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SOME REMARKS ON PRIMAL SUBMODULES

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Abstract. In this paper, we study the primal submodules of a module over a commutative ring with non-zero identity. We generalize the primal decomposition of ideals (see [2]) to that of submodules. Let R be a commutative ring, M an R-module and N a submodule of M. We establish a decomposition of N as an intersection of primal submodules of M. We show that if R is a Prüfer domain of finite character, then N has a primal decomposition. Also we prove that the representation of submodules as reduced intersections of primal submodules is unique.

1. Introduction

In this paper, R will denote a commutative ring with nonzero identity and all modules are unitary. We wish to study properties of submodules of a module over a certain Prüfer domain, in particular, their decomposition into intersections of primal submodules. So far, the literature on this subject is sparse and mostly restricted to the question of when or which submodules admit decompositions as intersections of finitely many primary submodules. We know that every submodule of a Noetherian module can be expressed as a finite intersection of irreducible submodules. Furthermore, in a Noetherian module, every irreducible submodule is primary. Hence if N is a proper submodule of the Noetherian module M, then N has a decomposition as an intersection of a finite number of primary submodules. This happens rarely in non-noetherian modules, because in general, modules irreducible submodules fail to be primary. Therefore we look for another decomposition for submodules. We investigate decompositions of submodules of a module over a Prüfer domain into intersections of primal submodules. As we intend to restrict our considerations to finite intersections, we assume to start with that our domain are of finite character; i.e. every non-zero element is contained but in a finite number of maximal ideals.

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The concept of primal ideals in a commutative ring was introduced by L. Fuchs in [1]. Let R be a commutative ring and let I be an ideal of R. An element $a \in R$ is called prime to I if $ra \in I$ $(r \in R)$ implies $r \in I$. Denote by $S(I)$ the set of all elements of R that are not prime to I. A proper ideal I of R is said to be primal if $S(I)$ forms an ideal of R; this ideal is always a prime ideal, called the adjoint ideal P of I. In this case we also say that I is a P-primal ideal [1]. Fuchs also has given a theory of the representation of an ideal as an intersection of primal ideals. Moreover, the theory of primal decomposition of ideals is studied extensively in [2].

Let us introduce some definitions that we will use. An R-submodule N of M is said to be irreducible if N is not the intersection of two submodules of M that properly contain it. Let N be a submodule of the R-module M, $a \in R$ and $f_a : M/N \to M/N$ the canonical homomorphism produced by multiplication by a. We say that N is a primary submodule of M if N is proper and for every $a \in R$, f_a is either injective or nilpotent, so N is primary if and only if whenever $am \in N$, for some $a \in R$, $m \in M$, then either $m \in N$ or $a^n M \subseteq N$ for some positive integer n. We say that $r \in R$ is a zero-divisor for the R-module M if $rm = 0$ for some $0 \neq m \in M$, and otherwise that r is M-regular. The set of zero-divisors of M is written $Zdv_R(M)$. By an arithmetical ring is understood a commutative ring R with identity for which the ideals form a distributive lattice. Also, a Prüfer domain is an arithmetical integral domain.

We shortly summarize the content of the paper. In Section 2 we give some preliminary results about primal submodules. For example, we show that if R is a commutative ring, P a prime ideal of R and M an R-module, then a submodule N of M is P-primal if and only if $(N :_R M) \subseteq P$ and $Zdv_{R/(N:_R M)}(M/N) = P/(N :_R M)$ (Theorem 2.3). Also it is shown in Proposition 2.11 that over a valuation ring, every submodule of a module is primal. For every prime ideal P of R, let $S_P = R \backslash P$. Then, as a result, we will show that over an arithmetical ring R, the S_P-component N_{S_P} of N is primal for every maximal ideal P of R containing $(N :_R M)$ (see Theorem 2.13). In section 3 we give some results about the intersection of primal submodules. It is shown that over a Prüfer domain of finite character, every submodule has a decomposition as an intersection of a finite number of primal submodules (Theorem 3.2). In Theorem 3.6 it is shown that if $N = N_1 \cap N_2 \cap \cdots \cap N_k$ is a reduced representation of N by P_i-primal submodules N_i, then N is a primal submodule of M if and only if one P_j divides all the others, in which case P_j is the adjoint prime ideal of N. Also in Theorem 3.9 we characterize the maximal not-prime-to-N ideals via the adjoint prime ideals in a reduced representation of N by primal submodules.

