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MORE ON HURWITZ AND TASOEV CONTINUED FRACTIONS

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Abstract. Several new types of Hurwitz continued fractions have been studied. Most basic Hurwitz continued fractions can be expressed by using confluent hypergeometric functions ${}_0F_1(;c;z)$. This expression enables us to find some more general Hurwitz continued fractions. A contrast between Tasoev continued fractions and Hurwitz ones yields some more general Tasoev continued fractions. Some Ramanujan continued fractions are also discussed.

1. Hurwitz continued fractions

$\alpha = [a_0; a_1, a_2, \dots]$ denotes the regular (or simple) continued fraction expansion of a real α, where

$$\alpha = a_0 + \theta_0, \quad a_0 = \lfloor \alpha \rfloor ,$$
$$1/\theta_{n-1} = a_n + \theta_n, \quad a_n = \lfloor 1/\theta_{n-1} \rfloor \quad (n \geq 1) .$$

Hurwitz continued fraction expansions have the form

$$[a_0; a_1, \dots, a_n, \overline{Q_1(k), \dots, Q_p(k)}]_{k=1}^{\infty}$$
$$= [a_0; a_1, \dots, a_n, Q_1(1), \dots, Q_p(1), Q_1(2), \dots, Q_p(2), Q_1(3), \dots] ,$$

where a_0 is an integer, $a_1, \dots, a_n$ are positive integers, $Q_1, \dots, Q_p$ are polynomials with rational coefficients which take positive integral values for $k = 1, 2, \dots$ and at least one of the polynomials is not constant. Up to the

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present, some basic known examples are the following.

$$e^{1/s} = [1; \overline{(2k-1)s-1, 1, 1}\,]_{k=1}^{\infty} \quad (s \in \mathbb{Z}, s > 1)\,.$$

$$ae^{1/a} = [a+1; \overline{2a-1, 2k, 1}\,]_{k=1}^{\infty} \quad (a \in \mathbb{Z}_+)\,.$$

$$\frac{1}{a}e^{1/a} = [0; a-1, 2a, \overline{1, 2k, 2a-1}\,]_{k=1}^{\infty} \quad (a \in \mathbb{Z}, a > 1)\,.$$

$$e^2 = [7; \overline{3k-1, 1, 1, 3k, 12k+6}\,]_{k=1}^{\infty}\,.$$

$$e^{2/s} = \left[1; \overline{3ks - \frac{5s+1}{2}, 12ks-6s, 3ks - \frac{s+1}{2}, 1, 1}\right]_{k=1}^{\infty} \quad (s : \text{odd}, s \ge 3)\,.$$

$$\sqrt{\frac{v}{u}} \tanh \frac{1}{\sqrt{uv}} = [0; \overline{(4k-3)u, (4k-1)v}\,]_{k=1}^{\infty} \quad (u, v \in \mathbb{Z}_+)\,.$$

$$\frac{I_{(a/b)+1}(\frac{2}{b})}{I_{a/b}(\frac{2}{b})} = [0; \overline{a+kb}]_{k=1}^{\infty}\,,$$

where $I_\lambda(z)$ are the modified Bessel functions of the first kind, defined by

$$I_\lambda(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{\lambda+2n}}{n!\Gamma(\lambda+n+1)}\,.$$

$$\sqrt{\frac{v}{u}} \tan \frac{1}{\sqrt{uv}} = [0; u-1, \overline{1, (4k-1)v-2, 1, (4k+1)u-2}\,]_{k=1}^{\infty}\,.$$

$$\frac{J_{(a/b)+1}(\frac{2}{b})}{J_{a/b}(\frac{2}{b})} = [0; a+b-1, \overline{1, a+(k+1)b-2}\,]_{k=1}^{\infty}\,,$$

where $J_\lambda(z)$ are the Bessel functions of the first kind, defined by

$$J_\lambda(z) = \left(\frac{z}{2}\right)^{\lambda} \sum_{n=0}^{\infty} \frac{(iz/2)^{2n}}{n!\Gamma(\lambda+n+1)}\,.$$

It seems that each one of the above belongs to one of the types, e-type, *tanh*-type, *tan*-type and e^2-type. No concrete example where the degree of any polynomial exceeds 1 has been known.

Recently, the author [5] obtained a generalized *tanh*-type Hurwitz continued fraction as

$$[0; \overline{u(a+(2k-1)b), v(a+2kb)}\,]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty}(n!)^{-1}u^{-n-1}(vb)^{-n}\prod_{i=1}^{n+1}(a+bi)^{-1}}{\sum_{n=0}^{\infty}(n!)^{-1}(uvb)^{-n}\prod_{i=1}^{n}(a+bi)^{-1}}\,,$$

which includes the cases of $\sqrt{\frac{v}{u}}\tanh\frac{1}{\sqrt{uv}}$ and $\frac{I_{(a/b)+1}(\frac{2}{b})}{I_{a/b}(\frac{2}{b})}$. The author also obtained a generalized *tan*-type Hurwitz continued fraction as

$$\begin{aligned}&[0; u(a+b)-1, \overline{1, v(a+2kb)-2, 1, u(a+(2k+1)b)-2}\,]\\&\qquad = \frac{\sum_{n=0}^\infty (-1)^n (n!)^{-1} u^{-n-1}(vb)^{-n}\prod_{i=1}^{n+1}(a+bi)^{-1}}{\sum_{n=0}^\infty (-1)^n (n!)^{-1}(uvb)^{-n}\prod_{i=1}^{n}(a+bi)^{-1}}\,,\end{aligned}$$

which includes the cases of $\sqrt{\frac{v}{u}}\tan\frac{1}{\sqrt{uv}}$ and and $\frac{J_{(a/b)+1}(\frac{2}{b})}{J_{a/b}(\frac{2}{b})}$.

In [8], the author constituted more general forms of Hurwitz continued fractions of e-type, namely, some extended forms of the continued fractions of $e^{1/s}$, $ae^{1/a}$ and $\frac{1}{a}e^{1/a}$.

$$\begin{aligned}&[0; \overline{u(a+kb)-1, 1, v-1}]_{k=1}^\infty\\&= \frac{\sum_{n=0}^\infty u^{-2n-1}v^{-2n}b^{-n}(n!)^{-1}\prod_{i=1}^{n+1}(a+bi)^{-1}}{\sum_{n=0}^\infty b^{-n}(n!)^{-1}\big((uv)^{-2n}\prod_{i=1}^{n}(a+bi)^{-1}-(uv)^{-2n-1}\prod_{i=1}^{n+1}(a+bi)^{-1}\big)}\end{aligned}$$

and

$$\begin{aligned}&[0; \overline{v-1, 1, u(a+kb)-1}]_{k=1}^\infty\\&= \frac{\sum\limits_{n=0}^\infty b^{-n}(n!)^{-1}\big(u^{-2n}v^{-2n-1}\prod\limits_{i=1}^{n}(a+bi)^{-1}+u^{-2n-1}v^{-2n-2}\prod\limits_{i=1}^{n+1}(a+bi)^{-1}\big)}{\sum\limits_{n=0}^\infty (uv)^{-2n}b^{-n}(n!)^{-1}\prod\limits_{i=1}^{n}(a+bi)^{-1}}.\end{aligned}$$

There are still more known Hurwitz continued fractions which may not belong to any of the above categories. Most of them are easily derived from one of the basic types (see e.g. [5, Props. 1 and 2]).

2. Confluent hypergeometric functions

Some generalized Hurwitz continued fractions which the author obtained become elegant to look at with the aid of hypergeometric functions (see e.g. [16]). Using the confluent hypergeometric limit functions defined by

$${}_0F_1(;c;z) = \sum_{n=0}^\infty \frac{1}{(c)_n}\frac{z^n}{n!}$$

with $(c)_n = c(c+1)\dots(c+n-1)$ $(n \ge 1)$ and $(c)_0 = 1$, we have the following identities.

