SARAJEVO JOURNAL OF MATHEMATICS
Vol.4 (17) (2008), 207–214

ON A SOBOLEV TYPE THEOREM FOR THE GENERALIZED RIESZ POTENTIAL GENERATED BY THE GENERALIZED SHIFT OPERATOR ON MORREY SPACE

MEHMET ZEKI SARIKAYA AND HÜSEYIN YILDIRIM

Abstract. In this paper, we give a generalized definition of Morrey space for Lebesgue measure. In this space, the inequality of Hardy-Sobolev type is established for the generalized Riesz potentials generated by the generalized shift operator.

1. Introduction

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ are vectors in $\mathbb{R}^n$, $x.y = x_1y_1 + \cdots + x_ny_n$, $|x| = (x.x)^{\frac{1}{2}}$,

$$\mathbb{R}_n^+ = \{x : \ x = (x_1, \ldots, x_n), \ x_1 > 0, \ldots, x_n > 0\}.$$

The Bessel differential operator is defined by

$$B_i = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}, \quad i = 1, 2, \ldots, n,$$

$v = (v_1, \ldots, v_n), \ v_1 > 0, \ldots, v_n > 0, |v| = v_1 + \cdots + v_n$.

For $1 \le p < \infty$ let $L_{p,v}(\mathbb{R}_n^+) = L_{p,v}(\mathbb{R}_n^+, \left(\prod_{i=1}^n x_i^{2v_i}\right)dx)$ be the space of functions measurable on $\mathbb{R}_n^+$ with the norm

$$\|f\|_{p,v} = \left(\int_{\mathbb{R}_n^+} |f(x)|^p \left(\prod_{i=1}^n x_i^{2v_i}\right)dx\right)^{\frac{1}{p}}.$$

2000 *Mathematics Subject Classification.* 31B10, 44A15.
Key words and phrases. Riesz potential, shift operator and Morrey space.

Denote by T^y the generalized shift operator acting according to the law

$$T_x^y F(x) = C_v \int_0^\pi \dots \int_0^\pi F\Big(\sqrt{x_1^2 + y_1^2 - 2x_1 y_1 \cos \alpha_1}, \dots,$$
$$\sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos \alpha_n}\Big)\Big(\prod_{i=1}^n \sin_i^{2v_i - 1} \alpha_i\Big)\, d\,\alpha_1 \dots d\,\alpha_n$$

where $x, y \in \mathbb{R}_n^+$, $v_i > \frac{1}{2}$, $i = 1, \dots, n$, $C_v = \prod_{i=1}^n \frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$ [2] and [8]. Let f be in $L_{p,v}(\mathbb{R}_n^+), 1 \le p < \infty$. Then $T^y f$ belongs to $L_{p,v}(\mathbb{R}_n^+)$, and

$$\|T^y f\|_{p,v} \le \|f\|_{p,v} .$$

We remark that T^y is closely connected with the Bessel differential operator $B = (B_1, \dots, B_n)$.

Definition. *Let* $1 \le p < \infty$. *By* $L_{p,w,v}(\mathbb{R}_n^+) = L_{p,w,v}(\mathbb{R}_n^+, \big(\prod_{i=1}^n x_i^{2v_i}\big)dx)$ *we denote the generalized Morrey space which are sets of functions* f *locally integrable on* $\mathbb{R}_n^+$, *with finite norm*

$$\|f\|_{p,w,v} = \sup_Q \Big(\frac{1}{w(Q)} \int_{\mathbf{Q}} T^y \left|f(x)\right|^p \Big(\prod_{i=1}^n x_i^{2v_i}\Big) dx\Big)^{\frac{1}{p}}$$

Then $L_{p,w,v}(\mathbb{R}_n^+)$ *is a Banach space with norm* $\|f\|_{p,w,v}$. *If* $w(0,\rho) = 1$ *and* $T^y \left|f(x)\right|^p = \left|f(x)\right|^p$, *then* $L_{p,w,v} = L_{p,v}$. *If* $w(0,\rho) = \rho^{n+2|v|}$ *and* $T^y \left|f(x)\right|^p = \left|f(x)\right|^p$, *then* $L_{p,w,v} = L_{\infty,v}$. *And if* $w(0,\rho) = \rho^\lambda$, $0 < \lambda < n + 2\left|v\right|$ *and* $T^y \left|f(x)\right|^p = \left|f(x)\right|^p$ *then* $L_{p,w,v}$ *is the Morrey space introduced in* [4] *which is denoted simply by* $L_{p,\lambda,v}$.