In Theorem 3.11 we prove that the short primal reduced representation of a submodule of an R-module is unique.

2. Basic Results

Let R be a commutative ring, M an R-module and N a submodule of M. An element $a \in R$ is called prime to N if $am \in N$ $(m \in M)$ implies that $m \in N$. Denote by $S(N)$ the set of all elements of R that are not prime to N. A proper submodule N of M is said to be primal if $S(N)$ forms an ideal of R; this ideal is called the adjoint ideal P of N. In this case we also say that N is a P-primal ideal. Note that if N is a primal submodule of M, then the ideal $P = S(N)$ is a prime ideal, for if $ab \in P$ with $a \notin P$, there exists $m \in M \backslash N$ with $abm \in N$; so $bm \in N$ implies that $b \in P$.

Lemma 2.1. *Let P be a prime ideal of a commutative ring R, M an R-module and N a submodule of M. Then N is a P-primal submodule of M if and only if for every $r \in R\backslash P$, the canonical homomorphism $f_r : M/N \to M/N$ (sending $m + N$ to $rm + N$) is injective and, for every $r \in P$, f_r is not injective.*

Proof. First assume that N is P-primal. Let $r \in R\backslash P$ and let $f_r(m+N) = 0$ for some $m + N \in M/N$. Then, $rm \in N$ and r is prime to N implies that $m \in N$; hence f_r is injective. Now assume that $r \in P$. Then there exists $m \in M\backslash N$ such that $rm \in N$. Thus $m + N \neq 0$ with $f_r(m + N) = 0$ implies that f_r is not injective.

Conversely, if the desired conditions hold, it is easy to check that P is exactly the set of elements of R that are not prime to N. Thus N is a P-primal submodule of M. □

Lemma 2.2. *Let N be a submodule of an R-module M. Then:*

(1) *If N is a proper submodule of M, then $(N :_R M) \subseteq S(N)$.*
(2) *If N is a P-primal submodule of M, then $(N :_R M) \subseteq P$.*

Proof. (1) Let $r \in (N :_R M)$. Then N proper gives there exists $m \in M\backslash N$ such that $rm \in N$, so r is not prime to N; hence $r \in S(N)$.

(2) This follows from (1) because every primal submodule is proper. □

Theorem 2.3. *Let R be a commutative ring, P a prime ideal of R, and M an R-module. A submodule N of M is P-primal if and only if $(N :_R M) \subseteq P$ and $Zdv_{R/(N:_R M)}(M/N) = P/(N :_R M)$.*

Proof. First assume that N is a P-primal submodule of M. Then, by Lemma 2.2, $(N :_R M) \subseteq P$. Set $I = (N :_R M)$. Assume that $r + I$ is an element of $Zdv_{R/I}(M/N)$. Then $rm + N = 0$ for some non-zero $m + N \in M/N$. This implies that r is not prime to N, so $r \in P$; hence $r + I \in P/I$. Therefore

$Zdv_{R/I}(M/N) \subseteq P/I$. Now pick an element $a+I$ in P/I. As $a \in P$, it follows that $am \in N$ for some $m \in M \backslash N$. Thus $a+I \in Zdv_{R/I}(M/N)$, and hence $P/I \subseteq Zdv_{R/I}(M/N)$ as required.

