2}{z}\sin z(\pi-2\tau),\quad z\in\mathbb{C}, \tag{4.11}$$

$$S(z)=\hat{b}(z)-b^{IQ^{(2)}}(z),\quad z\in\mathbb{C}. \tag{4.12}$$

From the identity system (4.11), (4.12) we obtain the infinite system of equations by taking $z=m$, $m\in\mathbb{N}_0$.

$$C(0)=\hat{a}(0)-a^{IQ^{(2)}}(0)+I_2(\pi-2\tau) \tag{4.13}$$

$$C(m)=\hat{a}(m)-a^{IQ^{(2)}}(m)+\frac{I_2}{m}\sin(\pi-2\tau)m,\quad m\in\mathbb{N}, \tag{4.14}$$

$$S(m)=\hat{b}(m)-b^{IQ^{(2)}}(m),\quad m\in\mathbb{N}. \tag{4.15}$$

Let $C_0=\frac{2}{\pi}C(0)$, $\hat{a}_0=\frac{2}{\pi}\hat{a}(0)$, $a_0^{IQ^{(2)}}=\frac{2}{\pi}a^{IQ^{(2)}}(0)$, $\Delta_0=\frac{2(\pi-2\tau)}{\pi}$,

$$\hat{a}_{2m}=\frac{2}{\pi}\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\hat{q}(\theta)\cos 2m\theta d\theta,\quad \hat{b}_{2m}=\frac{2}{\pi}\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\hat{q}(\theta)\sin 2m\theta d\theta,$$

$$a_{2m}^{IQ^{(2)}}=\frac{2}{\pi}\int_{\tau}^{\pi-\tau}\left(\int_{\tau}^{\theta}Q^{(2)}(\theta_1)d\theta_1\right)\cos 2m\theta d\theta,\quad b_{2m}^{IQ^{(2)}}=\frac{2}{\pi}\int_{\tau}^{\pi-\tau}\left(\int_{\tau}^{\theta}Q^{(2)}(\theta_1)d\theta_1\right)\sin 2m\theta d\theta,$$

$$C_{2m} = (-1)^m \frac{2}{\pi} C(m), \quad S_{2m} = (-1)^{m+1} \frac{2}{\pi} S(m).$$

Now, the system (4.13), (4.14), (4.15) becomes

$$\frac{1}{2}\hat{a}_0 = \frac{1}{2}C_0 + \frac{1}{2}a_0^{IQ^{(2)}} - \frac{1}{2}\Delta_0 \tag{4.16}$$

$$\hat{a}_{2m} = C_{2m} + a_{2m}^{IQ^{(2)}} - I_2 \frac{\sin 2m\tau}{m} \tag{4.17}$$

$$\hat{b}_{2m} = S_{2m} + b_{2m}^{IQ^{(2)}} \tag{4.18}$$

From (2.6) it is easy to prove that $C_{2m} \in l_2$ and $S_{2m} \in l_2$.
So there exists a function $f \in L_2[0,\pi]$ such that

$$f(\theta) = \frac{C_0}{2} + \sum_{m=1}^{\infty} C_{2m} \cos 2m\theta + S_{2m} \sin 2m\theta.$$

Let

$$\Delta(\theta) = \frac{\Delta_0}{2} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\sin 2m\tau \cos 2m\theta}{m}$$

Based on (4.16), (4.17), (4.18), we get

$$\hat{q}(\theta) = f(\theta) + \int_{\tau}^{\theta} Q^{(2)}(\theta_1) d\theta_1 - I_2 \Delta(\theta), \theta \in \left[\frac{\tau}{2}, \pi - \frac{\tau}{2}\right],$$

or

$$q\left(\theta + \frac{\tau}{2}\right) = f(\theta) + \int_{\frac{3\tau}{2}}^{\theta+\frac{\tau}{2}} Q^{(2)}\left(\theta_1 - \frac{\tau}{2}\right) d\theta_1 - I_2 \Delta(\theta).$$

$$q(x) = f\left(x - \frac{\tau}{2}\right) + \int_{\frac{3\tau}{2}}^{x} Q^{(2)}\left(x_1 - \frac{\tau}{2}\right) dx_1 - I_2 \Delta\left(x - \frac{\tau}{2}\right), x \in [\tau, \pi]. \tag{4.19}$$

