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A SPECIAL CLASS OF HARMONIC UNIVALENT FUNCTIONS

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Abstract. We define and investigate a special class of Salagean-type harmonic univalent functions in the open unit disk. We obtain coefficient conditions, extreme points, distortion bounds, convex combinations for the above class of harmonic univalent functions.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$.

Denote by S_H the class of functions $f = h+\bar{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

In 1984 Clunie and Sheil-Small [1] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses. See [2,6,7]. Jahangiri et al. [4] make use of the Alexander integral transforms of certain analytic functions (which are starlike or convex of positive order) with a view to investigating the construction of sense-preserving, univalent, and close to convex harmonic functions.

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The differential operator D^n was introduced by Salagean [5]. For $f = h + \bar{g}$ given by (1.1), Jahangiri and et al. [3] defined the modified Salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)} \tag{1.2}$$

where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad \text{and} \quad D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k.$$

In [8,9] one finds generalization of [3].

For $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $n \in N_0$ and $z \in U$, we let $S_H^n(\alpha, \beta)$ denote the family of harmonic functions f of the form (1.1) such that

$$\operatorname{Re}\left\{ \frac{D^{n+1} f(z) - D^n f(z)}{D^{n+1} f(z) + (1 - 2\alpha) D^n f(z)} \right\} < \beta \tag{1.3}$$

where $D^n f$ is defined by (1.2).

We let the subclass $\bar{S}_H^n(\alpha, \beta)$consist of harmonic functions $f_n = h + \bar{g}_n$ in $\bar{S}_H^n(\alpha, \beta)$ so that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k; \quad a_k, b_k \geq 0. \tag{1.4}$$

In this note, we extend the above result to the class $S_H^n(\alpha, \beta)$ and $\bar{S}_H^n(\alpha, \beta)$. We also obtain extreme points, distortion bounds, convolution conditions, and convex combinations for $\bar{S}_H^n(\alpha, \beta)$.

2. Main Results

We begin with a sufficient coefficient condition for functions in $\bar{S}_H^n(\alpha, \beta)$.

Theorem 1. *Let $f = h + \bar{g}$ be so that h and g are given by* (1.1)*. Furthermore, let*

$$\sum_{k=1}^{\infty} k^n \left\{ \left[k - 1 + \beta(k + 1 - 2\alpha) \right] |a_k| + \left[k + 1 + \beta(k - 1 + 2\alpha) \right] |b_k| \right\} \leq 4\beta\,(1 - \alpha) \tag{2.1}$$

where $a_1 = 1, n \in N_0$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$; then f is sense-preserving, harmonic univalent in U and $f \in \bar{S}_H^n(\alpha, \beta)$.

Proof. If $z_1 \neq z_2$, then

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g\left(z_1\right)-g\left(z_2\right)}{h\left(z_1\right)-h(z_2)}\right| \\ &= 1-\left|\frac{\sum_{k=1}^{\infty} b_k\left(z_1^k-z_2^k\right)}{(z_1-z_2)+\sum_{k=2}^{\infty} a_k\left(z_1^k-z_2^k\right)}\right| \\ &> 1-\frac{\sum_{k=1}^{\infty} k|b_k|}{1-\sum_{k=2}^{\infty} k|a_k|} \\ &> 1-\frac{\sum_{k=1}^{\infty}\frac{k^n[k+1+\beta(k-1+2\alpha)]}{2\beta(1-\alpha)}|b_k|}{1-\sum_{k=2}^{\infty}\frac{k^n[k-1+\beta(k+1-2\alpha)]}{2\beta(1-\alpha)}|a_k|} \geq 0\end{aligned}$$

which proves univalence. Note that f is sense-preserving in U. This is because

$$\begin{aligned}|h'(z)| &\geq 1-\sum_{k=2}^{\infty} k|a_k||z|^{k-1} > 1-\sum_{k=2}^{\infty}\frac{k^n\left[k-1+\beta\left(k+1-2\alpha\right)\right]}{2\beta\left(1-\alpha\right)}|a_k| \\ &> \sum_{k=1}^{\infty}\frac{k^n\left[k+1+\beta\left(k-1+2\alpha\right)\right]}{2\beta\left(1-\alpha\right)}|b_k||z|^{k-1} \\ &\geq \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|.\end{aligned}$$