Conversely, assume that $I \subseteq P$ and $Zdv_{R/I}(M/N) = P/I$. For every $a \in P$, we have $a+I \in Zdv_{R/I}(M/N)$. Thus there exists a nonzero element $m+N$ of M/N such that $(a+I)(m+N) = 0$. Thus $am \in N$ with $m \in M \backslash N$ shows that a is not prime to N. Now assume that $r \in R$ is not prime to N. Then, $rm \in N$ for some $m \in M \backslash N$. Therefore $(r+I)(m+N) = 0$ with $m + N \neq 0$. That is $r + I \in Zdv_{R/I}(M/N) = P/I$. Hence $r \in P$. We have already shown that P consists exactly of those elements of R that are not prime to N. Hence N is P-primal. □

Let M be an R-module and N a submodule of M. Set $I = (N :_R M)$ and let $U(N)$ be the set of those elements r of R such that $r+I$ is a regular element on the R/I-module M/N. It is easy to see that N is a P-primal submodule of M if and only if $P := R \backslash U(N)$ is an ideal of R. Also we have the following result:

Corollary 2.4. *Let N be a primal submodule of the R-module M. Then the adjoint ideal P of N is the unique ideal maximal with respect to the properties $(N :_R M) \subseteq P$ and $U(N) \cap P = \emptyset$.*

Let R be a commutative ring, S a multiplicatively closed subset of R and M an R-module. Consider the $S^{-1}R$-module $S^{-1}M$; the module of fractions of M with respect to S. A natural question is: "What is the relation between the primal submodules of M and primal submodules of $S^{-1}M$?" In the following theorems we answer this question.

Lemma 2.5. *Let S be a multiplicatively closed subset of a ring R, M an R-module and N a P-primal submodule of M with $P \cap S = \emptyset$. If $m/s \in S^{-1}N$, then $m \in N$.*

Proof. Suppose that $m/s \in S^{-1}N$ but $m \notin N$. There exists $m' \in N$ and $t \in S$ such that $m/s = m'/t$. So $utm = usm' \in N$ for some $u \in S$. It follows that ut is not prime to N; hence $ut \in P \cap S$ which is a contradiction. □

Let S be a multiplicatively closed subset of a ring R, M an R-module and N a submodule of M. We know that if M is finitely generated, then $S^{-1}(N :_R M) = (S^{-1}N :_{S^{-1}R} S^{-1}M)$ (See [4, Lemma 9.12]). Also we know that if M is not finitely generated, this equality is not necessarily true. But for primal submodules we have the following theorem:

Theorem 2.6. *Let S be a multiplicatively closed subset of R, M an R-module and N a P-primal submodule of M with $P \cap S = \emptyset$. Then*

$$S^{-1}(N :_R M) = (S^{-1}N :_{S^{-1}R} S^{-1}M).$$

Proof. Since the inclusion $S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$ is clear, we will prove the reverse inclusion. Assume that $r/s \in (S^{-1}N :_{S^{-1}R} S^{-1}M)$ and let $m \in M$. As $(rm)/s = (r/s).(m/1) \in S^{-1}N$ we get $rm \in N$ by Lemma 2.5. So $r \in (N :_R M)$ and hence $r/s \in S^{-1}(N :_R M)$, as required. □

Proposition 2.7. *Let S be a multiplicatively closed subset of R, M an R-module and N a P-primal submodule of M with $P \cap S = \emptyset$. Then $S^{-1}N$ is a $S^{-1}P$-primal submodule of $S^{-1}M$.*

Proof. Clearly $S^{-1}P$ is a prime ideal of $S^{-1}R$. It is enough to show that $S^{-1}P$ is exactly the set of elements of $S^{-1}R$ which are not prime to $S^{-1}N$. Let $r/s \in S^{-1}P$. Then r is not prime to N, so there exists $m \in M\backslash N$ with $rm \in N$. Since $P \cap S = \emptyset$, we get $sm \notin N$, hence $(sm)/1 \notin S^{-1}N$ by Lemma 2.5. As $(r/s)(sm/1) \in S^{-1}N$, we obtain r/s is not prime to $S^{-1}N$. Now assume that r/s is not prime to $S^{-1}N$. Then there exists $m/t \notin S^{-1}N$ with$(r/s)(m/t) \in S^{-1}N$, hence $rm \in N$ by Lemma 2.5. Since $m \notin N$, it follows that r is not prime to N. Thus $r \in P$ and hence $r/s \in S^{-1}P$ as required. □

Let R be a commutative ring, M an R-module and S a multiplicatively closed set in R. If K is a submodule of $S^{-1}M$, define $K \cap M = v^{-1}(K)$ where $v : M \to S^{-1}M$ is the natural homomorphism. Clearly, $K \cap M$ is a submodule of M.