For $x \in \left[\tau, \frac{3\tau}{2}\right)$ we have $\Delta(x - \frac{\tau}{2}) \equiv 0$ and the equation (4.19) becomes the identity

$$q(x) = f\left(x - \frac{\tau}{2}\right).$$

If q is continuous from the left at $x = \frac{3\tau}{2}$, we have

$$q\left(\frac{3\tau}{2}\right) = \lim_{x \nearrow \frac{3\tau}{2}} q(x),$$

and from (4.19) we get

$$I_2 = \frac{1}{\Delta(\tau)} \left(f(\tau) - q\left(\frac{3\tau}{2}\right)\right).$$

Let

$$f_1(x) = f\left(x - \frac{\tau}{2}\right) - I_2 \Delta\left(x - \frac{\tau}{2}\right), x \in \left[\frac{3\tau}{2}, \pi - \frac{\tau}{2}\right].$$

Then the equation (4.19) takes the form

$$q(x) = f(x) + \int_{\frac{3\tau}{2}}^{x} Q^{(2)}\left(x_1 - \frac{\tau}{2}\right) dx_1.$$

Since $\int\limits_{\frac{3\tau}{2}}^{\pi-\frac{\tau}{2}} Q^{(2)}\left(x_1-\frac{\tau}{2}\right)dx_1 = I_2$, the equation (4.19) for $x\in\left(\pi-\frac{\tau}{2},\pi\right]$ takes the form

$$q(x) = f\left(x-\frac{\tau}{2}\right)+I_2,$$

because

$$\int\limits_{\frac{3\tau}{2}}^{x} Q^{(2)}\left(x_1-\frac{\tau}{2}\right)dx_1 = I_2, x\in\left[\pi-\frac{\tau}{2},\pi\right].$$

So, the function q has the unique solution in L_2 on $\left[\tau,\frac{3\tau}{2}\right]\cup\left[\pi-\frac{\tau}{2},\pi\right]$.

5. Solution of Integral Equation on $\left[\frac{3\tau}{2},\pi-\frac{\tau}{2}\right]$

Let us denote by $q_1(x)$ the solution on the interval $\left(\tau,\frac{3\tau}{2}\right)$, and with $q_2(x)$ the solution on the interval $\left(\pi-\frac{\tau}{2},\pi\right)$. For $\tau\in\left(\frac{\pi}{3},\frac{2\pi}{5}\right)$ we have $\left(\frac{3\tau}{2},\pi-\frac{\tau}{2}\right)=\left(\frac{3\tau}{2},\pi-\tau\right)\cup[\pi-\tau,2\tau]\cup\left(2\tau,\pi-\frac{\tau}{2}\right)$. For $\tau=\frac{\pi}{3}$ we have $\pi-\tau=2\tau$ and $[\pi-\tau,2\tau]=\left\{\frac{2\pi}{3}\right\}$ so that interval degenerates into a point.

Theorem 5.1. *The equation*

$$q(x) = f_1(x)+\int\limits_{\frac{3\tau}{2}}^{x} Q^{(2)}\left(x_1-\frac{\tau}{2}\right)dx_1, \quad x\in\left[\frac{3\tau}{2},\pi-\frac{\tau}{2}\right] \tag{5.1}$$

is a linear integral equation of the Volterra type with a deviating argument.