It remains to show that $f \in \bar{S}_H^n(\alpha,\beta)$. Suppose that the inequality (2.1) holds true and let $z \in \partial U = \{z : z \in C \text{ and } |z| = 1\}$. Then we find from definition (1.2) that

$$\left|\frac{D^{n+1}f(z)-D^nf(z)}{D^{n+1}f(z)+(1-2\alpha)D^nf(z)}\right|$$

$$\begin{aligned}&= \left|\frac{\sum_{k=2}^{\infty} k^n\left(k-1\right)a_kz^k-(-1)^n\sum_{k=1}^{\infty} k^n\left(k+1\right)\overline{b_kz^k}}{2(1-\alpha)z+\sum_{k=2}^{\infty} k^n\left(k+1-2\alpha\right)a_kz^k-(-1)^n\sum_{k=1}^{\infty} k^n(k-1+2\alpha)\overline{b_kz^k}}\right| \\ &\leq \frac{\sum_{k=2}^{\infty} k^n(k-1)|a_k||z|^k+\sum_{k=1}^{\infty} k^n\left(k+1\right)|b_k||z|^k}{2(1-\alpha)|z|-\sum_{k=2}^{\infty} k^n\left(k+1-2\alpha\right)|a_k||z|^k-\sum_{k=1}^{\infty} k^n(k-1+2\alpha)|b_k||z|^k} \\ &\leq \beta\end{aligned}$$

provided that the inequality (2.1) is satisfied. Hence, by the maximum modulus theorem, we have $f(z) \in S_H^n(\alpha,\beta)$. □

The harmonic function

$$f(z) = z + \sum_{k=2}^{\infty} \frac{2\beta(1-\alpha)}{k^n[k-1+\beta(k+1-2\alpha)]} x_k\ z^k + \sum_{k=1}^{\infty} \frac{2\beta(1-\alpha)}{k^n[k+1+\beta(k-1+2\alpha)]} \overline{y_k\ z^k} \quad (2.2)$$

where $\sum_{k=2}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k| = 1$ shows that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.2) are in $S_H^n(\alpha,\beta)$ because

$$\sum_{k=1}^{\infty}\left\{\frac{k^n[k-1+\beta(k+1-2\alpha)]}{2\beta(1-\alpha)}|a_k| + \frac{k^n[k+1+\beta(k-1+2\alpha)]}{2\beta(1-\alpha)}|b_k|\right\} = 1 + \sum_{k=2}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k| = 2.$$

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f_n = h + \bar{g}_n$ where h and $\bar{g}_n$ are of the form (1.4).

Theorem 2. *Let $f_n = h + \bar{g}_n$ be given by (1.4). Then $f_n \in \bar{S}_H^n(\alpha,\beta)$ if and only if*

$$\sum_{k=1}^{\infty} k^n\{[k-1+\beta(k+1-2\alpha)]a_k + [k+1+\beta(k-1+2\alpha)]b_k\} \leq 4\beta(1-\alpha). \quad (2.3)$$

Proof. Since $\bar{S}_H^n(\alpha,\beta) \subset S_H^n(\alpha,\beta)$, we only need to prove the "only if" part of the theorem. To this end, for functions f_n of the form (1.4), we notice that the condition

$$\operatorname{Re}\left\{\left[\frac{D^{n+1}f(z) - D^n f(z)}{D^{n+1}f(z) + (1-2\alpha)D^n f(z)}\right]\right\} < \beta$$

is equivalent to

$$\operatorname{Re}\left\{\frac{-\sum_{k=2}^{\infty}k^n(k-1)a_k z^k - (-1)^{2n}\sum_{k=1}^{\infty}k^n(k+1)b^k\bar{z}^k}{2(1-\alpha)z - \sum_{k=2}^{\infty}k^n(k+1-2\alpha)a_k z^k - (-1)^{2n}\sum_{k=1}^{\infty}k^n(k-1+2\alpha)b^k\bar{z}^k}\right\} > -\beta$$