Proposition 2.8. *Let S be a multiplicatively closed subset of a ring R, M an R-module and B a Q-primal submodule of the $S^{-1}R$-module $S^{-1}M$. Then the following hold:*

(1) *$B \cap M$ is a primal submodule of M with adjoint prime ideal $Q \cap R$.*
(2) *$S^{-1}(B \cap M) = B$.*

Proof. (1) Clearly, $Q \cap R$ is a prime ideal of R. It only remains to show that $Q \cap R$ is exactly the set of elements non-prime to $B \cap M$. If $a \notin Q \cap R$, then $a/1 \notin Q$, so $(B :_{S^{-1}M} a/1) = B$. It follows that $(B \cap M :_M a) = B \cap M$; hence such an a is prime to $B \cap M$. If $a \in Q \cap R$, then $a/1 \in P$, so there exists $m/s \in S^{-1}M$ such that $(am)/s \in B$, but $m/s \notin B$. So $am \in B \cap M$ implies that a is not prime to $B \cap M$.

(2) Clearly $B \subseteq S^{-1}(B \cap M)$. For the reverse inclusion, assume that $m/s \in S^{-1}(B \cap M)$. Then, by Lemma 2.5, $m \in B \cap M$; hence $m/1 \in B$. Now $(s/1).(m/s) = m/1 \in B$ and $s/1 \notin Q$ implies that $m/s \in B$, as needed. □

Proposition 2.9. *Let S be a multiplicatively closed subset of a ring R, M an R-module and N a P-primal submodule of M. Then the following hold:*

(1) *If* $P \cap S = \emptyset$, *then* $N = (S^{-1}N) \cap M$.
(2) *If* $P \cap S \neq \emptyset$, *then* $N \subsetneqq (S^{-1}N) \cap M$.

Proof. (1) Since $N \subseteq (S^{-1}N) \cap M$ is trivial, we will prove the reverse inclusion. Pick $m \in (S^{-1}N) \cap M$. As $m/1 \in S^{-1}N$, there exist $n \in N$ and $s \in S$ such that $m/1 = n/s$, so $tsm = tn \in N$ for some $t \in S$; hence $m \in N$ since $ts \notin P$ and N is P-primal.

(2) Since $P \cap S \neq \emptyset$, there is an element $t \in P \cap S$, so t is not prime to N; hence $(N :_M t) \supsetneqq N$. Suppose that $y \in (N :_M t) - N$. Then we have $y/1 = (ty)/t \in S^{-1}N$. Thus $y \in (S^{-1}N) \cap M$, as required. □

Theorem 2.10. *Let P be a prime ideal of a ring R, S a multiplicatively closed subset of R with $P \cap S = \emptyset$, and let M be an R-module. Then there exists a one-to-one correspondence between the P-primal submodules of M and the $S^{-1}P$-primal submodules of of $S^{-1}M$.*

Proof. This follows from Propositions 2.7, 2.8 and 2.9. □

It is proved in [3] that the ring R is arithmetical if and only if R_P is a valuation ring for any maximal ideal P of R. By using this fact we have the following results.