Proof. From (2.3) we have

$$Q^{(2)}\left(x_1-\frac{\tau}{2}\right) = q\left(x_1+\frac{\tau}{2}\right)\int\limits_{\tau}^{x_1-\frac{\tau}{2}} q(t_1)dt_1+\int\limits_{x_1+\frac{\tau}{2}}^{\pi} q\left(t_1-x_1+\frac{\tau}{2}\right)q(t_1)dt_1-$$
$$-q\left(x_1-\frac{\tau}{2}\right)\int\limits_{x_1+\frac{\tau}{2}}^{\pi} q(t_1)dt_1,\ \ x_1\in\left[\frac{3\tau}{2},x\right]\subset\left[\frac{3\tau}{2},\pi-\frac{\tau}{2}\right].$$

For $x\in\left[\frac{3\tau}{2},\pi-\tau\right]$ we have

$$Q^{(2)}\left(x_1-\frac{\tau}{2}\right) = \left[q\left(x_1+\frac{\tau}{2}\right)\int\limits_{\tau}^{x_1-\frac{\tau}{2}} q(t_1)dt_1+\int\limits_{\frac{3\tau}{2}}^{\pi-x_1+\frac{\tau}{2}} q(t_1)q_2\left(t_1+x_1-\frac{\tau}{2}\right)dt_1-\right.$$
$$\left.-q_1\left(x_1-\frac{\tau}{2}\right)\int\limits_{x_1+\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} q(t_1)dt_1\right]+\left[\int\limits_{\tau}^{\frac{3\tau}{2}} q_1(t_1)q_2\left(t_1+x_1-\frac{\tau}{2}\right)dt_1-\right.$$
$$\left.-q_1\left(x_1-\frac{\tau}{2}\right)\int\limits_{\pi-\frac{\tau}{2}}^{\pi} q_2(t_1)dt_1\right].$$

The difference in the second squared bracket is known, and the expression in the first squared bracket linearly depends on the unknown function q. For $\tau \in \left(\frac{\pi}{2}, \frac{2\pi}{5}\right)$ the kernel $Q^{(2)}\left(x_1 - \frac{\tau}{2}\right)$, $x_1 \in (\pi - \tau, 2\tau)$ is known. For $x \in \left[2\tau, \pi - \frac{\tau}{2}\right]$ we have

$$Q^{(2)}\left(x_1 - \frac{\tau}{2}\right) = \left[q_2\left(x_1 + \frac{\tau}{2}\right) \int_{\frac{3\tau}{2}}^{x_1 - \frac{\tau}{2}} q(t_1)dt_1 - q\left(x_1 - \frac{\tau}{2}\right) \int_{x_1 + \frac{\tau}{2}}^{\pi} q_2(t_1)dt_1\right] + \\ + \left[q_2\left(x_1 + \frac{\tau}{2}\right) \int_{\tau}^{\frac{3\tau}{2}} q_1(t_1)dt_1 + \int_{\tau}^{\pi - x_1 + \frac{\tau}{2}} q_1(t_1)q_2\left(t_1 + x_1 - \frac{\tau}{2}\right)dt_1\right].$$

The expression in the second squared bracket is known and the expression in the first squared bracket is linearly dependent on the unknown function q. Let

$$K_2^{(x)}(x_1, q(x_1)) = \begin{cases} q\left(x_1 + \frac{\tau}{2}\right) \int_{\tau}^{x_1 - \frac{\tau}{2}} q_1(t_1)dt_1 + \int_{\frac{3\tau}{2}}^{\pi - x_1 + \frac{\tau}{2}} q(t_1)q_2\left(t_1 + x_1 - \frac{\tau}{2}\right)dt_1 - \\ \quad -q_1\left(x_1 - \frac{\tau}{2}\right) \int_{x_1 + \frac{\tau}{2}}^{\pi - \frac{\tau}{2}} q(t_1)dt_1, \quad x \in \left[\frac{3\tau}{2}, \pi - \tau\right],\ x_1 \in \left[\frac{3\tau}{2}, x\right] \\ 0, \quad x \in [\pi - \tau, 2\tau], x_1 \in [\pi - \tau, x] \\ q_2\left(x_1 + \frac{\tau}{2}\right) \int_{\frac{3\tau}{2}}^{x_1 - \frac{\tau}{2}} q(t_1)dt_1 - q\left(x_1 - \frac{\tau}{2}\right) \int_{x_1 + \frac{\tau}{2}}^{\pi} q_2(t_1)dt_1, \\ \quad x \in \left[2\tau, \pi - \frac{\tau}{2}\right],\ x_1 \in [2\tau, x]. \end{cases}$$