Lemma 1.

$$[0; \overline{u(a+(2k-1)b), v(a+2kb)}\,]_{k=1}^\infty = \frac{{}_0F_1(;\frac{a}{b}+2;\frac{1}{uvb^2})}{u(a+b){}_0F_1(;\frac{a}{b}+1;\frac{1}{uvb^2})}\,. \tag{1}$$

$$[0; u(a+b)-1, \overline{1, v(a+2kb)-2, 1, u(a+(2k+1)b)-2}\,] = \frac{{}_0F_1(;\frac{a}{b}+2;\frac{-1}{uvb^2})}{u(a+b){}_0F_1(;\frac{a}{b}+1;\frac{-1}{uvb^2})}\,. \quad (2)$$

Proof of Lemma 1. By (91.4) in Chapter 18 of [17]

$$\frac{{}_0F_1(;c+1;z)}{{}_0F_1(;c;z)} = \frac{1}{1+}\frac{\frac{z}{c(c+1)}}{1}\frac{\frac{z}{(c+1)(c+2)}}{1}\frac{\frac{z}{(c+2)(c+3)}}{1}\,{+}\cdots\,.$$

Then we transform it to the simple continued fraction by using the following formula in [4, (2.3.23) (p. 35)] (*Cf.* [6, Lemma 1]):

$$[a_0; a_1, a_2, \dots] = a_0^* + \frac{b_1^*}{a_1^*+}\frac{b_2^*}{a_2^*+}\frac{b_3^*}{a_3^*+}\cdots$$

iff $a_0 = a_0^*$, $a_1 = a_1^*/b_1^*$ and for $k = 1, 2, \dots$

$$a_{2k} = \frac{b_{2k-1}^* b_{2k-3}^* \cdots b_1^*}{b_{2k}^* b_{2k-2}^* \cdots b_2^*} a_{2k}^* \quad \text{and} \quad a_{2k+1} = \frac{b_{2k}^* b_{2k-2}^* \cdots b_2^*}{b_{2k+1}^* b_{2k-1}^* \cdots b_1^*} a_{2k+1}^*\,.$$

If we put $c = (a/b) + 1$ and $z = 1/(uvb^2)$, then

$$\begin{aligned}\frac{{}_0F_1(;\frac{a}{b}+2;\frac{1}{uvb^2})}{{}_0F_1(;\frac{a}{b}+1;\frac{1}{uvb^2})} &= \frac{1}{1+}\frac{u^{-1}v^{-1}(a+b)^{-1}(a+2b)^{-1}}{1}\\ &\quad+\frac{u^{-1}v^{-1}(a+2b)^{-1}(a+3b)^{-1}}{1}+\frac{u^{-1}v^{-1}(a+3b)^{-1}(a+4b)^{-1}}{1}+\cdots\\ &= \Big[0; 1, u^2v^2(a+b)(a+2b), \frac{a+3b}{a+b}, u^2v^2(a+b)(a+4b), \frac{a+5b}{a+b},\\ &\qquad u^2v^2(a+b)(a+6b), \frac{a+7b}{a+b}, \dots\Big]\,.\end{aligned}$$

Dividing by $u(a+b)$ yields

$$\frac{{}_0F_1(;\frac{a}{b}+2;\frac{1}{uvb^2})}{u(a+b){}_0F_1(;\frac{a}{b}+1;\frac{1}{uvb^2})} = [0; u(a+b), v(a+2b), u(a+3b), v(a+4b), u(a+5b), v(a+6b), \dots]\,.$$

The proof of Lemma 1 (2) is similar and omitted. □

Lemma 1 can be used to produce varieties of continued fractions including e-type Hurwitz continued fractions. For example, the author [10] obtained the following extended Hurwitz continued fractions by using the confluent hypergeometric functions ${}_0F_1(;c;z)$. The rational numbers u, u', a and b are chosen so that each partial quotient of the continued fraction is a positive integer.

Proposition 1. *If v is an integer with $v > 1$, then*

$$[0; \overline{u(a+(2k-1)b)-1, 1, v-1, u'(a+2kb)-1, 1, v-1}]_{k=1}^{\infty} = \frac{v\,{}_0F_1(;\frac{a}{b}+2;\frac{1}{uu'v^2b^2})}{uv(a+b)\,{}_0F_1(;\frac{a}{b}+1;\frac{1}{uu'v^2b^2}) - {}_0F_1(;\frac{a}{b}+2;\frac{1}{uu'v^2b^2})} \tag{3}$$

and

$$[0; \overline{v-1, 1, u(a+(2k-1)b)-1, v-1, 1, u'(a+2kb)-1}]_{k=1}^{\infty} = \frac{{}_0F_1(;\frac{a}{b}+2;\frac{1}{uu'v^2b^2})}{uv^2(a+b)\,{}_0F_1(;\frac{a}{b}+1;\frac{1}{uu'v^2b^2})} + \frac{1}{v}. \tag{4}$$

If v is an integer with $v > 2$, then

$$[0; \overline{u(a+(2k-1)b)-1, 1, v-2, 1, u'(a+2kb)-1, v}]_{k=1}^{\infty} = \frac{v\,{}_0F_1(;\frac{a}{b}+2;\frac{-1}{uu'v^2b^2})}{uv(a+b)\,{}_0F_1(;\frac{a}{b}+1;\frac{-1}{uu'v^2b^2}) - {}_0F_1(;\frac{a}{b}+2;\frac{-1}{uu'v^2b^2})} \tag{5}$$

and

$$[0; v-1, \overline{1, u(a+(2k-1)b)-1, v, u'(a+2kb)-1, 1, v-2}]_{k=1}^{\infty} = \frac{{}_0F_1(;\frac{a}{b}+2;\frac{-1}{uu'v^2b^2})}{uv^2(a+b)\,{}_0F_1(;\frac{a}{b}+1;\frac{-1}{uu'v^2b^2})} + \frac{1}{v}. \tag{6}$$

3. Main results about Hurwitz continued fractions

First, we shall show some more general and new Hurwitz continued fractions by using the confluent hypergeometric functions. As such generalizations, we get the Hurwitz continued fractions with much longer period. The rational numbers u, u', a and b are chosen so that each partial quotient of the continued fraction is a positive integer.

Theorem 1. *If $l|(v-1)$ and $v > l+1 > 2$, then*

$$\left[0; \overline{u(a+(2k-1)b)-1, 1, \frac{v-l-1}{l}, l-1, 1, u'(a+2kb)-1, l, \frac{v-1}{l}}\right]_{k=1}^{\infty} = \frac{v\,{}_0F_1(;\frac{a}{b}+2;\frac{1}{uu'v^2b^2})}{uv(a+b)\,{}_0F_1(;\frac{a}{b}+1;\frac{1}{uu'v^2b^2}) - l\,{}_0F_1(;\frac{a}{b}+2;\frac{1}{uu'v^2b^2})}. \tag{7}$$

If $l|(v+1)$ *and* $v>2l-1>3$, *then*

$$\Big[0;\overline{u(a+(2k-1)b)-1,1,\frac{v-2l+1}{l},1,l-1,}$$
$$\overline{u'(a+2kb)-1,1,l-2,1,\frac{v-l+1}{l}}\Big]_{k=1}^{\infty}$$
$$=\frac{v\,{}_0F_1(;\frac{a}{b}+2;\frac{1}{uu'v^2b^2})}{uv(a+b)\,{}_0F_1(;\frac{a}{b}+1;\frac{1}{uu'v^2b^2})-l\,{}_0F_1(;\frac{a}{b}+2;\frac{1}{uu'v^2b^2})}\,. \quad (8)$$

Remark. If the case $l=1$ is allowed in (7), by the rule $[\dots,\frac{v-l-1}{l},l-1,1,\dots]=[\dots,v-1,\dots]$ the identity (7) is reduced to Proposition 1 (3). If the case $l=2$ is allowed in (8), by $[\dots,1,l-2,1,\dots]=[\dots,2,\dots]$ this becomes the same as (7).

Theorem 2. *If* $l|(v-1)$ *and* $v>l+1>2$, *then*

$$\Big[0;\overline{\frac{v-1}{l},l,u(a+(2k-1)b)-1,1,l-1,\frac{v-l-1}{l},1,u'(a+2kb)-1}\Big]_{k=1}^{\infty}$$
$$=\frac{{}_0F_1(;\frac{a}{b}+2;\frac{1}{uu'v^2b^2})}{uv^2(a+b)\,{}_0F_1(;\frac{a}{b}+1;\frac{1}{uu'v^2b^2})}+\frac{l}{v}\,. \quad (9)$$

If $l(v+1)$ *and* $v>2l-1>3$, *then*

$$\Big[0;\overline{\frac{v-l+1}{l},1,l-2,1,u(a+(2k-1)b)-1,l-1,1,}$$
$$\overline{\frac{v-2l+1}{l},1,u'(a+2kb)-1}\Big]_{k=1}^{\infty}$$
$$=\frac{{}_0F_1(;\frac{a}{b}+2;\frac{1}{uu'v^2b^2})}{uv^2(a+b)\,{}_0F_1(;\frac{a}{b}+1;\frac{1}{uu'v^2b^2})}+\frac{l}{v}\,. \quad (10)$$

Remark. If $l=1$ in (9), by the rule $[\dots,1,l-1,\frac{v-l-1}{l},\dots]=[\dots,v-1,\dots]$ this is reduced to Proposition 1 (4). If $l=2$ in (10), by $[\dots,1,l-2,1,\dots]=[\dots,2,\dots]$ this becomes the same as (9).

Theorem 3. *If* $l|(v-1)$ *and* $v>l+1>2$, *then*

$$\Big[0;\overline{u(a+(2k-1)b)-1,1,\frac{v-l-1}{l},l,u'(a+2kb)-1,1,l-1,\frac{v-1}{l}}\Big]_{k=1}^{\infty}$$
$$=\frac{v\,{}_0F_1(;\frac{a}{b}+2;\frac{-1}{uu'v^2b^2})}{uv(a+b)\,{}_0F_1(;\frac{a}{b}+1;\frac{-1}{uu'v^2b^2})-l\,{}_0F_1(;\frac{a}{b}+2;\frac{-1}{uu'v^2b^2})}\,. \quad (11)$$

If $l|(v+1)$ *and* $v>2l-1>3$, *then*

$$\Big[0;\overline{u(a+(2k-1)b)-1,1,\frac{v-2l+1}{l},1,l-2,1,}$$
$$\overline{u'(a+2kb)-1,l-1,1,\frac{v-l+1}{l}}\Big]_{k=1}^{\infty}$$
$$=\frac{v_0F_1(;\frac{a}{b}+2;\frac{-1}{uu'v^2b^2})}{uv(a+b)_0F_1(;\frac{a}{b}+1;\frac{-1}{uu'v^2b^2})-l_0F_1(;\frac{a}{b}+2;\frac{-1}{uu'v^2b^2})}\,, \quad (12)$$

Remark. If $l=1$ in (11), by the rule $[\dots,1,l-1,\frac{v-1}{l},\dots]=[\dots,v,\dots]$ this is reduced to Proposition 1 (5). If $l=2$ in (12), by the rule $[\dots,1,l-2,1,\dots]=[\dots,l,\dots]$ this becomes the same as (11).