If we choose z on the real axis and $z \to 1^-$ we get

$$\frac{\sum_{k=2}^{\infty}k^n(k-1)a_k + \sum_{k=1}^{\infty}k^n(k+1)b^k}{2(1-\alpha) - \sum_{k=2}^{\infty}k^n(k+1-2\alpha)a_k - \sum_{k=1}^{\infty}k^n(k-1+2\alpha)b^k} < \beta$$

whence

$$\sum_{k=2}^{\infty} k^n (k-1) a_k + \sum_{k=1}^{\infty} k^n (k+1) b^k$$
$$< 2\beta (1-\alpha) - \beta \sum_{k=2}^{\infty} k^n (k+1-2\alpha) a_k - \beta \sum_{k=1}^{\infty} k^n (k-1+2\alpha) b_k$$

and so

$$\sum_{k=2}^{\infty} k^n [k-1+\beta (k+1-2\alpha)] a_k$$
$$+ \sum_{k=1}^{\infty} k^n [k+1+\beta (k-1+2\alpha)] b_k < 2\beta (1-\alpha)$$

which is equivalent to (2.3). □

Next we determine the extreme points of closed convex hulls of $\bar{S}_H^n(\alpha,\beta)$ denoted by clco $\bar{S}_H^n(\alpha,\beta)$.

Theorem 3. *Let $f_n = h + \bar{g}_n$ be given by (1.4). Then $f_n \in \bar{S}_H^n(\alpha,\beta)$ if and only if*

$$f_n(z) = \sum_{k=1}^{\infty} \left(x_k h_k(z) + y_k g_{n_k}(z) \right), \quad \textit{where} \quad h_1(z) = z,$$
$$h_k(z) = \frac{2\beta (1-\alpha)}{k^n [k-1+\beta (k+1-2\alpha)]} z^k, \quad (k = 2, 3, \dots) \ \textit{and}$$
$$g_{n_k}(z) = z + (-1)^n \frac{2\beta (1-\alpha)}{k^n [k+1+\beta (k-1+2\alpha)]} \bar{z}^k, \ \ (k = 1, 2, \dots)$$
$$\sum_{k=1}^{\infty} (x_k + y_k) = 1, \quad x_n \geq 0, \ \ y_n \geq 0.$$

In particular, the extreme points of $\bar{S}_H^n(\alpha,\beta)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof. Suppose

$$f_n(z) = \sum_{k=1}^{\infty} \left(x_k h_k(z) + y_k g_{nk}(z) \right)$$
$$= \sum_{k=1}^{\infty} (x_k + y_k) z - \sum_{k=2}^{\infty} \frac{2\beta (1-\alpha)}{k^n [k-1+\beta (k+1-2\alpha)]} x_k z^k$$
$$+ (-1)^n \sum_{k=1}^{\infty} \frac{2\beta (1-\alpha)}{k^n [k+1+\beta (k-1+2\alpha)]} y_k \bar{z}^k.$$

Then

$$\sum_{k=2}^{\infty}\frac{k^{n}\left[k-1+\beta\left(k+1-2\alpha\right)\right]}{2\beta\left(1-\alpha\right)}\left(\frac{2\beta\left(1-\alpha\right)}{k^{n}\left[k+1+\beta\left(k-1+2\alpha\right)\right]}x_{k}\right)$$
$$+\sum_{k=1}^{\infty}\frac{k^{n}\left[k+1+\beta\left(k-1+2\alpha\right)\right]}{2\beta\left(1-\alpha\right)}\left(\frac{2\beta\left(1-\alpha\right)}{k^{n}\left[k+1+\beta\left(k-1+2\alpha\right)\right]}y_{k}\right)$$
$$=\sum_{k=2}^{\infty}x_{k}+\sum_{k=1}^{\infty}y_{k}=1-x_{1}\leq 1$$