The known function differs on given intervals. If $x \in \left[\frac{3\tau}{2}, \pi - \tau\right]$ and if we denote the starting function by $f_2^{(1)}(x)$, we have

$$f_2^{(1)}(x) = f_1(x) + \int_{\frac{3\tau}{2}}^{x} \left[\int_{\tau}^{\frac{3\tau}{2}} q_1(t_1)q_2\left(t_1 + x_1 - \frac{\tau}{2}\right)dt_1 - q_1\left(x_1 - \frac{\tau}{2}\right) \int_{\pi - \frac{\tau}{2}}^{\pi} q_2(t_1)dt_1\right]dx_1.$$

For $x \in [\pi - \tau, 2\tau]$, we have

$$f_2^{(2)}(x) = f_1(x) + \int_{\frac{3\tau}{2}}^{x} \left[\int_{\tau}^{\frac{3\tau}{2}} q_1(t_1)q_2\left(t_1 + x_1 - \frac{\tau}{2}\right)dt_1 - \\ - q_1\left(x_1 - \frac{\tau}{2}\right) \int_{\pi - \frac{\tau}{2}}^{\pi} q_2(t_1)dt_1\right]dx_1 + \int_{\pi - \tau}^{x} Q^{(2)}\left(x_1 - \frac{\tau}{2}\right)dx_1.$$

For $x \in \left[2\tau, \pi - \frac{\tau}{2}\right]$ the following relation holds

$$f_2^{(3)}(x) = f_1(x) + \int_{\frac{3\tau}{2}}^{\pi - \tau} \left[\int_{\tau}^{\frac{3\tau}{2}} q_1(t_1)q_2\left(t_1 + x_1 - \frac{\tau}{2}\right)dt_1 - \\ - q_1\left(x_1 - \frac{\tau}{2}\right) \int_{\pi - \frac{\tau}{2}}^{\pi} q_2(t_1)dt_1\right]dx_1 + \int_{\pi - \tau}^{2\tau} Q^{(2)}\left(x_1 - \frac{\tau}{2}\right)dx_1 +$$

$$+\int_{2\tau}^{x}\left[q_2\left(x_1+\tfrac{\tau}{2}\right)\int_{\tau}^{\frac{3\tau}{2}}q_1(t_1)dt_1+\int_{\tau}^{\pi-x_1+\frac{\tau}{2}}q_1(t_1)q_2\left(t_1+x_1-\tfrac{\tau}{2}\right)dt_1\right]dx_1.$$

Let

$$f_2(x)=\begin{cases} f_2^{(1)}, & x\in\left[\frac{3\tau}{2},\pi-\tau\right],\\ f_2^{(2)}, & x\in(\pi-\tau,2\tau],\\ f_2^{(3)}, & x\in\left(2\tau,\pi-\frac{\tau}{2}\right].\end{cases}$$

Now, the equation (5.1) becomes

$$q(x)=f_2(x)+\int_{\frac{3\tau}{2}}^{x}K_2^{(x)}\left(x_1,q(x_1)\right)dx_1.$$

Herewith, the theorem is proved. □

Let us transform the equation (5.1) on the interval $\left[2\tau,\pi-\frac{\tau}{2}\right]$ into a new form. Because the equation (5.1) is equivalent to the equation

$$q(x)=f_1(x)+I_2-\int_{x}^{\pi-\frac{\tau}{2}}Q^{(2)}\left(x_1-\frac{\tau}{2}\right)dx_1, x\in\left[2\tau,\pi-\frac{\tau}{2}\right], \tag{5.2}$$