Theorem 4. *If* $l|(v-1)$ *and* $v>l+1>2$, *then*

$$\Big[0;\frac{v-1}{l},\overline{l,u(a+(2k-1)b)-1,1,l-1,}$$
$$\overline{\frac{v-1}{l},u'(a+2kb)-1,1,\frac{v-l-1}{l}}\Big]_{k=1}^{\infty}$$
$$=\frac{{}_0F_1(;\frac{a}{b}+2;\frac{-1}{uu'v^2b^2})}{uv^2(a+b)_0F_1(;\frac{a}{b}+1;\frac{-1}{uu'v^2b^2})}+\frac{l}{v}\,. \quad (13)$$

If $l|(v+1)$ *and* $v>2l-1>3$, *then*

$$\Big[0;\frac{v-l+1}{l},\overline{1,l-2,1,u(a+(2k-1)b)-1,l-1,1,\frac{v-l+1}{l},}$$
$$\overline{u'(a+2kb)-1,1,\frac{v-2l+1}{l}}\Big]_{k=1}^{\infty}$$
$$=\frac{{}_0F_1(;\frac{a}{b}+2;\frac{-1}{uu'v^2b^2})}{uv^2(a+b)_0F_1(;\frac{a}{b}+1;\frac{-1}{uu'v^2b^2})}+\frac{l}{v}\,. \quad (14)$$

Remark. If $l=1$ in (13), by the rule $[\dots,1,l-1,\frac{v-1}{l},\dots]=[\dots,v,\dots]$ this is reduced to Proposition 1 (6). If $l=2$ in (14), by the rule $[\dots,1,l-2,1,\dots]=[\dots,l,\dots]$ this becomes the same as (13).

4. Proof of the results

We use a Hurwitz's method to obtain the continued fraction expansion $(a\alpha+b)/d$ from the continued fraction expansion of α. In most practical cases it is enough to consider the rational linear forms of α. According to Satz 4.1 (p. 111) in [13], which is essentially from Hurwitz [3] and Châtelet [2], it says

Lemma 2. *Let* $[a_0; a_1, a_2, \dots]$ *be the regular continued fraction of an irrational number* α *and denote its* n*-th convergent by* $p_n/q_n = [a_0; a_1, a_2, \dots, a_n]$. *Moreover, let* $\beta = (r_0\alpha + t_0)/s_0$, *where* r_0, s_0 *and* t_0 *are integers with* $r_0 > 0$, $s_0 > 0$ *and* $r_0 s_0 = N > 1$. *For an arbitrary index* $\nu \ge 1$ *we have*

$$\frac{r_0[a_0; a_1, \dots, a_{\nu-1}] + t_0}{s_0} = \frac{r_0 p_{\nu-1} + t_0 q_{\nu-1}}{s_0 q_{\nu-1}} = [b_0; b_1, \dots, b_{\mu-1}]$$

where the index μ *is adjusted such that* $\mu \equiv \nu \pmod 2$. *Denote its convergent by*

$$\frac{p'_{\mu-1}}{q'_{\mu-1}} = [b_0; b_1, \dots, b_{\mu-1}] .$$

Then three integers t_1, r_1 *and* s_1 *are uniquely given satisfying the matrix formula*

$$\begin{pmatrix} r_0 & t_0 \\ 0 & s_0 \end{pmatrix} \begin{pmatrix} p_{\nu-1} & p_{\nu-2} \\ q_{\nu-1} & q_{\nu-2} \end{pmatrix} = \begin{pmatrix} p'_{\mu-1} & p'_{\mu-2} \\ q'_{\mu-1} & q'_{\mu-2} \end{pmatrix} \begin{pmatrix} r_1 & t_1 \\ 0 & s_1 \end{pmatrix} ,$$

where $r_1 > 0$, $s_1 > 0$, $r_1 s_1 = N$, $-s_1 \le t_1 \le r_1$ *and* $\beta = [b_0; b_1, \dots, b_{\mu-1}, \beta_\mu]$ *with* $\beta_\mu = (r_1\alpha_\nu + t_1)/s_1$.

Proof of Theorem 1. Replacing v by $u'v^2$ and taking its reciprocal in Lemma 1 (1), we have

$$(\alpha :=) u(a+b) \cdot \frac{{}_0F_1(; \frac{a}{b}+1; \frac{1}{uu'v^2b^2})}{{}_0F_1(; \frac{a}{b}+2; \frac{1}{uu'v^2b^2})} = [\overline{u(a+(2k-1)b), u'v^2(a+2kb)}]_{k=1}^{\infty} .$$

Applying Lemma 2 to the continued fraction of $\alpha - l/v$ with $v = lv' + 1$. For $k = 1, 2, \dots$ we have

$$\begin{aligned} \frac{v[u(a+(2k-1)b)] - l}{v} &= u(a+(2k-1)b) - \frac{l}{v} \\ &= [u(a+(2k-1)b) - 1; 1, v'-1, l-1, 1] \end{aligned}$$

and

$$\begin{aligned} &\begin{pmatrix} v & -l \\ 0 & v \end{pmatrix} \begin{pmatrix} u(a+(2k-1)b) & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} uv(a+(2k-1)b) - l & u((l-1)v'+1)(a+(2k-1)b) - (l-1) \\ v & (l-1)v'+1 \end{pmatrix} \\ &\qquad \times \begin{pmatrix} 1 & -v((l-1)v'+1) \\ 0 & v^2 \end{pmatrix} , \end{aligned}$$

then

$$\frac{[u'v^2(a+2kb)] - v((l-1)v'+1)}{v^2} = [u'(a+2kb) - 1; l, v']$$

and

$$\begin{pmatrix} 1 & -v((l-1)v'+1) \\ 0 & v^2 \end{pmatrix} \begin{pmatrix} u'v^2(a+2kb) & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} u'v(a+2kb)-((l-1)v'+1) & lu'(a+2kb)-l+1 \\ v & l \end{pmatrix} \begin{pmatrix} v & -l \\ 0 & v \end{pmatrix}.$$

Therefore, we obtain the desired continued fraction as

$$\frac{1}{\alpha - l/v} = \frac{v{}_0F_1(;\frac{a}{b}+2;\frac{1}{uu'v^2b^2})}{uv(a+b){}_0F_1(;\frac{a}{b}+1;\frac{1}{uu'v^2b^2}) - l{}_0F_1(;\frac{a}{b}+2;\frac{1}{uu'v^2b^2})}$$
$$= \Big[0; \overline{u(a+(2k-1)b)-1, 1, \frac{v-l-1}{l}, l-1, 1, u'(a+2kb)-1, l, \frac{v-1}{l}}\Big]_{k=1}^{\infty}.$$

In order to prove (8), let

$$\alpha = u(a+b)\cdot \frac{{}_0F_1(;\frac{a}{b}+1;\frac{1}{uu'v^2b^2})}{{}_0F_1(;\frac{a}{b}+2;\frac{1}{uu'v^2b^2})} = [\overline{u(a+(2k-1)b), u'v^2(a+2kb)}]_{k=1}^{\infty}$$

and consider the continued fraction of $\alpha - l/v$ with $v = lv' + (l-1)$. For $k = 1, 2, \dots$ we have

$$\frac{v[u(a+(2k-1)b)]-l}{v} = [u(a+(2k-1)b)-1; 1, v'-1, 1, l-1]$$

and

$$\begin{pmatrix} v & -l \\ 0 & v \end{pmatrix} \begin{pmatrix} u(a+(2k-1)b) & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} uv(a+(2k-1)b)-l & u(v'+1)(a+(2k-1)b)-1 \\ v & v'+1 \end{pmatrix} \begin{pmatrix} 1 & -v(v'+1) \\ 0 & v^2 \end{pmatrix},$$

then

$$\frac{[u'v^2(a+2kb)]-v(v'+1)}{v^2} = [u'(a+2kb)-1; 1, l-2, 1, v']$$

and

$$\begin{pmatrix} 1 & -v(v'+1) \\ 0 & v^2 \end{pmatrix} \begin{pmatrix} u'v^2(a+2kb) & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} u'v(a+2kb)-(v'+1) & lu'(a+2kb)-1 \\ v & l \end{pmatrix} \begin{pmatrix} v & -l \\ 0 & v \end{pmatrix}.$$