and so $f_n \in \bar{S}_H^n(\alpha,\beta)$. Conversely, if $f_n \in \operatorname{clco} \bar{S}_H^n(\alpha,\beta)$, then

$$a_k \leq \frac{2\beta\left(1-\alpha\right)}{k^{n}\left[k-1+\beta\left(k+1-2\alpha\right)\right]} \quad \text{and} \quad b_k \leq \frac{2\beta\left(1-\alpha\right)}{k^{n}\left[k+1+\beta\left(k-1+2\alpha\right)\right]}.$$

Set

$$x_k = \frac{k^{n}\left[k-1+\beta\left(k+1-2\alpha\right)\right]}{2\beta\left(1-\alpha\right)}a_k, \quad (k=2,3,\dots)$$

and

$$y_k = \frac{k^{n}\left[k+1+\beta\left(k-1+2\alpha\right)\right]}{2\beta\left(1-\alpha\right)}b_k, \quad (k=1,2,\dots).$$

Then note that by Theorem 2, $0 \leq x_k \leq 1$, $(k=2,3,\dots)$ and $0 \leq y_k \leq 1$, $(k=1,2,\dots)$. We define

$$x_1 = 1-\sum_{k=2}^{\infty}x_k-\sum_{k=1}^{\infty}y_k$$

and note that, by Theorem 2, $x_1 \geq 0$. Consequently, we obtain $f_n(z) = \sum_{k=1}^{\infty}\left(x_k h_k(z)+y_k g_{n_k}(z)\right)$ as required. □

The following theorem gives the distortion bounds for functions in $\bar{S}_H^n(\alpha,\beta)$ which yields a covering result for this class.

Theorem 4. *Let $f_n \in \bar{S}_H^n(\alpha,\beta)$. Then for $|z|=r<1$ we have*

$$|f_n(z)| \leq (1+b_1)\,r+\frac{1}{2^{n-1}}\left(\frac{\beta\left(1-\alpha\right)-\left(1+\beta\alpha\right)b_1}{1+\beta\left(3-2\alpha\right)}\right)r^2, \quad |z|=r<1$$

and

$$|f_n(z)| \geq (1-b_1)\,r-\frac{1}{2^{n-1}}\left(\frac{\beta\left(1-\alpha\right)-\left(1+\beta\alpha\right)b_1}{1+\beta\left(3-2\alpha\right)}\right)r^2, \quad |z|=r<1.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_n \in \bar{S}_H^n(\alpha, \beta)$. Taking the absolute value of f_n we have

$$\begin{aligned}
|f_n(z)| &\leq (1+b_1)\, r + \sum_{k=2}^{\infty}(a_k+b_k)r^k \leq (1+b_1)\, r + \sum_{k=2}^{\infty}(a_k+b_k)r^2 \\
&= (1+b_1)\, r + \frac{2\beta(1-\alpha)}{2^n\left[1+\beta\,(3-2\alpha)\right]}\sum_{k=2}^{\infty}\frac{2^n\left[1+\beta\,(3-2\alpha)\right]}{2\beta(1-\alpha)}(a_k+b_k)r^2 \\
&\leq (1+b_1)\, r + \frac{\beta(1-\alpha)}{2^{n-1}\left[1+\beta\,(3-2\alpha)\right]}\left(1-\frac{2+2\beta\alpha}{2\beta(1-\alpha)}b_1\right)r^2 \\
&= (1+b_1)\, r + \frac{1}{2^{n-1}}\left(\frac{\beta(1-\alpha)-(1+\beta\alpha)b_1}{1+\beta\,(3-2\alpha)}\right)r^2.
\end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 4. □