and if we define the function $f_2^-(x)$ by

$$f_2^-(x)=f_1(x)+I_2-\int_{x}^{\pi-\frac{\tau}{2}}\left[q_2\left(x_1+\frac{\tau}{2}\right)\int_{\tau}^{\frac{3\tau}{2}}q_1(t_1)dt_1+\right.$$
$$\left.+\int_{\tau}^{\pi-x+\frac{\tau}{2}}q_1(t_1)q_2\left(t_1+x_1-\frac{\tau}{2}\right)dt_1\right]dx_1,$$

then the equation (5.2) takes the form

$$q(x)=f_2^-(x)-\int_{x}^{\pi-\frac{\tau}{2}}\left[q_2\left(x_1+\frac{\tau}{2}\right)\int_{\frac{3\tau}{2}}^{x_1-\frac{\tau}{2}}q(t_1)dt_1-q\left(x_1-\frac{\tau}{2}\right)\int_{x_1+\frac{\tau}{2}}^{\pi}q_2(t_1)dt_1\right]dx_1. \tag{5.3}$$

Let

$$\Delta I_1=I_1-\int_{\tau}^{\frac{3\tau}{2}}q_1(t_1)dt_1-\int_{\pi-\frac{\tau}{2}}^{\pi}q_2(t_1)dt_1.$$

Now, we have that

$$\int_{\frac{3\tau}{2}}^{x_1-\frac{\tau}{2}}q(t_1)dt_1=\Delta I_1-\int_{x_1-\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}q(t_1)dt_1. \tag{5.4}$$

Using (5.4), from (5.3) it follows

$$q(x)=f_2^-(x)-\Delta I_1\int_x^{\pi-\frac{\tau}{2}}q_2\left(x_1+\frac{\tau}{2}\right)dx_1+$$
$$+\int_x^{\pi-\frac{\tau}{2}}\left[q_2\left(x_1+\frac{\tau}{2}\right)\int_{x_1-\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}q(t_1)dt_1+q\left(x_1-\frac{\tau}{2}\right)\int_{x_1+\frac{\tau}{2}}^{\pi}q_2(t_1)dt_1\right]dx_1. \tag{5.5}$$

Let

$$f_2^*(x)=f_2^-(x)-\Delta I_1\int_x^{\pi-\frac{\tau}{2}}q_2\left(x_1+\frac{\tau}{2}\right)dx_1.$$

Now, the equation (5.5) becomes

$$q(x)=f_2^*(x)+\int_x^{\pi-\frac{\tau}{2}}\left[q_2\left(x_1+\frac{\tau}{2}\right)\int_{x_1-\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}q(t_1)dt_1+q\left(x_1-\frac{\tau}{2}\right)\int_{x_1+\frac{\tau}{2}}^{\pi}q_2(t_1)dt_1\right]dx_1. \tag{5.6}$$

Since $q(x)=q\left(x+\frac{\tau}{2}\right)$, we have that $q\left(x_1-\frac{\tau}{2}\right)=q(x_1)$ and it holds

$$\int_{x_1-\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}q(t_1)dt_1=\int_{x_1-\frac{\tau}{2}}^{\pi-\tau}q(t_1)dt_1+\int_{\pi-\tau}^{2\tau}q(t_1)dt_1+\int_{2\tau}^{\pi-\frac{\tau}{2}}q(t_1)dt_1=$$
$$=\int_{x_1}^{\pi-\frac{\tau}{2}}q(t_1)dt_1+\int_{\pi-\tau}^{2\tau}q(t_1)dt_1+\frac{1}{2}\left(\Delta I_1-\int_{\pi-\tau}^{2\tau}q(t_1)dt_1\right)=$$
$$=\int_{x_1}^{\pi-\frac{\tau}{2}}q(t_1)dt_1+\frac{1}{2}\left(\Delta I_1+\int_{\pi-\tau}^{2\tau}q(t_1)dt_1\right).$$