Therefore, we obtain the desired continued fraction as

$$\frac{1}{\alpha - l/v} = \Big[0; \overline{u(a+(2k-1)b)-1, 1, \frac{v-2l+1}{l}, 1, l-1,} \\ \overline{u'(a+2kb)-1, 1, l-2, 1, \frac{v-l+1}{l}}\Big]_{k=1}^{\infty}.$$

□

Proof of Theorem 2. Replacing u by uv^2 and v by u' in Lemma 1 (1), we have

$$(\alpha :=) \frac{1}{uv^2(a+b)} \frac{{}_0F_1(;\frac{a}{b}+2;\frac{1}{uu'v^2b^2})}{{}_0F_1(;\frac{a}{b}+1;\frac{1}{uu'v^2b^2})} \\ = [0; \overline{uv^2(a+(2k-1)b), u'(a+2kb)}\,]_{k=1}^{\infty}.$$

Applying Lemma 2 to the continued fraction of $\alpha + l/v$ with $v = lv' + 1$. First,

$$\frac{v[0]+l}{v} = [0; v', l]$$

and

$$\begin{pmatrix} v & l \\ 0 & v \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} l & 1 \\ v & v' \end{pmatrix} \begin{pmatrix} 1 & -vv' \\ 0 & v^2 \end{pmatrix}.$$

Next, for $k = 1, 2, \dots$ we have

$$\frac{[uv^2(a+(2k-1)b)] - vv'}{v^2} = [u(a+(2k-1)b)-1; 1, l-1, 1, v'-1, 1]$$

and

$$\begin{pmatrix} 1 & -vv' \\ 0 & v^2 \end{pmatrix} \begin{pmatrix} uv^2(a+(2k-1)b) & 1 \\ 1 & 0 \end{pmatrix} \\ = \begin{pmatrix} uv(a+(2k-1)b)-v' & u(lv'-l+1)(a+(2k-1)b)-v'+1 \\ v & lv'-l+1 \end{pmatrix} \\ \times \begin{pmatrix} v & -lv'+l-1 \\ 0 & v \end{pmatrix},$$

then

$$\frac{v[u'(a+2kb)] - lv' + l - 1}{v} = [u'(a+2kb)-1; v', l]$$

and

$$\begin{pmatrix} v & -lv'+l-1 \\ 0 & v \end{pmatrix} \begin{pmatrix} u'(a+2kb) & 1 \\ 1 & 0 \end{pmatrix} \\ = \begin{pmatrix} u'v(a+2kb)-lv+l-1 & u'v'(a+2kb)-v'+1 \\ v & v' \end{pmatrix} \begin{pmatrix} 1 & -vv' \\ 0 & v^2 \end{pmatrix}.$$

Therefore,

$$\alpha + \frac{l}{v} = \Big[0; \frac{v-1}{l}, l, \overline{u(a+(2k-1)b)-1, 1, l-1,}$$
$$\overline{\frac{v-l-1}{l}, 1, u'(a+2kb)-1, \frac{v-1}{l}, l}\Big]_{k=1}^{\infty}$$
$$= \Big[0; \overline{\frac{v-1}{l}, l, u(a+(2k-1)b)-1, 1, l-1, \frac{v-l-1}{l}, 1, u'(a+2kb)-1}\Big]_{k=1}^{\infty}.$$

In order to prove (10), let

$$\alpha = [0; \overline{uv^2(a+(2k-1)b), u'(a+2kb)}\,]_{k=1}^{\infty}$$

and consider the continued fraction of $\alpha + l/v$ with $v = lv' + l - 1$. First,

$$\frac{v[0]+l}{v} = [0; v', 1, l-2, 1]$$

and

$$\begin{pmatrix} v & l \\ 0 & v \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} l & l-1 \\ v & (l-1)v' + (l-2) \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & -v((l-1)v' + (l-2)) \\ 0 & v^2 \end{pmatrix}.$$

Next, for $k = 1, 2, \dots$ we have

$$\frac{[uv^2(a+(2k-1)b)] - v((l-1)v' + l-2)}{v^2} = [u(a+(2k-1)b)-1; l-1, 1, v'-1, 1]$$

and

$$\begin{pmatrix} 1 & -v((l-1)v' + l - 2) \\ 0 & v^2 \end{pmatrix}\begin{pmatrix} uv^2(a+(2k-1)b) & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} uv(a+(2k-1)b) - (l-1)v' - (l-2) & u(lv'-1)(a+(2k-1)b) - (l-1)v' + 1 \\ v & lv'-1 \end{pmatrix}$$
$$\times \begin{pmatrix} v & -lv'-1 \\ 0 & v \end{pmatrix},$$

then

$$\frac{v[u'(a+2kb)] - lv' + 1}{v} = [u'(a+2kb)-1; v', 1, l-2, 1]$$

and

$$\begin{pmatrix} v & -lv'+1 \\ 0 & v \end{pmatrix} \begin{pmatrix} u'(a+2kb) & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} u'v(a+2kb)-lv'+1 & u'((l-1)v'+l-2)(a+2kb)-(l-1)v'+1 \\ v & (l-1)v'+l-2 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & -v((l-1)v'+l-2) \\ 0 & v^2 \end{pmatrix}.$$

Therefore,

$$\alpha + \frac{l}{v} = \Big[0; \frac{v-l+1}{l}, 1, l-2, 1, \overline{u(a+(2k-1)b)-1, l-1, 1, \frac{v-2l+1}{l},}$$

$$\overline{1, u'(a+2kb)-1, \frac{v-l+1}{l}, 1, l-2, 1}\Big]_{k=1}^{\infty}$$

$$= \Big[0; \overline{\frac{v-l+1}{l}, 1, l-2, 1, u(a+(2k-1)b)-1, l-1, 1,}$$

$$\overline{\frac{v-2l+1}{l}, 1, u'(a+2kb)-1}\Big]_{k=1}^{\infty}.$$

□

Proof of Theorem 3. Replacing v by $u'v^2$ and taking its reciprocal in Lemma 1 (2), we have

$$(\alpha :=) u(a+b) \cdot \frac{{}_0F_1(; \frac{a}{b}+1; \frac{-1}{uu'v^2b^2})}{{}_0F_1(; \frac{a}{b}+2; \frac{-1}{uu'v^2b^2})}$$

$$= [u(a+b)-1; \overline{1, u'v^2(a+2kb)-2, 1, u(a+(2k+1)b)-2}\,]_{k=1}^{\infty}.$$

And consider the continued fraction of $1/(\alpha - l/v)$ with $v = lv'+1$.

In order to prove (12), let

$$\alpha = [u(a+b)-1; \overline{1, u'v^2(a+2kb)-2, 1, u(a+(2k+1)b)-2}\,]_{k=1}^{\infty}$$

and consider the continued fraction of $1/(\alpha - l/v)$ with $v = lv'+l-1$. □

Proof of Theorem 4. Replacing u by uv^2 and v by u' in Lemma 1 (2), we have

$$(\alpha :=) \frac{1}{uv^2(a+b)} \frac{{}_0F_1(; \frac{a}{b}+2; \frac{-1}{uu'v^2b^2})}{{}_0F_1(; \frac{a}{b}+1; \frac{-1}{uu'v^2b^2})}$$

$$= [0; uv^2(a+b)-1, \overline{1, u'(a+2kb)-2, 1, uv^2(a+(2k+1)b)-2}\,]_{k=1}^{\infty}.$$

And consider the continued fraction of $\alpha + l/v$ with $v = lv'+1$.

In order to prove (14), let

$$\alpha = [0; uv^2(a+b)-1, \overline{1, u'(a+2kb)-2, 1, uv^2(a+(2k+1)b)-2}\,]_{k=1}^{\infty}$$

and consider the continued fraction of $\alpha + l/v$ with $v = lv' + l - 1$. □

5. Tasoev continued fractions and their generalizations

Tasoev continued fractions, which have the form $[a_0; a_1, \dots, a_n, \overline{Q_1(k), \dots, Q_p(k)}]_{k=1}^{\infty}$, are also quasi-periodic but $Q_j(k)$ includes exponentials in k instead of polynomials ([5], [7], [8], [9], [11]). They are as systematic as Hurwitz ones, but have seldom been known before. Corresponding with Hurwitz continued fractions, *tanh*-type, *tan*-type and *e*-type Tasoev continued fractions were obtained by the author.