Let $w : \mathbb{R}_n^+ \times \mathbb{R}_n^+ \to \mathbb{R}^+$, $1 \le p < \infty$ and $Q(0,\rho)$ be the cube

$$\{x \in \mathbb{R}_n^+ : \left|x_i\right| \le \frac{\rho}{2}, i = 1, 2, \dots, n\}$$

whose edges have length ρ and are parallel to the coordinate axes. For $Q = Q(0,\rho)$, let $kQ = Q(0, k\rho)$ and $w(Q) = w(0,\rho)$.

Assume that there is a constant $C > 0$ such that, for any $\rho > 0$

$$\rho \le t \le 2\rho \Rightarrow C^{-1} \le \frac{w(0,t)}{w(0,\rho)} \le C \tag{1}$$

and

$$\int_\rho^\infty \frac{w(0,t)}{t^{n+2|v|-\alpha p+1}}\, dt \le C \frac{w(0,\rho)}{\rho^{n+2|v|-\alpha p}}, \ \ \alpha p < n + 2\left|v\right| . \tag{2}$$

The convolution operator determined by the T^y is as follows.

$$(f * \varphi)(x) = \int_{\mathbb{R}_n^+} f(y) T^y \varphi(x) \Big(\prod_{i=1}^n y_i^{2v_i}\Big)\, dy.$$

This convolution is known as a B-convolution. We note the following properties of the B-convolution and T^y [3] and [7].

a. $f * \varphi = \varphi * f$

b. $\|f * \varphi\|_{r,v} \le \|f\|_{p,v} \|\varphi\|_{q,v}$, $1 \le p,\ r \le \infty,\ \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$

c. $T^y.1 = 1$

d. If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function all $x \in \mathbb{R}_n^+$ and

$$\int_{\mathbb{R}_n^+} |f(x)| \Big(\prod_{i=1}^n x_i^{2v_i} \Big) dx < \infty$$

then

$$\int_{\mathbb{R}_n^+} T^y f(x) g(y) \Big(\prod_{i=1}^n y_i^{2v_i} \Big) dy = \int_{\mathbb{R}_n^+} f(y) T^y g(x) \Big(\prod_{i=1}^n y_i^{2v_i} \Big) dy$$

e. $|T^y f(x)| \le \sup\limits_{x \ge 0} |f(x)|$.

In this work, we are considering the Hardy-Littlewood radial maximal type function

$$Mf(x) = \sup_{\rho > 0} \frac{1}{|Q(0,\rho)|_v} \int_{\mathbf{Q}} T^y f(x) \Big(\prod_{i=1}^n y_i^{2v_i} \Big) dy.$$

where

$$|Q(0,\rho)|_v = \int_{Q(0,\rho)} \Big(\prod_{i=1}^n x_i^{2v_i} \Big) dx.$$

Let $0 < \alpha < n + 2|v|$. We define the generalized Riesz potentials generated by the generalized shift operator as

$$I_{\alpha,v} f(x) = \int_{\mathbb{R}_n^+} f(y) T^y |x|^{\alpha - n - 2|v|} \Big(\prod_{i=1}^n y_i^{2v_i} \Big) dy. \tag{3}$$

The important properties of the classical Riesz potentials on the Morrey space were studied by Chiarenza et all [1] and Nakai [5]. Furthermore, it is well known that the Riesz potentials $I^\alpha f = C_{n,\alpha} f * |x|^{\alpha - n}$, where $C_{n,\alpha} = \frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})}$, are bounded operators from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ [6,7].

The boundedness of the Riesz potentials generated by the generalized shift operator from $L_{p,v}(\mathbb{R}_n^+)$ to $L_{q,v}(\mathbb{R}_n^+)$ ($\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n+2|v|}$) was proved in [8]. In this work, an inequality of Hardy-Sobolev type is established for these potentials on Morrey space.

Lemma 1. *Let* $1 \le p < \infty,\ \ f \in L_{p,w,v}(\mathbb{R}_n^+)$. *Then the inequality*

$$|T^y f(x)|^p \le T^y |f(x)|^p$$

holds.

Proof. Let

$$F(\alpha, x, y) = f(\sqrt{x_1^2 + y_1^2 - 2x_1 y_1 \cos\alpha_1}, \ldots, \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\alpha_n}).$$

From Hölder's inequality, we have the following inequality with $\frac{1}{p} + \frac{1}{p'} = 1$.

$$\begin{aligned} |T^y f(x)|^p &= \left| C_v \int_0^\pi \ldots \int_0^\pi F(\alpha, x, y) \prod_{i=1}^n (\sin^{2v_i - 1} \alpha_i \, d\alpha_i) \right|^p \\ &\leq \left(C_v \int_0^\pi \ldots \int_0^\pi |F(\alpha, x, y)|^p \prod_{i=1}^n (\sin^{2v_i - 1} \alpha_i \, d\alpha_i) \right) \\ &\qquad \cdot \left[\left(C_v \int_0^\pi \ldots \int_0^\pi \prod_{i=1}^n (\sin^{2v_i - 1} \alpha_i \, d\alpha_i) \right)^{\frac{1}{p'}} \right]^p \\ &\leq T^y \left(|f(x)|^p \right). \end{aligned}$$