Using this we get

$$\int_x^{\pi-\frac{\tau}{2}}\left[q_2\left(x_1+\frac{\tau}{2}\right)\int_{x_1-\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}q(t_1)dt_1+q\left(x_1-\frac{\tau}{2}\right)\int_{x_1+\frac{\tau}{2}}^{\pi}q_2(t_1)dt_1\right]dx_1=$$
$$=\int_x^{\pi-\frac{\tau}{2}}\left[q_2\left(x_1+\frac{\tau}{2}\right)\int_{x_1}^{\pi-\frac{\tau}{2}}q(t_1)dt_1+q(x_1)\int_{x_1+\frac{\tau}{2}}^{\pi}q_2(t_1)dt_1\right]dx_1+$$
$$+\frac{1}{2}\left(\Delta I_1+\int_{\pi-\tau}^{2\tau}q(t_1)dt_1\right)\int_x^{\pi-\frac{\tau}{2}}q_2\left(x_1+\frac{\tau}{2}\right)dx_1.$$

Let

$$f_2^{**}(x) = f_2^*(x) + \frac{1}{2}\left(\Delta I_1 + \int_{\pi-\tau}^{2\tau} q(t_1)dt_1\right) \int_x^{\pi-\frac{\tau}{2}} q_2\left(x_1 + \frac{\tau}{2}\right) dx_1.$$

Now, the equation (5.1) on the interval $\left[2\tau, \pi - \frac{\tau}{2}\right]$ can be written in the form

$$q(x) = f_2^{**}(x) + \int_x^{\pi-\frac{\tau}{2}} \left[q_2\left(x_1 + \frac{\tau}{2}\right) \int_{x_1}^{\pi-\frac{\tau}{2}} q(t_1)dt_1 + q(x_1) \int_{x_1+\frac{\tau}{2}}^{\pi} q_2(t_1)dt_1 \right] dx_1.$$

Hereby, we have proved the following result:

Theorem 5.2. *If the number* $\int_{\pi-\tau}^{2\tau} q(t_1)dt_1$, $\tau \in \left(\frac{\pi}{3}, \frac{2\pi}{5}\right)$ *is known, and if* $q(x_1) = q\left(x_1 + \frac{\tau}{2}\right)$, $x_1 \in \left[\frac{3\tau}{2}, \pi - \tau\right]$, *then the equation* (5.1) *is equivalent to the linear Volterra equation without deviation.*

Corollary 5.1. *The equation* (5.1) *on the interval* $\left[2\tau, \pi - \frac{\tau}{2}\right]$ *has the unique solution* $q_3(x)$.

If $\tau = \frac{\pi}{3}$ then the potential q is determined on the interval $\left[\frac{\pi}{3}, \pi\right]$ and it holds

$$q(x) = \begin{cases} q_1(x), & x \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right], \\ q_3(x), & x \in \left(\frac{\pi}{2}, \frac{5\pi}{6}\right], \\ q_2(x), & x \in \left(\frac{5\pi}{6}, \pi\right]. \end{cases}$$

For $\tau \in \left(\frac{\pi}{3}, \frac{2\pi}{5}\right)$ on the interval $[\pi - \tau, 2\tau]$ the equation becomes an identity and its solution we denote by $q_4(x)$. So

$$q(x) = \begin{cases} q_1(x), & x \in \left[\tau, \frac{3\tau}{2}\right), \\ q_3(x), & x \in \left[\frac{3\tau}{2}, \pi - \tau\right] \cup \left[2\tau, \pi - \frac{\tau}{2}\right], \\ q_4(x), & x \in (\pi - \tau, 2\tau) \\ q_2(x), & x \in \left(\pi - \frac{\tau}{2}, \pi\right]. \end{cases}$$

Therefore, we have proved the main result.

Theorem 5.3. *If two sequences of eigenvalues* λ_{nj}, $n \in \mathbb{N}_0$, $j = 1,2$ *of the operator* $D^{(2)}$ *are obtained by varying the boundary conditions at the right end, then the operators* $D_j^{(2)} = D^{(2)}(\tau, H_j, q)$ *are uniquely determined, where* $\int_\tau^\pi q(t)dt \neq 0$, $\int_{\pi-\tau}^{2\tau} q(t_1)dt_1$ *is a known number, and* $q(x_1) = q\left(x_1 + \frac{\tau}{2}\right)$, $x \in \left[\frac{3\tau}{2}, \pi - \tau\right]$.