tanh-type Tasoev continued fractions obtained by the author [6] were

$$[0; \overline{ua^k}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-2n-1} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i} - 1)^{-1}}{\sum_{n=0}^{\infty} u^{-2n} a^{-n^2} \prod_{i=1}^{n} (a^{2i} - 1)^{-1}}$$

and

$$[0; \overline{ua^k, va^k}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-n-1} v^{-n} a^{-(n+1)(n+2)/2} \prod_{i=1}^{n} (a^{i} - 1)^{-1}}{\sum_{n=0}^{\infty} u^{-n} v^{-n} a^{-n(n+1)/2} \prod_{i=1}^{n} (a^{i} - 1)^{-1}} .$$

tan-type Tasoev continued fractions obtained by the author [6] were

$$[0; ua - 1, \overline{1, ua^{k+1} - 2}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} (-1)^n u^{-2n-1} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i} - 1)^{-1}}{\sum_{n=0}^{\infty} (-1)^n u^{-2n} a^{-n^2} \prod_{i=1}^{n} (a^{2i} - 1)^{-1}}$$

and

$$[0; ua - 1, 1, va - 2, \overline{1, ua^{k+1} - 2, 1, va^{k+1} - 2}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} (-1)^n u^{-n-1} v^{-n} a^{-(n+1)(n+2)/2} \prod_{i=1}^{n} (a^{i} - 1)^{-1}}{\sum_{n=0}^{\infty} (-1)^n u^{-n} v^{-n} a^{-n(n+1)/2} \prod_{i=1}^{n} (a^{i} - 1)^{-1}} .$$

e-type Tasoev continued fractions obtained by the author [8] were

$$[0; \overline{ua^k - 1, 1, v - 1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-2n-1} v^{-2n} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i} - 1)^{-1}}{\sum_{n=0}^{\infty} \big((uv)^{-2n} a^{-n^2} - (uv)^{-2n-1} a^{-(n+1)^2} \big) \prod_{i=1}^{n} (a^{2i} - 1)^{-1}}$$

and

$$[0; \overline{v - 1, 1, ua^k - 1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} \big(u^{-2n} v^{-2n-1} a^{-n^2} + u^{-2n-1} v^{-2n-2} a^{-(n+1)^2} \big) \prod_{i=1}^{n} (a^{2i} - 1)^{-1}}{\sum_{n=0}^{\infty} (uv)^{-2n} a^{-n^2} \prod_{i=1}^{n} (a^{2i} - 1)^{-1}} .$$

Some more Tasoev continued fractions with a general construction can be seen in [9].

As pointed in [7, Fact 1], Tasoev continued fractions are geometric and Hurwitz continued fractions are arithmetic. From such a point of view, we can have the following eight theorems. Proving of Theorem 5 to Theorem 8 is similar to proving of Theorem 1 to Theorem 4, respectively. Simply replace $u(a+(2k-1)b)$ by ua^k and $u'(a+2kb)$ by wa^k etc. The rational numbers u, w and a are chosen so that each partial quotient of the continued fraction is a positive integer.

Theorem 5. *If* $l|(v-1)$ *and* $v>l+1>2$, *then*

$$\Big[0;\overline{ua^k-1,1,\frac{v-l-1}{l},l-1,1,wa^k-1,l,\frac{v-1}{l}}\Big]_{k=1}^\infty$$
$$=\frac{\sum_{n=0}^\infty u^{-n-1}w^{-n}v^{-2n}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^n(a^i-1)^{-1}}{\sum_{n=0}^\infty(u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}-lu^{-n-1}w^{-n}v^{-2n-1}a^{-\frac{(n+1)(n+2)}{2}})\prod_{i=1}^n(a^i-1)^{-1}}. \tag{15}$$

If $l|(v+1)$ *and* $v>2l-1>3$, *then*

$$\Big[0;\overline{ua^k-1,1,\frac{v-2l+1}{l},1,l-1,wa^k-1,1,l-2,1,\frac{v-l+1}{l}}\Big]_{k=1}^\infty$$
$$=\frac{\sum_{n=0}^\infty u^{-n-1}w^{-n}v^{-2n}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^n(a^i-1)^{-1}}{\sum_{n=0}^\infty(u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}-lu^{-n-1}w^{-n}v^{-2n-1}a^{-\frac{(n+1)(n+2)}{2}})\prod_{i=1}^n(a^i-1)^{-1}}. \tag{16}$$

Theorem 6. *If* $l|(v-1)$ *and* $v>l+1>2$, *then*

$$\Big[0;\overline{\frac{v-1}{l},l,ua^k-1,1,l-1,\frac{v-l-1}{l},1,wa^k-1}\Big]_{k=1}^\infty$$
$$=\frac{\sum_{n=0}^\infty u^{-n-1}w^{-n}v^{-2n-2}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^n(a^i-1)^{-1}}{\sum_{n=0}^\infty u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}\prod_{i=1}^n(a^i-1)^{-1}}+\frac{l}{v}. \tag{17}$$

If $l|(v+1)$ *and* $v>2l-1>3$, *then*

$$\Big[0;\overline{\frac{v-l+1}{l},1,l-2,1,ua^k-1,l-1,1,\frac{v-2l+1}{l},1,wa^k-1}\Big]_{k=1}^\infty$$
$$=\frac{\sum_{n=0}^\infty u^{-n-1}w^{-n}v^{-2n-2}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^n(a^i-1)^{-1}}{\sum_{n=0}^\infty u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}\prod_{i=1}^n(a^i-1)^{-1}}+\frac{l}{v}. \tag{18}$$

Theorem 7. *If* $l|(v-1)$ *and* $v>l+1>2$*, then*

$$\Big[0;\overline{ua^k-1,1,\frac{v-l-1}{l},l,wa^k-1,1,l-1,\frac{v-1}{l}}\Big]_{k=1}^\infty$$
$$=\frac{\sum_{n=0}^\infty(-1)^n u^{-n-1}w^{-n}v^{-2n}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^n(a^i-1)^{-1}}{\sum_{n=0}^\infty(-1)^n(u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}-lu^{-n-1}w^{-n}v^{-2n-1}a^{-\frac{(n+1)(n+2)}{2}})\prod_{i=1}^n(a^i-1)^{-1}}. \tag{19}$$

If $l|(v+1)$ *and* $v>2l-1>3$*, then*

$$\Big[0;\overline{ua^k-1,1,\frac{v-2l+1}{l},1,l-2,1,wa^k-1,l-1,1,\frac{v-l+1}{l}}\Big]_{k=1}^\infty$$
$$=\frac{\sum_{n=0}^\infty(-1)^n u^{-n-1}w^{-n}v^{-2n}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^n(a^i-1)^{-1}}{\sum_{n=0}^\infty(-1)^n(u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}-lu^{-n-1}w^{-n}v^{-2n-1}a^{-\frac{(n+1)(n+2)}{2}})\prod_{i=1}^n(a^i-1)^{-1}}. \tag{20}$$

Theorem 8. *If* $l|(v-1)$ *and* $v>l+1>2$*, then*

$$\Big[0;\frac{v-1}{l},\overline{l,ua^k-1,1,l-1,\frac{v-1}{l},wa^k-1,1,\frac{v-l-1}{l}}\Big]_{k=1}^\infty$$
$$=\frac{\sum_{n=0}^\infty(-1)^n u^{-n-1}w^{-n}v^{-2n-2}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^n(a^i-1)^{-1}}{\sum_{n=0}^\infty(-1)^n u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}\prod_{i=1}^n(a^i-1)^{-1}}+\frac{l}{v}. \tag{21}$$

If $l|(v+1)$ *and* $v>2l-1>3$*, then*

$$\Big[0;\frac{v-l+1}{l},\overline{1,l-2,1,ua^k-1,l-1,1,\frac{v-l+1}{l},wa^k-1,1,\frac{v-2l+1}{l}}\Big]_{k=1}^\infty$$
$$=\frac{\sum_{n=0}^\infty(-1)^n u^{-n-1}w^{-n}v^{-2n-2}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^n(a^i-1)^{-1}}{\sum_{n=0}^\infty(-1)^n u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}\prod_{i=1}^n(a^i-1)^{-1}}+\frac{l}{v}. \tag{22}$$

6. Different types of Tasoev continued fractions

The different types of Tasoev continued fractions with period 3 were obtained in [7]:

$$[0;\overline{ua^{2k-1}-1,1,va^{2k}-1}]_{k=1}^\infty$$
$$=\frac{\sum_{n=0}^\infty u^{-n-1}v^{-n}a^{-(n+1)^2}\prod_{i=1}^n(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^\infty(-1)^n u^{-n}v^{-n}a^{-n^2}\prod_{i=1}^n(a^{2i}-(-1)^i)^{-1}},$$

$$[0; ua, \overline{va^{2k}-1, 1, ua^{2k+1}-1}]_{k=1}^{\infty}$$
$$= \frac{\sum_{n=0}^{\infty}(-1)^n u^{-n-1} v^{-n} a^{-(n+1)^2} \prod_{i=1}^n (a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty} u^{-n} v^{-n} a^{-n^2} \prod_{i=1}^n (a^{2i}-(-1)^i)^{-1}},$$

$$[0; \overline{ua^k-1, 1, va^k-1}]_{k=1}^{\infty}$$
$$= \frac{\sum_{n=0}^{\infty} u^{-n-1} v^{-n} a^{-(n+1)(n+2)/2} \prod_{i=1}^n (a^i-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n} v^{-n} a^{-n(n+1)/2} \prod_{i=1}^n (a^i-(-1)^i)^{-1}}$$

and

$$[0; ua, \overline{va^k-1, 1, ua^{k+1}-1}]_{k=1}^{\infty}$$
$$= \frac{\sum_{n=0}^{\infty}(-1)^n u^{-n-1} v^{-n} a^{-(n+1)(n+2)/2} \prod_{i=1}^n (a^i-(-1)^i)^{-1}}{\sum_{n=0}^{\infty} u^{-n} v^{-n} a^{-n(n+1)/2} \prod_{i=1}^n (a^i-(-1)^i)^{-1}}.$$

If we use the first relation as a basic continued fraction, then by methods similar to the ones used in the previous sections we get the following Theorem 9. If we use the second, third, and fourth relations, then we get Theorem 10, Theorem 11, and Theorem 12, respectively. The rational numbers u, w and a are chosen so that each partial quotient of the continued fraction is a positive integer.