Corollary 1. *Let f_n of the form* (1.4) *be such that $f_n \in \bar{S}_H^n(\alpha, \beta)$. Then*

$$\left\{w : |w| < \frac{2^{n-1}\left[1+\beta\,(3-2\alpha)\right]-\beta\,(1-\alpha)}{2^{n-1}\left[1+\beta\,(3-2\alpha)\right]} - \frac{2^{n-1}\left[1+\beta\,(3-2\alpha)\right]-(1+\beta\alpha)}{2^{n-1}\left[1+\beta\,(3-2\alpha)\right]}b_1\right\} \subset f_n(U).$$

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f_n(z) = z - \sum_{k=2}^{\infty}a_k z^k + (-1)^n\sum_{k=1}^{\infty}b_k\bar{z}^k$$

and

$$F_n(z) = z - \sum_{k=2}^{\infty}A_k z^k + (-1)^n\sum_{k=1}^{\infty}B_k\bar{z}^k$$

we define the convolution of two harmonic functions f_n and F_n as

$$\left(f_n * F_n\right)(z) = f_n(z) * F_n(z) = z - \sum_{k=2}^{\infty}a_k A_k z^k + (-1)^n\sum_{k=1}^{\infty}b_k B_k\bar{z}^k \qquad (2.4)$$

Using this definition, we show that the class $\bar{S}_H^n(\alpha, \beta)$ is closed under convolution.

Theorem 5. *For $0 \leq \alpha_1 \leq \alpha_2 < 1$ let $f_n \in \bar{S}_H^n(\alpha_2, \beta)$, and $F_n \in \bar{S}_H^n(\alpha_1, \beta)$. Then the convolution $f_n * F_n \in \bar{S}_H^n(\alpha_2, \beta) \subset \bar{S}_H^n(\alpha_1, \beta)$.*

Proof. For f_n and F_n as the Theorem 5, write

$$f_n(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k \bar{z}^k$$

and

$$F_n(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^n \sum_{k=1}^{\infty} B_k \bar{z}^k.$$

Then the convolution $f_n * F_n$ is given by (2.4). We wish to show that the coefficients of $f_n * F_n$ satisfy the required condition given in Theorem 2. For $F_n \in \bar{S}_H^n(\alpha_2, \beta)$ we note that $A_k < 1$ and $B_k < 1$. Now, for the convolution function $f_n * F_n$ we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{k^n \left[k-1+\beta\left(k+1-2\alpha_1\right)\right]}{2\beta\left(1-\alpha_1\right)} a_k A_k + \sum_{k=1}^{\infty} \frac{k^n \left[k+1+\beta\left(k-1+2\alpha_1\right)\right]}{2\beta\left(1-\alpha_1\right)} b_k B_k \\
&\leq \sum_{k=2}^{\infty} \frac{k^n \left[k-1+\beta\left(k+1-2\alpha_1\right)\right]}{2\beta\left(1-\alpha_1\right)} a_k + \sum_{k=1}^{\infty} \frac{k^n \left[k+1+\beta\left(k-1+2\alpha_1\right)\right]}{2\beta\left(1-\alpha_1\right)} b_k \\
&\leq \sum_{k=2}^{\infty} \frac{k^n \left[k-1+\beta\left(k+1-2\alpha_2\right)\right]}{2\beta\left(1-\alpha_2\right)} a_k + \sum_{k=1}^{\infty} \frac{k^n \left[k+1+\beta\left(k-1+2\alpha_2\right)\right]}{2\beta\left(1-\alpha_2\right)} b_k \\
&\leq 1
\end{aligned}$$

since $0 \leq \alpha_1 \leq \alpha_2 < 1$ and $f_n \in \bar{S}_H^n(\alpha_2, \beta)$. Therefore $f_n * F_n \in \bar{S}_H^n(\alpha_2, \beta) \subset \bar{S}_H^n(\alpha_1, \beta)$. □

Now we show that $\bar{S}_H^n(\alpha, \beta)$ is closed under convex combinations of its members.

