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AN UPPER BOUND ESTIMATE FOR H. ALZER'S INTEGRAL INEQUALITY

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ABSTRACT. We get an upper bound estimate for H. Alzer's integral inequality. As applications, we obtain some inequalities for the logarithmic mean.

1. INTRODUCTION

For $b > a > 0$, the logarithmic mean $L(a, b)$ of a and b is defined as

$$L(a,b) = \frac{b-a}{\log b - \log a}. \tag{1.1}$$

The logarithmic mean has numerous applications in physics. Many properties and inequalities are obtained by many mathematicians (see [1-8] and the references therein).

In 1989, H. Alzer [9] proved the following result.

Theorem A. *Suppose $b > a > 0$, and $f \in C[a, b]$ is a strictly increasing function. If $\frac{1}{f^{-1}}$ is strictly convex, then*

$$\int_a^b f(x)\,dx > (b-a)f(L(a,b)). \tag{1.2}$$

The main purpose of this paper is to get the upper bound estimate for $\int_a^b f(x)\,dx$. Our main result is the following Theorem 1.

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Theorem 1. *Suppose $b > a > 0$, and $f \in C[a,b]$ is a strictly increasing function. If $\frac{1}{f^{-1}}$ is strictly convex, then*

$$\int_a^b f(x)\,dx < \frac{b(L(a,b)-a)f(b) + a(b-L(a,b))f(a)}{L(a,b)}. \tag{1.3}$$

As applications for Theorem A and Theorem 1, we shall give some inequalities for the logarithmic mean in Section 3.

2. Proof of Theorem 1

Proof of Theorem 1. Put $c = f(a)$ and $d = f(b)$. Then $f^{-1} : [c,d] \to [a,b]$ is strictly increasing. For any $x \in [0,1]$, the strict convexity of $\frac{1}{f^{-1}}$ implies

$$\begin{aligned}\frac{1}{f^{-1}[xf(a)+(1-x)f(b)]} &= \frac{1}{f^{-1}[xc+(1-x)d]}\\ &< \frac{x}{f^{-1}(c)} + \frac{1-x}{f^{-1}(d)} = \frac{x}{a} + \frac{1-x}{b}.\end{aligned} \tag{2.1}$$

Since f is strictly increasing, (2.1) leads to

$$xf(a) + (1-x)f(b) > f\left(\frac{ab}{xb+(1-x)a}\right). \tag{2.2}$$

Next, for any $t \in [a,b]$, taking $x = \frac{a(b-t)}{t(b-a)}$, then $0 \le x \le 1$ and $t = \frac{ab}{xb+(1-x)a}$. The inequality (2.2) and the transformation of the variable of integration yield

$$\begin{aligned}\int_a^b f(t)\,dt &= ab(b-a)\int_0^1 \frac{f\left(\frac{ab}{xb+(1-x)a}\right)}{[xb+(1-x)a]^2}\,dx\\ &< ab(b-a)\int_0^1 \frac{f(a)x+f(b)(1-x)}{[(b-a)x+a]^2}\,dx\\ &= \frac{b(L(a,b)-a)f(b)+a(b-L(a,b))f(a)}{L(a,b)}.\end{aligned}$$

□

3. Applications

In this section, we shall prove a number of inequalities for logarithmic mean underlying H. Alzer's inequality.

Theorem 2. *If $b > a > 0$ and $\alpha > 0$, then*

$$L(a,b) > \frac{\alpha+1}{\alpha}\frac{ab(b^\alpha - a^\alpha)}{b^{\alpha+1}-a^{\alpha+1}} \tag{3.1}$$

and

$$L(a,b) < \left[\frac{b^{\alpha+1}-a^{\alpha+1}}{(\alpha+1)(b-a)}\right]^{\frac{1}{\alpha}}. \tag{3.2}$$

Proof. Taking $f(x) = x^\alpha$, then $f : [a,b] \to [a^\alpha, b^\alpha]$ is strictly increasing, and $g = \frac{1}{f^{-1}} : [a^\alpha, b^\alpha] \to [\frac{1}{b}, \frac{1}{a}]$ satisfies

$$g''(x) = \frac{1}{\alpha}(\frac{1}{\alpha}+1)x^{-\frac{1}{\alpha}-2} > 0 \tag{3.3}$$

for $x \in [a^\alpha, b^\alpha]$.