Theorem 9. *If* $l|(v-1)$ *and* $v \ge 2l+1 \ge 5$*, then*

$$[\,\overline{ua^{4k-3}-1; 1, \frac{v-l-1}{l}, l, wa^{4k-2}-1, 1, l-1, \frac{v-l-1}{l}, 1,}$$
$$\overline{ua^{4k-1}-1, \frac{v-1}{l}, l-1, 1, wa^{4k}-1, l, \frac{v-1}{l}}\,]_{k=1}^{\infty}$$
$$= \frac{\sum_{n=0}^{\infty}(-1)^n u^{-n} w^{-n} v^{-2n} a^{-n^2} \prod_{i=1}^n (a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty} u^{-n-1} w^{-n} v^{-2n} a^{-(n+1)^2} \prod_{i=1}^n (a^{2i}-(-1)^i)^{-1}} - \frac{l}{v}. \tag{23}$$

If $l|(v+1)$ *and* $v \ge 3l-1 \ge 8$*, then*

$$[\,\overline{ua^{4k-3}-1; 1, \frac{v-2l+1}{l}, 1, l-2, 1, wa^{4k-2}-1, l-1, 1, \frac{v-2l+1}{l},}$$
$$\overline{1, ua^{4k-1}-1, \frac{v-l+1}{l}, 1, l-1, wa^{4k}-1, 1, l-2, 1, \frac{v-l+1}{l}}\,]_{k=1}^{\infty}$$
$$= \frac{\sum_{n=0}^{\infty}(-1)^n u^{-n} w^{-n} v^{-2n} a^{-n^2} \prod_{i=1}^n (a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty} u^{-n-1} w^{-n} v^{-2n} a^{-(n+1)^2} \prod_{i=1}^n (a^{2i}-(-1)^i)^{-1}} - \frac{l}{v}. \tag{24}$$

If $l|(v-1)$ *and* $v\ge 2l+1\ge 5$, *then*

$$[0;\overline{\frac{v-1}{l},l,ua^{4k-3}-1,1,l-1,\frac{v-1}{l},wa^{4k-2}-1,1,\frac{v-l-1}{l},}$$
$$\overline{l-1,1,ua^{4k-1}-1,l,\frac{v-l-1}{l},1,wa^{4k}-1}\,]_{k=1}^\infty$$
$$=\frac{\sum_{n=0}^\infty u^{-n-1}w^{-n}v^{-2n-2}a^{-(n+1)^2}\prod_{i=1}^n(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^\infty(-1)^n u^{-n}w^{-n}v^{-2n}a^{-n^2}\prod_{i=1}^n(a^{2i}-(-1)^i)^{-1}}+\frac{l}{v}\,. \tag{25}$$

If $l|(v+1)$ *and* $v\ge 3l-1\ge 8$, *then*

$$[0;\overline{\frac{v-l+1}{l},1,l-2,1,ua^{4k-3}-1,l-1,1,\frac{v-l+1}{l},wa^{4k-2}-1,1,}$$
$$\overline{\frac{v-2l+1}{l},1,l-1,ua^{4k-1}-1,1,l-2,1,\frac{v-2l+1}{l},1,wa^{4k}-1}\,]_{k=1}^\infty$$
$$=\frac{\sum_{n=0}^\infty u^{-n-1}w^{-n}v^{-2n-2}a^{-(n+1)^2}\prod_{i=1}^n(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^\infty(-1)^n u^{-n}w^{-n}v^{-2n}a^{-n^2}\prod_{i=1}^n(a^{2i}-(-1)^i)^{-1}}+\frac{l}{v}\,. \tag{26}$$

Theorem 10. *If* $l|(v-1)$ *and* $v\ge 2l+1\ge 5$, *then*

$$\Big[\overline{ua^{4k-3}-1;1,\frac{v-l-1}{l},l-1,1,wa^{4k-2}-1,l,\frac{v-l-1}{l},1,}$$
$$\overline{ua^{4k-1}-1,\frac{v-1}{l},l,wa^{4k}-1,1,l-1,\frac{v-1}{l}}\Big]_{k=1}^\infty$$
$$=\frac{\sum_{n=0}^\infty u^{-n}w^{-n}v^{-2n}a^{-n^2}\prod_{i=1}^n(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^\infty(-1)^n u^{-n-1}w^{-n}v^{-2n}a^{-(n+1)^2}\prod_{i=1}^n(a^{2i}-(-1)^i)^{-1}}-\frac{l}{v}\,. \tag{27}$$

If $l|(v+1)$ *and* $v\ge 3l-1\ge 8$, *then*

$$\Big[\overline{ua^{4k-3}-1;1,\frac{v-2l+1}{l},1,l-1,wa^{4k-2}-1,1,l-2,1,\frac{v-2l+1}{l},}$$
$$\overline{1,ua^{4k-1}-1,\frac{v-l+1}{l},1,l-2,1,wa^{4k}-1,l-1,1,\frac{v-l+1}{l}}\Big]_{k=1}^\infty$$
$$=\frac{\sum_{n=0}^\infty u^{-n}w^{-n}v^{-2n}a^{-n^2}\prod_{i=1}^n(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^\infty(-1)^n u^{-n-1}w^{-n}v^{-2n}a^{-(n+1)^2}\prod_{i=1}^n(a^{2i}-(-1)^i)^{-1}}-\frac{l}{v}\,. \tag{28}$$

If $l|(v-1)$ *and* $v\ge 2l+1\ge 5$*, then*

$$\Big[0;\frac{v-1}{l},\overline{l,ua^{4k-3}-1,1,l-1,\frac{v-l-1}{l},1,wa^{4k-2}-1,\frac{v-1}{l},}\\ \overline{l-1,1,ua^{4k-1}-1,l,\frac{v-1}{l},wa^{4k}-1,1,\frac{v-l-1}{l}}\Big]_{k=1}^{\infty}\\ =\frac{\sum_{n=0}^{\infty}(-1)^n u^{-n-1}w^{-n}v^{-2n-2}a^{-(n+1)^2}\prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}u^{-n}w^{-n}v^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}+\frac{l}{v}\,. \quad (29)$$

If $l|(v+1)$ *and* $v\ge 3l-1\ge 8$*, then*

$$\Big[0;\frac{v-l+1}{l},\overline{1,l-2,1,ua^{4k-3}-1,l-1,1,\frac{v-2l+1}{l},1,wa^{4k-2}-1,}\\ \overline{\frac{v-l+1}{l},1,l-1,ua^{4k-1}-1,1,l-2,1,\frac{v-l+1}{l},wa^{4k}-1,1,\frac{v-2l+1}{l}}\Big]_{k=1}^{\infty}\\ =\frac{\sum_{n=0}^{\infty}(-1)^n u^{-n-1}w^{-n}v^{-2n-2}a^{-(n+1)^2}\prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}u^{-n}w^{-n}v^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-(-1)^i)^{-1}}+\frac{l}{v}\,. \quad (30)$$

Theorem 11. *If* $l|(v-1)$ *and* $v\ge 2l+1\ge 5$*, then*

$$\Big[\overline{ua^{2k-1}-1;1,\frac{v-l-1}{l},l,wa^{2k-1}-1,1,l-1,\frac{v-l-1}{l},}\\ \overline{1,ua^{2k}-1,\frac{v-1}{l},l-1,1,wa^{2k}-1,l,\frac{v-1}{l}}\Big]_{k=1}^{\infty}\\ =\frac{\sum_{n=0}^{\infty}(-1)^n u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}\prod_{i=1}^{n}(a^{i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}u^{-n-1}w^{-n}v^{-2n}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^{n}(a^{i}-(-1)^i)^{-1}}-\frac{l}{v}\,. \quad (31)$$

If $l|(v+1)$ *and* $v\ge 3l-1\ge 8$*, then*

$$\Big[\overline{ua^{2k-1}-1;1,\frac{v-2l+1}{l},1,l-2,1,wa^{2k-1}-1,l-1,1,\frac{v-2l+1}{l},}\\ \overline{1,ua^{2k}-1,\frac{v-l+1}{l},1,l-1,wa^{2k}-1,1,l-2,1,\frac{v-l+1}{l}}\Big]_{k=1}^{\infty}\\ =\frac{\sum_{n=0}^{\infty}(-1)^n u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}\prod_{i=1}^{n}(a^{i}-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}u^{-n-1}w^{-n}v^{-2n}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^{n}(a^{i}-(-1)^i)^{-1}}-\frac{l}{v}\,. \quad (32)$$

If $l|(v-1)$ *and* $v\geq 2l+1\geq 5$*, then*

$$\begin{aligned}&\Big[0;\overline{\frac{v-1}{l},l,ua^{2k-1}-1,1,l-1,\frac{v-1}{l},wa^{2k-1}-1,1,\frac{v-l-1}{l},}\\&\qquad\overline{l-1,1,ua^{2k}-1,l,\frac{v-l-1}{l},1,wa^{2k}-1}\Big]_{k=1}^{\infty}\\&=\frac{\sum_{n=0}^{\infty}u^{-n-1}w^{-n}v^{-2n-2}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}+\frac{l}{v}\,.\end{aligned}\tag{33}$$