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References

[1] V. Ambarzumjan, *Uber eine Frage der Eigenwerttheorie*, Zeitshr.fr Physik, Bd.53., S., 690–695 (1929)

[2] N.P. Bondarenko, V.A. Yurko, *An inverse problem for Sturm-Liouville differential operators with deviating argument*. Applied Mathematics Letters, vol.83 (2018), 140–144.

[3] N.P. Bondarenko, V.A. Yurko, *Partial inverse problems for the Sturm-Liouville equation with deviating argument*. Mathematical Methods in the Applied Sciences, 41, no.17 (2018), 8350–8354.

[4] G. Borg, *Umkehrung der Sturm-Liouvillschen Eigenwertaufgabe*, Acta Math. Bd.78. No.1. S., 1–96 (1946)

[5] S.A. Buterin, V.A. Yurko, *An inverse spectral problem for Sturm-Liouville operators with a large constant delay*. Analysis and Mathematical Physics, vol.9, no.1 (2019), 17–27. DOI 10.1007/s13324-017-0176-6 (2017).

[6] G. Freiling, V.A. Yurko, *Inverse problems for Sturm-Liouville differential operators with a constant delay*, Appl. Math. Lett. 25 (2012), 1999—2004.

[7] И.М. Гельфанд, Б.М. Левитан, Об определении дифференциального уравнения по его спектралной функции, Изв. АН СССР, Сер.мат(1951)Т.15, 309–360

[8] D. Nedić, M. Pikula, *Rješenje inverznog zadatka za operator sa homogenim kašnjenjem asimptotskim metodom*,Fourth Mathematical Conference of the Republic of Srpska, Proceedings, Trebinje (2015.), 347–362

[9] М. Пикула, Определениие дифференциального оператора Штурма-Лиувилляс запаздывающим аргументом по двум спектрам, Матемтички весник 43,(1991),159–171.

[10] M. Pikula, E. Čatrnja, I. Kalčo, *Spectral problems for operators with deviating arguments*, Hacettepe Journal of Mathematics and Statistics, Volume 47(5)(2018), 1172–1183

[11] M. Pikula, V. Vladičić, B. Vojvodić, *Inverse spectral problems for Sturm–Liouville operators with a constant delay less than half the length of the interval and Robin boundary conditions*. Results Math. 74 (2019), no. 1, 74:45.

[12] F. Vajzović, *Functional Analysis III, (Mimeographs lectures)*, Faculty of Natural Sciences and Mathematics, Sarajevo (1973), p. 146

[13] V. Vladičić, M. Pikula, *An inverse problems for Sturm-Liouville-type differential equation with a constant delay*, Sarajevo Journal of Mathematics, DOI: 10.5644/SJM.12.1.06 Vol.12 (24), No.1, 83–88, (2016)

[14] V. Vladičić, N. Djurić, *Incomplete Inverse Problem for Sturm–Liouville Type Differential Equation with Constant Delay*, Results Math, Springer, 74(4) 161(2019)

[15] V.A. Yurko, S.A. Buterin, M. Pikula, *Sturm-Liouville differential operators with deviating argument* , Tamkang Journal of Mathematics, 48(1), 49–59, (2017)

[16] V.A. Yurko, *An inverse spectral problem for second order differential operators with retarded argument.* Results in Mathematics, vol. 74, no. 2 (2019), article 71. DOI: 10.1007/s00025-019-0999-6 (2019).

[17] V.A. Yurko, *Recovering Differential Operators with a Retarded Argument*, Differential Equations, (2019), Vol. 55, No. 4, pp. 510–514

[18] V.A. Yurko, *Solution of Inverse Problems for Differential Operators with Delay*. Transmutation operators and Applications. Trends in Mathematics. Springer, 2020, pp.467–475. https://www.springer.com/gp/book/9783030359133

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