Equation (3.3) implies that $\frac{1}{f^{-1}}$ is strictly convex on $[a^\alpha, b^\alpha]$. Thus Theorem A and Theorem 1 imply

$$(b-a)L^\alpha(a,b) < \int_a^b x^\alpha\,dx < \frac{b^{\alpha+1}(L(a,b)-a)+a^{\alpha+1}(b-L(a,b))}{L(a,b)}. \tag{3.4}$$

Equations (3.1) and (3.2) follow from equation (3.4). □

The following result is well-known.

Theorem 3. *If* $b > a > 0$, *then*

$$\sqrt{ab} < L(a,b) < \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}. \tag{3.5}$$

Proof. Taking $f(x) = \log x$, then $f : [a,b] \to [\log a, \log b]$ is strictly increasing, and $g = \frac{1}{f^{-1}} : [\log a, \log b] \to [\frac{1}{b}, \frac{1}{a}]$ satisfies

$$g''(x) = e^{-x} > 0 \tag{3.6}$$

for $x \in [\log a, \log b]$.

Equation (3.6), Theorem A and Theorem 1 yield

$$(b-a)\log L(a,b) < \int_a^b \log x\,dx < \frac{b(L(a,b)-a)\log b + a(b-L(a,b))\log a}{L(a,b)}. \tag{3.7}$$

Equation (3.5) follows from equation (3.7). □

Theorem 4. *If* $b > a > 0$, *then*

$$\log\frac{e^b-e^a}{b-a} < L(a,b) < \frac{ab(e^b-e^a)}{(b-1)e^b-(a-1)e^a}. \tag{3.8}$$

Proof. Taking $f(x) = e^x$, then $f : [a,b] \to [e^a, e^b]$ is strictly increasing, and $g = \frac{1}{f^{-1}} : [e^a, e^b] \to [\frac{1}{b}, \frac{1}{a}]$ satisfies

$$g''(x) = \frac{\log x + 2}{x^2(\log x)^3} > 0 \tag{3.9}$$

for $x \in [e^a, e^b]$.

Equation (3.9), Theorem A and Theorem 1 yield

$$(b-a)e^{L(a,b)} < \int_a^b e^x\,dx < \frac{b(L(a,b)-a)e^b + a(b-L(a,b))e^a}{L(a,b)}. \tag{3.10}$$

Equation (3.8) follows from equation (3.10). □

Theorem 5. *If $\frac{\pi}{2} > b > a > 0$, then*

$$\tan L(a,b) < \frac{\log\cos a - \log\cos b}{b-a} \tag{3.11}$$

and

$$L(a,b) > \frac{ab(\tan b - \tan a)}{(b\tan b + \log\cos b) - (a\tan a + \log\cos a)}. \tag{3.12}$$

Proof. Taking $f(x) = \tan x$, then $f : [a,b] \to [\tan a, \tan b] \subset (0,+\infty)$ is strictly increasing, and $g = \frac{1}{f^{-1}} : [\tan a, \tan b] \to [\frac{1}{b}, \frac{1}{a}]$ satisfies

$$g''(x) = \frac{2(1 + x\arctan x)}{(1+x^2)^2(\arctan x)^3} > 0 \tag{3.13}$$

for $x \in [\tan a, \tan b]$.