Proposition 2.11. *Let R be a valuation ring, and let M be an R-module. Then every proper submodule of M is primal.*

Proof. Let N be a proper submodule of M. Assume that $a, b \in R$ are not prime to N. We can assume that $b = ra$ for some $r \in R$. There exists $m \in M \backslash N$ such that $am \in N$. Then $(a - b)m \in N$ gives $a - b$ is not prime to N; hence N is a primal submodule of M. □

Corollary 2.12. *Let R be an arithmetical ring, P a maximal ideal of R and M an R-module. Then every proper submodule of M_P is primal.*

Let R be a commutative ring, M an R-module and S a multiplicatively closed subset of R. For every submodule N of M, let

$$N_S = \{m \in M : sm \in N \text{ for some } s \in S\}.$$

It is clear that N_S is a submodule of M containing N. Also if $(N :_R M) \cap S \neq \emptyset$, then $N_S = M$. N_S is called the S-component of N. Let P be a prime ideal of a commutative ring R and set $S_P = R \backslash P$. Then $m \in N_{S_P}$ if and only if $(N :_R m) \nsubseteq P$. Furthermore $N_{S_P} = N_P \cap M$ where N_P is the localization of N at P.

Theorem 2.13. *Let R be an arithmetical ring, and let M be an R-module. Then, for every non-zero submodule N of M and every maximal ideal P containing $(N :_R M)$, N_{S_P} is a primal submodule of M.*

Proof. Clearly, N_P is a proper submodule of M_P. As R_P is a valuation ring, N_P is a primal submodule of M_P by Proposition 2.11. Then Proposition 2.8 gives $N_{S_P} = N_P \cap M$ is a primal submodule of M. □

3. Primal decomposition of submodules of a module

In this section we investigate the primal decomposition of submodules of a module. Although Lemma 3.1 is known, we do not know an appropriate reference, and so we include a proof.

Lemma 3.1. *Let R be a commutative ring, M an R-module and N an R-submodule of M. Then $N = \bigcap_{P \in Max(R)} N_{S_P}$.*

Proof. Clearly $N \subseteq \bigcap_{P \in Max(R)} N_{S_P}$. For the other containment, assume that $m \in \bigcap_{P \in Max(R)} N_{S_P}$. Set $I = (N :_R m)$. Then I is an ideal of R. For every $P \in Max(R)$, as $m \in N_{S_P}$, there exists an element $s \in S \backslash P$ such that $sm \in N$. This implies that $I \nsubseteq P$ for every $P \in Max(R)$, so $I = R$; hence $m \in N$, and so we have the equality. □

Theorem 3.2. *Let R be a Prüfer domain of finite character, M an R-module and N a proper submodule of M. Then N is the intersection of a finite number of primal submodules.*

Proof. We know that $N = \bigcap_{P \in Max(R)} N_{S_P}$. As R is a domain of finite character, there are only a finite number of maximal ideals, say $P_1, P_2, \ldots, P_k$, containing $(N :_R M)$. Moreover if P is a maximal ideal of R not containing $(N :_R M)$, $N_{S_P} = M$, and if P contains $(N :_R M)$, then N_{S_P} is a primal submodule of M by Theorem 2.13. Therefore $N = N_{S_{P_1}} \cap N_{S_{P_2}} \cap \cdots \cap N_{S_{P_k}}$, as required. □

Definition 3.3. *Let M be an R-module and N a submodule of M. An ideal I of R is said to be prime to N if $(N :_M I) = N$. Otherwise I is not prime to N. In the other word I is not prime to N if every element of I is not prime to N.*

Definition 3.4. *Let M be an R-module and N a submodule of M. A representation*

$$N = N_1 \cap N_2 \cap \cdots \cap N_k \tag{1}$$

will be called irredundant if no N_i contains the intersection of the remaining ones, and it is called reduced if no component may be replaced by one of its proper divisors.

Lemma 3.5. *Assume that* (1) *is a reduced representation of N by P_i-primal submodules N_i. Then an ideal I of R is not prime to N if and only if $I \subseteq P_j$ for some $1 \leq j \leq k$.*

Proof. First assume that I is not prime to N. For every $a \in I$, there exists $m \in M\backslash N$ with $am \in N \subseteq N_i$ for every $1 \leq i \leq k$. As $m \notin N$, there exists $1 \leq j \leq k$ with $m \in M\backslash N_j$. Therefore a is not prime to N_j and $a \in P_j$ since N_j is P_j-primal. Therefore $I \subseteq \cup_{i=1}^{k} P_i$ and hence $I \subseteq P_i$ for some $1 \leq i \leq k$ by the Prime Avoidance Theorem.