Theorem 6. *The class $\bar{S}_H^n(\alpha, \beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, \ldots$ let $f_{n_i} \in \bar{S}_H^n(\alpha, \beta)$, where f_{n_i} is given by

$$f_{n_i}(z) = z - \sum_{k=2}^{\infty} a_{k_i} z^k + (-1)^n \sum_{k=1}^{\infty} b_{k_i} \bar{z}^k.$$

Then by (2.3),

$$\sum_{k=1}^{\infty} \left(\frac{k^n \left[k-1+\beta\left(k+1-2\alpha\right)\right]}{\beta\left(1-\alpha\right)} a_{k_i} + \frac{k^n \left[k+1+\beta\left(k-1+2\alpha\right)\right]}{\beta\left(1-\alpha\right)} b_{ki} \right) \leq 4 \tag{2.5}$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \le t_i \le 1$, the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{k_i}\right) z^k + (-1)^n \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i b_{k_i}\right)\bar{z}^k$$

Then by (2.5),

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left[\frac{k^n\left[k-1+\beta\left(k+1-2\alpha\right)\right]}{\beta\left(1-\alpha\right)}\sum_{i=1}^{\infty} t_i a_{k_i} + \frac{k^n\left[k+1+\beta\left(k-1+2\alpha\right)\right]}{\beta\left(1-\alpha\right)}\sum_{i=1}^{\infty} t_i b_{k_i}\right] \\
&= \sum_{i=1}^{\infty} t_i \left\{\sum_{k=1}^{\infty}\left(\frac{k^n\left[k-1+\beta\left(k+1-2\alpha\right)\right]}{\beta\left(1-\alpha\right)} a_{k_i}\right.\right. \\
&\qquad\qquad \left.\left. + \frac{k^n\left[k+1+\beta\left(k-1+2\alpha\right)\right]}{\beta\left(1-\alpha\right)} b_{k_i}\right)\right\} \le 4\sum_{i=1}^{\infty} t_i = 4
\end{aligned}$$

This is the condition required by (2.3) and so $\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in \bar{S}_H^n(\alpha,\beta)$. □

Theorem 7. *If $f_n \in \bar{S}_H^n(\alpha,\beta)$ then f_n is convex in the disc*

$$|z| \le \min_k \left\{\frac{\beta\left(1-\alpha\right)\left(1-b_1\right)}{k\left[\beta\left(1-\alpha\right)-\left(1+\beta\alpha\right)b_1\right]}\right\}^{\frac{1}{k-1}}, \quad k = 2, 3, \ldots.$$

Proof. Let $f_n \in \bar{S}_H^n(\alpha,\beta)$, and let $r,\ 0 < r < 1$, be fixed. Then $r^{-1} f_n(rz) \in \bar{S}_H^n(\alpha,\beta)$ and we have

$$\begin{aligned}
\sum_{k=2}^{\infty} k^2\left(a_k+b_k\right) &= \sum_{k=2}^{\infty} k\left(a_k+b_k\right)\left(kr^{k-1}\right) \\
&\le \sum_{k=2}^{\infty}\left(\frac{k^n\left[k-1+\beta\left(k+1-2\alpha\right)\right]}{2\beta\left(1-\alpha\right)} a_k\right. \\
&\quad \left. + \sum_{k=1}^{\infty}\frac{k^n\left[k+1+\beta\left(k-1+2\alpha\right)\right]}{2\beta\left(1-\alpha\right)} b_k\right)\left(kr^{k-1}\right) \le 1 - b_1
\end{aligned}$$

provided

$$kr^{k-1} \le \frac{1-b_1}{1-\frac{1+\beta\alpha}{\beta(1-\alpha)}}$$

which is true if

$$r \le \min_k \left\{\frac{\beta\left(1-\alpha\right)\left(1-b_1\right)}{k\left[\beta\left(1-\alpha\right)-\left(1+\beta\alpha\right)b_1\right]}\right\}^{\frac{1}{k-1}}, \quad k = 2, 3, \ldots.$$

□

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