Equation (3.13), Theorem A and Theorem 1 yield

$$(b-a)\tan L(a,b) < \int_a^b \tan x\,dx < \frac{b(L(a,b)-a)\tan b + a(b-L(a,b))\tan a}{L(a,b)}. \tag{3.14}$$

Equations (3.11) and (3.12) follow from equation (3.14). □

Theorem 6. *If $b > a > 0$, then*

$$\arctan L(a,b) < \frac{b\arctan b + \frac{1}{2}\log(1+a^2) - a\arctan a - \frac{1}{2}\log(1+b^2)}{b-a} \tag{3.15}$$

and

$$L(a,b) > \frac{2ab(\arctan b - \arctan a)}{\log(1+b^2) - \log(1+a^2)}. \tag{3.16}$$

Proof. Taking $f(x) = \arctan x$, then $f : [a,b] \to [\arctan a, \arctan b]$ is strictly increasing, and $g = \frac{1}{f^{-1}} : [\arctan a, \arctan b] \to [\frac{1}{b}, \frac{1}{a}]$ satisfies

$$g''(x) = 2\csc x \cot x > 0 \tag{3.17}$$

for $x \in [\arctan a, \arctan b]$.

Equation (3.17), Theorem A and Theorem 1 yield

$$(b-a)\arctan L(a,b) < \int_a^b \arctan x\,dx < \frac{b(L(a,b)-a)\arctan b + a(b-L(a,b))\arctan a}{L(a,b)}. \quad (3.18)$$

Equations (3.15) and (3.16) follow from equations (3.17) and (3.18). □

Theorem 7. *If* $\frac{\pi}{3} \geq b > a > 0$, *then*

$$\sin L(a,b) < \frac{\cos a - \cos b}{b-a} \quad (3.19)$$

and

$$L(a,b) > \frac{ab(\sin b - \sin a)}{(b\sin b + \cos b) - (a\sin a + \cos a)}. \quad (3.20)$$

Proof. Taking $f(x) = \sin x$, then $f : [a,b] \subset (0, \frac{\pi}{3}] \to [\sin a, \sin b] \subset (0, \frac{\sqrt{3}}{2}]$ is strictly increasing, and $g = \frac{1}{f^{-1}} : [\sin a, \sin b] \subset (0, \frac{\sqrt{3}}{2}] \to [\frac{1}{b}, \frac{1}{a}]$ satisfies

$$g''(x) = \frac{2\sqrt{1-x^2} - x\arcsin x}{(1-x^2)^{\frac{3}{2}}(\arcsin x)^3} > 0 \quad (3.21)$$

for $x \in [\sin a, \sin b] \subset (0, \frac{\sqrt{3}}{2}]$.

Equation (3.21), Theorem A and Theorem 1 yield

$$(b-a)\sin L(a,b) < \int_a^b \sin x\,dx < \frac{b(L(a,b)-a)\sin b + a(b-L(a,b))\sin a}{L(a,b)}. \quad (3.22)$$

Equations (3.19) and (3.20) follow from equation (3.22). □

Theorem 8. *If* $1 \geq b > a > 0$, *then*

$$\arcsin L(a,b) < \frac{b\arcsin b + \sqrt{1-b^2} - a\arcsin a - \sqrt{1-a^2}}{b-a} \quad (3.23)$$

and

$$L(a,b) > \frac{ab(\arcsin b - \arcsin a)}{\sqrt{1-a^2} - \sqrt{1-b^2}}. \quad (3.24)$$

Proof. Taking $f(x) = \arcsin x$, then $f : [a,b] \subset (0,1] \to [\arcsin a, \arcsin b] \subset (0, \frac{\pi}{2}]$ is strictly increasing, and $g = \frac{1}{f^{-1}} : [\arcsin a, \arcsin b] \subset (0, \frac{\pi}{2}] \to [\frac{1}{b}, \frac{1}{a}]$ satisfies

$$g''(x) = \csc x(1 + 2\cot^2 x) > 0 \quad (3.25)$$

for $x \in [\arcsin a, \arcsin b]$.

Equation (3.25), Theorem A and Theorem 1 yield

$$(b-a)\arcsin L(a,b) < \int_a^b \arcsin x\,dx < \frac{b(L(a,b)-a)\arcsin b + a(b-L(a,b))\arcsin a}{L(a,b)}. \quad (3.26)$$

Equations (3.23) and (3.24) follow from equation (3.26). □

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