Conversely, assume that $I \subseteq P_i$ for some i. For every $a \in I \subseteq P_i$, there exists $m \in M\backslash N_i$ with $am \in N_i$. Since $N_i \subsetneqq N_i + Rm$ and (1) is reduced, there exists $m' \in N_1 \cap N_2 \cap \cdots N_{i-1} \cap (N_i + Rm) \cap N_{i+1} \cap \cdots \cap N_k$ with $m \notin N$. So $m' = n_i + rm$ for some $n_i \in N_i$ and $r \in R$. In this case $am' = an_i + arm \in N_i$ and so $am' \in N_1 \cap N_2 \cap \cdots \cap N_k = N$ with $m' \in M\backslash N$. This implies that a is not prime to N; hence I is not prime to N. □

The intersection of primal submodules need not be necessarily primal; however, we have the following result:

Theorem 3.6. *Assume that* (1) *is a reduced representation of N by P_i-primal submodules N_i. Then N is a primal submodule of M if and only if one P_j divides all the others, in which case P_j is the adjoint prime ideal of N.*

Proof. First assume that there is $1 \leq j \leq k$ such that $P_i \subseteq P_j$ for all $1 \leq i \leq k$. Then $P_j = \cup_{i=1}^{k} P_i$. If $a \in R$ is not prime to N, then $a \in P_j$ by lemma 3.5. On the other hand, P_j is not prime to N by lemma 3.5. So that $S(N) = P_j$, that is N is P_j-primal.

Conversely, assume that N is a primal submodule of M with adjoint prime ideal P. Since P is not prime to N, $P \subseteq P_j$ for some $1 \leq j \leq k$ by Lemma 3.5. On the other hand every P_i is not prime to N by Lemma 3.5. As N is P-primal, $P_i \subseteq P$ for every $1 \leq i \leq k$. Thus $P_j = P \supseteq P_i$; hence P_j divides the other $P_i's$. □

Definition 3.7. *Let M be an R-module and N a submodule of M. The maximal not-prime-to-N ideal is an ideal which is maximal in the - "inclusion ordered" set of prime ideal divisors of $(N :_R M)$ which are not prime to N.*

Remark 3.8. Let N be a submodule of an R-module M. Then, in general, there may be no the maximal not-prime-to-N ideal, since the union of an ascending chain of prime ideals need not be again prime.

Theorem 3.9. *Assume that* (1) *is a reduced representation of N as an intersection of P_i-primal submodules N_i of M. Then the maximal not prime to N ideals are the maximal not-prime-to-N ideals and are in fact the maximal elements of the "inclusion ordered" set $\{P_1, P_2, \cdots, P_k\}$.*

Proof. Let P be a maximal not prime to N ideal. There exists $1 \leq i \leq k$ such that $P \subseteq P_i$ by Lemma 3.5. Furthermore, by Lemma 3.5, P_i is not prime to N, so $P = P_i$ by the maximality of P; hence P is a prime ideal, as needed.

Conversely, assume that P_j is a maximal member of the set $\{P_1, P_2, \dots, P_k\}$ with respect to inclusion. In this case P_j is a maximal prime of N, otherwise, there exists an ideal Q of R that is not prime to N and $P_j \subset Q$. As Q is not prime to N, by Lemma 3.5, we have $Q \subset P_i$ for some $1 \leq i \leq k$. Hence $P_j \subset P_i$ which contradicts the maximality of P_j. □

Let N be a submodule of an R-module M. By a short primal representation $N = N_1 \cap \cdots \cap N_t$ of N we shall mean one where

(1) No component can be omitted, and
(2) Where the adjoint prime ideals $P_1, \dots, P_n$ of the primal components N_i are pairwise incomparable.