If $l|(v+1)$ *and* $v\geq 3l-1\geq 8$*, then*

$$\begin{aligned}&\Big[0;\overline{\frac{v-l+1}{l},1,l-2,1,ua^{2k-1}-1,l-1,1,\frac{v-l+1}{l},wa^{2k-1}-1,1,}\\&\qquad\overline{\frac{v-2l+1}{l},1,l-1,ua^{2k}-1,1,l-2,1,\frac{v-2l+1}{l},1,wa^{2k}-1}\Big]_{k=1}^{\infty}\\&=\frac{\sum_{n=0}^{\infty}u^{-n-1}w^{-n}v^{-2n-2}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}+\frac{l}{v}\,.\end{aligned}\tag{34}$$

Theorem 12. *If* $l|(v-1)$ *and* $v\geq 2l+1\geq 5$*, then*

$$\begin{aligned}&\Big[\overline{ua^{2k-1}-1;1,\frac{v-l-1}{l},l-1,1,wa^{2k-1}-1,l,\frac{v-l-1}{l},}\\&\qquad\overline{1,ua^{2k}-1,\frac{v-1}{l},l,wa^{2k}-1,1,l-1,\frac{v-1}{l}}\Big]_{k=1}^{\infty}\\&=\frac{\sum_{n=0}^{\infty}u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n-1}w^{-n}v^{-2n}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}-\frac{l}{v}\,.\end{aligned}\tag{35}$$

If $l|(v+1)$ *and* $v\geq 3l-1\geq 8$*, then*

$$\begin{aligned}&\Big[\overline{ua^{2k-1}-1;1,\frac{v-2l+1}{l},1,l-1,wa^{2k-1}-1,1,l-2,1,\frac{v-2l+1}{l},}\\&\qquad\overline{1,ua^{2k}-1,\frac{v-l+1}{l},1,l-2,1,wa^{2k}-1,l-1,1,\frac{v-l+1}{l}}\Big]_{k=1}^{\infty}\\&=\frac{\sum_{n=0}^{\infty}u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n-1}w^{-n}v^{-2n}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}-\frac{l}{v}\,.\end{aligned}\tag{36}$$

If $l|(v-1)$ *and* $v\geq 2l+1\geq 5$*, then*

$$\Big[0;\frac{v-1}{l},\overline{l,ua^{2k-1}-1,1,l-1,\frac{v-l-1}{l},1,wa^{2k-1}-1,\frac{v-1}{l},}$$
$$\overline{l-1,1,ua^{2k}-1,l,\frac{v-1}{l},wa^{2k}-1,1,\frac{v-l-1}{l}}\Big]_{k=1}^{\infty}$$
$$=\frac{\sum_{n=0}^{\infty}(-1)^n u^{-n-1}w^{-n}v^{-2n-2}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}+\frac{l}{v}\,. \tag{37}$$

If $l|(v+1)$ *and* $v\geq 3l-1\geq 8$*, then*

$$\Big[0;\frac{v-l+1}{l},\overline{1,l-2,1,ua^{2k-1}-1,l-1,1,\frac{v-2l+1}{l},1,wa^{2k-1}-1,}$$
$$\overline{\frac{v-l+1}{l},1,l-1,ua^{2k}-1,1,l-2,1,\frac{v-l+1}{l},wa^{2k}-1,1,\frac{v-2l+1}{l}}\Big]_{k=1}^{\infty}$$
$$=\frac{\sum_{n=0}^{\infty}(-1)^n u^{-n-1}w^{-n}v^{-2n-2}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}{\sum_{n=0}^{\infty}u^{-n}w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}\prod_{i=1}^{n}(a^i-(-1)^i)^{-1}}+\frac{l}{v}\,. \tag{38}$$

7. Ramanujan continued fractions

The regular continued fractions entailed from Ramanujan continued fractions were mentioned in [12]. These continued fractions can be seen as Tasoev type continued fractions. All the four formulas in [12, Theorem 4] are derived from our results as special cases. If we put $l=w=1$ in our Theorem 6 (17), then by the relation $[\dots,r,0,s,\dots]=[\dots,r+s,\dots]$ (see e.g. [14, p.223, (2)]) we have

$$[0,\overline{v-1,1,ua^k-1,v-1,1,a^k-1}\,]_{k=1}^{\infty}$$
$$=\frac{1}{v}+\frac{\sum_{n=0}^{\infty}u^{-n-1}v^{-2n-2}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^{n}(a^i-1)^{-1}}{\sum_{n=0}^{\infty}u^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}\prod_{i=1}^{n}(a^i-1)^{-1}}\,.$$

If uv^2 is replaced by u, this becomes the identity [12, (2.10)]. If we put $l=u=1$ in our Theorem 8 (21) and make some minor changes, then we have

$$[1,\overline{v,wa^k-1,1,v-2,1,a^k-1}\,]_{k=1}^{\infty}$$
$$=\frac{1}{v}+\frac{\sum_{n=0}^{\infty}(-1)^n u^{-n}a^{-\frac{n(n-1)}{2}}\prod_{i=1}^{n}(a^i-1)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n}a^{-\frac{n(n+1)}{2}}\prod_{i=1}^{n}(a^i-1)^{-1}}\,,$$

which is the identity [12, (2.11)]. If we put $l=u=1$ in our Theorem 7 (19), then we have

$$[0,\overline{a^k-1,1,v-2,1,wa^k-1,v}\,]_{k=1}^\infty$$
$$=\frac{\sum_{n=0}^\infty(-1)^n w^{-n}v^{-2n}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^n(a^i-1)^{-1}}{\sum_{n=0}^\infty(-1)^n(w^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}-w^{-n}v^{-2n-1}a^{-\frac{(n+1)(n+2)}{2}})\prod_{i=1}^n(a^i-1)^{-1}}\,.$$

After some minor changes, this becomes the identity [12, (2.12)]. If we put $l=w=1$ in our Theorem 12 (35), then we have

$$[\,\overline{ua^{2k-1}-1;1,v-1,a^{2k-1}-1,1,v-2,1,ua^{2k}-1,v-1,2,a^{2k}-1,v}\,]_{k=1}^\infty$$
$$=\frac{\sum_{n=0}^\infty u^{-n}v^{-2n}a^{-\frac{n(n+1)}{2}}\prod_{i=1}^n(a^i-(-1)^i)^{-1}}{\sum_{n=0}^\infty(-1)^n u^{-n-1}v^{-2n}a^{-\frac{(n+1)(n+2)}{2}}\prod_{i=1}^n(a^i-(-1)^i)^{-1}}-\frac{1}{v}\,.$$

After some minor changes, this is reduced to the identity [12, (2.13)].

Some more interesting formulas were treated in [12, Theorems 5,6,7,8]. Along the same line as previous sections, more extended results can be obtained from [12, Theorem 9]. Let

$$\phi(u,v,a)=\sum_{n=0}^\infty u^n a^{\frac{n(n+1)}{2}}\prod_{i=1}^n(1-a^i)^{-1}(1+va^i)^{-1}\,.$$

Theorem 13.

$$[1;\overline{uv^2a^{k-1},b^k}\,]_{k=1}^\infty=\frac{\phi(ab/uv^2,a/uv^2,1/ab)}{\phi(1/uv^2,a/uv^2,1/ab)}\,. \tag{39}$$

If $l|(v-1)$, *then*

$$\left[1;\overline{\frac{v-1}{l},l,ua^{k-1}-1,1,l-1,\frac{v-l-1}{l},1,b^k-1}\right]_{k=1}^\infty$$
$$=\frac{\phi(ab/uv^2,a/uv^2,1/ab)}{\phi(1/uv^2,a/uv^2,1/ab)}+\frac{l}{v} \tag{40}$$

and

$$\left[0;\overline{1,\frac{v-l-1}{l},l-1,1,ua^{k-1}-1,l,\frac{v-1}{l},b^k-1}\right]_{k=1}^\infty$$
$$=\frac{\phi(ab/uv^2,a/uv^2,1/ab)}{\phi(1/uv^2,a/uv^2,1/ab)}-\frac{l}{v}\,. \tag{41}$$

If $l|(v+1)$ *with* $l\ge 3$*, then*

$$\Big[1;\overline{\frac{v-l+1}{l},1,l-2,1,ua^{k-1}-1,l-1,1,\frac{v-2l+1}{l},1,b^k-1}\Big]_{k=1}^{\infty} = \frac{\phi(ab/uv^2,a/uv^2,1/ab)}{\phi(1/uv^2,a/uv^2,1/ab)}+\frac{l}{v} \quad (42)$$

and

$$\Big[0;\overline{1,\frac{v-2l+1}{l},1,l-1,ua^{k-1}-1,1,l-2,1,\frac{v-l+1}{l},b^k-1}\Big]_{k=1}^{\infty} = \frac{\phi(ab/uv^2,a/uv^2,1/ab)}{\phi(1/uv^2,a/uv^2,1/ab)}-\frac{l}{v}\,. \quad (43)$$

Remark. If $l=1$ in (40), this is reduced to the identity [12, Theorem 9 (2.44)]. If $l=2$ in (42), this is the same as (40). If $l=1$ in (41), this is reduced to the identity [12, Theorem 9 (2.46)]. If $l=2$ in (43), this is the same as (41).