Theorem 3.10. *Let* (1) *be a reduced representation of N as an intersection of P_i- primal submodules N_i of M. Then N has a short primal representation whose adjoint ideals are the maximal not-prime-to-N ideals.*

Proof. We can assume that the representation (1) is irredundant since otherwise we can eliminate some N_i's and the remaining intersection is again reduced. Without loss of generality we may assume that $P_1, P_2, \dots, P_r (r \leq k)$ are the maximal elements of the set $\{P_1, P_2, \dots, P_k\}$. Let

$$N'_1 = \cap\{N_i : P_i \subseteq P_1\}$$

and $N'_j = \cap\{N_i : P_i \subseteq P_j \text{ and } P_i \nsubseteq P_t \text{ if } t < j\}$. In this case $N = N'_1 \cap N'_2 \cap \cdots \cap N'_r$. Also, for every $1 \leq j \leq r$, N'_j is P_j-primal by Theorem 3.6. For every $j \neq k$, $N'_j \cap N'_k$ is a reduced intersection of primal submodules of M whose adjoint ideals are incomparable. Therefore $N'_j \cap N'_k$ is not primal by Theorem 3.6. Thus the representation $N = N'_1 \cap N'_2 \cap \cdots \cap N'_r$ is short. Moreover, since for every $1 \leq i \leq r$, P_i is a maximal member of the set $\{P_1, P_2, \dots, P_k\}$, we must have $P_1, \dots, P_r$ are the maximal not-prime-to-N ideals. □

Theorem 3.11. *Let N be a submodule of an R-module M. Then, for any short primal reduced representation of N, the adjoint ideals and the number of primal components are uniquely determined.*

Proof. Let $N = N_1 \cap N_2 \cap \cdots \cap N_k$ with adjoint prime ideals $P_1, P_2, \dots, P_k$ and $N = N'_1 \cap N'_2 \cap \cdots \cap N'_t$ with adjoint prime ideals $P'_1, P'_2, \dots, P'_t$ be two short primal reduced representation of N. Since both representations are short, neither P_i properly contains the another P_j and nor P'_i properly contains another P'_j. Thus by Theorem 3.9, both sets $\{P_1, P_2, \dots, P_k\}$ and $\{P'_1, P'_2, \dots, P'_k\}$ are the set of maximal not-prime-to-N ideals. Hence $k = t$ and $\{P_1, P_2, \dots, P_k\} = \{P'_1, P'_2, \dots, P'_k\}$ □

Proposition 3.12. *Let N be a submodule of an R-module M which has a primal decomposition $N = N_1 \cap N_2 \cap \cdots \cap N_k$ where N_i is P_i-primal for every $1 \leq i \leq k$. Let S be a multiplicatively closed subset of R. Assume that $P_i \cap S = \emptyset$ for all $i = 1, \ldots, h$, and that $(N_i :_R M) \cap S \neq \emptyset$ for the remaining i. Then $N_S = N_1 \cap N_2 \cap \cdots \cap N_h$.*

Proof. Let $x \in N_1 \cap N_2 \cap \cdots \cap N_h$. Then, for every $h + 1 \leq i \leq k$, there exists $s_i \in (N_i :_R M) \cap S$. As $((s_{h+1} \ldots s_k)M \subseteq N_{h+1} \cap \cdots \cap N_k$, we have $((s_{h+1} \ldots s_k)x \in N_1 \cap \cdots \cap N_k = N \subseteq N_S$. It follows that $N_1 \cap N_2 \cap \cdots \cap N_h \subseteq N_S$. Now let $x \in N_S$. There exists $s \in S$ with $sx \in N = N_1 \cap N_2 \cap \cdots \cap N_k$. For every $1 \leq j \leq h$, $s \notin P_j$, so s is prime to N_j. Therefore from $sx \in N_j$ we have $x \in N_j$. Consequently, $N_S \subseteq N_1 \cap N_2 \cap \cdots \cap N_h$. Thus $N_S = N_1 \cap N_2 \cap \cdots \cap N_h$. □

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