Theorem 14.

$$[0;\overline{1,uv^2a^{k-1}-2,1,b^k-2}\,]_{k=1}^{\infty}=\frac{\phi(-ab/uv^2,-a/uv^2,1/ab)}{\phi(-1/uv^2,-a/uv^2,1/ab)}\,. \quad (44)$$

If $l|(v-1)$*, then*

$$\Big[1;\overline{\frac{v-1}{l},l-1,1,ua^{k-1}-1,l,\frac{v-l-1}{l},1,b^k-1}\Big]_{k=1}^{\infty} = \frac{\phi(-ab/uv^2,-a/uv^2,1/ab)}{\phi(-1/uv^2,-a/uv^2,1/ab)}+\frac{l}{v} \quad (45)$$

If $l|(v+1)$ *with* $l\ge 3$*, then*

$$\Big[1;\overline{\frac{v-l+1}{l},1,l-1,ua^{k-1}-1,1,l-2,1,\frac{v-2l+1}{l},1,b^k-1}\Big]_{k=1}^{\infty} = \frac{\phi(-ab/uv^2,-a/uv^2,1/ab)}{\phi(-1/uv^2,-a/uv^2,1/ab)}+\frac{l}{v} \quad (46)$$

Remark. If $l=1$ in (45), this is reduced to the identity [12, Theorem 9 (2.45)]. If $l=2$ in (46), this is the same as (45).

Theorem 15.

$$[1;\overline{uv^2a^{2k-2}-1,1,b^{2k-1}-2,1,uv^2a^{2k-1}-1,b^{2k}}\,]_{k=1}^{\infty} = \frac{\phi(-ab/uv^2,a/uv^2,-1/ab)}{\phi(1/uv^2,a/uv^2,-1/ab)}\,. \quad (47)$$

If $l|(v-1)$, *then*

$$\Big[1;\overline{\frac{v-1}{l},l,ua^{2k-2}-1,1,l-1,\frac{v-1}{l},b^{2k-1}-1,1,\frac{v-l-1}{l},l,}$$
$$\overline{ua^{2k}-1,1,l-1,\frac{v-l-1}{l},1,b^{2k}-1}\Big]_{k=1}^{\infty}$$
$$=\frac{\phi(-ab/uv^2,a/uv^2,-1/ab)}{\phi(1/uv^2,a/uv^2,-1/ab)}+\frac{l}{v} \quad (48)$$

and

$$\Big[0;\overline{1,\frac{v-l-1}{l},l-1,1,ua^{2k-2}-1,l,\frac{v-l-1}{l},1,b^{2k-1}-1,\frac{v-1}{l},}$$
$$\overline{l-1,1,ua^{2k-1}-1,l,\frac{v-1}{l},b^{2k}-1}\Big]_{k=1}^{\infty}$$
$$=\frac{\phi(-ab/uv^2,a/uv^2,-1/ab)}{\phi(1/uv^2,a/uv^2,-1/ab)}-\frac{l}{v}\,. \quad (49)$$

If $l|(v+1)$ *with* $l\ge 3$, *then*

$$\Big[1;\overline{\frac{v-l+1}{l},1,l-2,1,ua^{2k-2}-1,l-1,1,\frac{v-l+1}{l},b^{2k-1}-1,1,}$$
$$\overline{\frac{v-2l+1}{l},1,l-2,1,ua^{2k}-1,l-1,1,\frac{v-2l+1}{l},1,b^{2k}-1}\Big]_{k=1}^{\infty}$$
$$=\frac{\phi(-ab/uv^2,a/uv^2,-1/ab)}{\phi(1/uv^2,a/uv^2,-1/ab)}+\frac{l}{v} \quad (50)$$

and

$$\Big[0;\overline{1,\frac{v-2l+1}{l},1,l-1,ua^{2k-2}-1,1,l-2,1,\frac{v-2l+1}{l},1,b^{2k-1}-1,}$$
$$\overline{\frac{v-l+1}{l},1,l-1,ua^{2k-1}-1,1,l-2,1,\frac{v-l+1}{l},b^{2k}-1}\Big]_{k=1}^{\infty}$$
$$=\frac{\phi(-ab/uv^2,a/uv^2,-1/ab)}{\phi(1/uv^2,a/uv^2,-1/ab)}-\frac{l}{v}\,. \quad (51)$$

Remark. If $l=1$ in (48), this is reduced to the identity [12, Theorem 9 (2.47)]. If $l=2$ in (50), this is the same as (48). If $l=2$ in (51), this is the same as (49).

Theorem 16.

$$[0;\overline{1,uv^2a^{2k-2}-2,1,b^{2k-1}-1,uv^2a^{2k-1},b^{2k}-1}\,]_{k=1}^\infty = \frac{\phi(ab/uv^2,a/uv^2,-1/ab)}{\phi(-1/uv^2,a/uv^2,-1/ab)}\,. \tag{52}$$

If $l|(v-1)$*, then*

$$\Big[1;\overline{\frac{v-1}{l},l-1,1,ua^{2k-2}-1,l,\frac{v-l-1}{l},1,b^{2k-1}-1,}\;\overline{\frac{v-1}{l},l,ua^{2k-1}-1,1,l-1,\frac{v-l-1}{l},1,b^{2k}-1}\Big]_{k=1}^\infty = \frac{\phi(ab/uv^2,a/uv^2,-1/ab)}{\phi(-1/uv^2,a/uv^2,-1/ab)}+\frac{l}{v} \tag{53}$$

If $l|(v+1)$ *with* $l\ge 3$*, then*

$$\Big[1;\overline{\frac{v-l+1}{l},1,l-1,ua^{2k-2}-1,1,l-2,1,\frac{v-2l+1}{l},1,}\;\overline{b^{2k-1}-1,\frac{v-l+1}{l},1,l-2,1,ua^{2k-1}-1,l-1,1,\frac{v-2l+1}{l},1,b^{2k}-1}\Big]_{k=1}^\infty = \frac{\phi(ab/uv^2,a/uv^2,-1/ab)}{\phi(-1/uv^2,a/uv^2,-1/ab)}+\frac{l}{v} \tag{54}$$

Remark. If $l=1$ in (53), this is reduced to the identity [12, Theorem 9 (2.48)]. If $l=2$ in (54), this is the same as (53).

We shall prove only the first parts of Theorem 13 because the rest is similar. From [1, Entry 17 (p.45)] by using the formula [4, (2.3.23) (p.35)] or [6, Lemma 1] we get

$$\begin{aligned}&\frac{\phi(ab/uv^2,a/uv^2,1/ab)}{\phi(1/uv^2,a/uv^2,1/ab)}\\ &\quad=1+\frac{(uv^2)^{-1}}{1}\,{}_{+}\,\frac{(buv^2)^{-1}}{1}\,{}_{+}\,\frac{(abuv^2)^{-1}}{1}\,{}_{+}\,\frac{(ab^2uv^2)^{-1}}{1}\,{}_{+}\cdots\\ &\quad=[1;\overline{uv^2a^{k-1},b^k}\,]_{k=1}^\infty\,.\end{aligned}$$

When $\alpha=[1;\overline{uv^2a^{k-1},b^k}\,]_{k=1}^\infty$, apply Lemma 2 to the continued fraction of $\alpha+l/v$ with $v=lv'+1$. First,

$$\frac{v[1]+l}{v}=[1;v',l]$$

and

$$\begin{pmatrix} v & l\\ 0 & v\end{pmatrix}\begin{pmatrix} 1 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix} v+l & v'+1\\ v & v'\end{pmatrix}\begin{pmatrix} 1 & -vv'\\ 0 & v^2\end{pmatrix}.$$

Next, for $k = 1, 2, \dots$ we have

$$\frac{[uv^2a^{k-1}] - vv'}{v^2} = [ua^{k-1} - 1; 1, l-1, 1, v'-1, 1]$$

and

$$\begin{pmatrix} 1 & -vv' \\ 0 & v^2 \end{pmatrix} \begin{pmatrix} uv^2a^{k-1} & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} uva^{k-1} - v' & u(v-l)(a+(2k-1)b) - v' + 1 \\ v & v-l \end{pmatrix} \begin{pmatrix} v & l-v \\ 0 & v \end{pmatrix},$$

then

$$\frac{v[b^k] + l - v}{v} = [b^k - 1; v', l]$$

and

$$\begin{pmatrix} v & l-v \\ 0 & v \end{pmatrix} \begin{pmatrix} b^k & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} vb^k - v + l & v'b^k - v' + 1 \\ v & v' \end{pmatrix} \begin{pmatrix} 1 & -vv' \\ 0 & v^2 \end{pmatrix}.$$

Therefore, we get (40) as

$$\alpha + \frac{l}{v} = [1; \overline{\frac{v-1}{l}, l, ua^{k-1} - 1, 1, l-1, \frac{v-l-1}{l}, 1, b^k - 1}]_{k=1}^{\infty}.$$

The rest of the parts is similarly proved and omitted.

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