□

Lemma 2. *Let* $f \in L_{p,w,v}(\mathbb{R}_n^+)$ *and* $1 < p < \infty$. *Then we have*

$$\|T^y f\|_{p,w,v} \leq \|f\|_{p,w,v}.$$

Proof. From Lemma 1 the following inequality holds.

$$\|T^y f\|_{p,w,v}^p \leq \frac{1}{w(Q)} \int_Q T^z |T^y f(x)|^p \left(\prod_{i=1}^n z_i^{2v_i} \right) dz.$$

If we consider the properties (**c**) and (**d**) of the shift operator, then we have the following inequality

$$\|T^y f\|_{p,w,v} \leq \left(\frac{1}{w(Q)} \int_Q T^z |f(x)|^p \left(\prod_{i=1}^n z_i^{2v_i} \right) dz \right)^{\frac{1}{p}} = \|f\|_{p,w,v}.$$

□

Lemma 3. *Let* $0 < \delta \leq 1$. *Assume that* w *satisfies* (1) *and*

$$\int_\rho^\infty \frac{w(0,t)}{t^{(n+2|v|)\delta + 1}} \, dt \leq C \frac{w(0,\rho)}{\rho^{(n+2|v|)\delta}}.$$

Then for $1 \leq p < \infty$ *there is a constant* $C > 0$ *such that*

$$\int T^y |f(x)|^p (M\chi_Q(y))^\delta \left(\prod_{i=1}^n y_i^{2v_i} \right) dy \leq Cw(Q) \|f\|_{p,w,v}^p, \ \textit{for } f \in L_{p,w,v}.$$

Proof. Let χ_Q be the characteristic function of $Q=Q(0,\rho)$. Then $M\chi_Q(x)\leq 1$. For $x\in 2^{k+1}Q\backslash 2^kQ$, $M\chi_Q(x)$ is comparable to $2^{-k(n+2|v|)}$, $k=1,2,\ldots$. Therefore

$$\begin{aligned}
&\int T^y\,|f(x)|^p\,(M\chi_Q(y))^\delta\left(\prod_{i=1}^n y_i^{2v_i}\right)dy\\
&\quad\leq\left\{\int_{2Q}T^y\,|f(x)|^p\left(\prod_{i=1}^n y_i^{2v_i}\right)dy\right.\\
&\qquad\left.+\sum_{k=1}^\infty\int_{2^{k+1}Q\backslash 2^kQ}2^{-k(n+2|v|)\delta}T^y\,|f(x)|^p\left(\prod_{i=1}^n y_i^{2v_i}\right)dy\right\}\\
&\quad\leq C\left\{w(2Q)+\sum_{k=1}^\infty 2^{-k(n+2|v|)\delta}w(2^{k+1}Q)\right\}\|f\|_{p,w,v}^p\\
&\quad\leq C\rho^{(n+2|v|)\delta}\sum_{k=0}^\infty\frac{w(0,2^kQ)}{(2^{-k}\rho)^{(n+2|v|)\delta}}\,\|f\|_{p,w,v}^p\,.
\end{aligned}$$

Since $\frac{w(0,2^kQ)}{(2^{-k}\rho)^{(n+2|v|)\delta}}$ is comparable to $\int_{2^k\rho}^{2^{k+1}\rho}\frac{w(0,t)}{t^{(n+2|v|)\delta+1}}\,dt$, we have

$$\begin{aligned}
\int T^y\,|f(x)|^p\,(M\chi_Q(y))^\delta\left(\prod_{i=1}^n y_i^{2v_i}\right)dy&\leq C\rho^{(n+2|v|)\delta}\int_\rho^\infty\frac{w(0,t)}{t^{(n+2|v|)\delta+1}}dt\,\|f\|_{p,w,v}^p\\
&\leq Cw(0,\rho)\,\|f\|_{p,w,v}^p\,.
\end{aligned}$$

□

For the generalized Riesz potentials generated by the generalized shift operator the following analogue of the Hardy-Sobolev theorem is valid.

Theorem. *Let* $0<\alpha<n+2\,|v|$, $1\leq p<\frac{n+2|v|}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n+2|v|}$. *Assume that* w *satisfies* (1) *and* (2).

i) *If* $p>1$, *then there is a constant* $C_{p,q}>0$ *such that*

$$\|I_{\alpha,v}f\|_{q,w,v}\leq C_{p,q}\,\|f\|_{p,w,v}\ \ \textit{for}\ f\in L_{p,w,v}(\mathbb{R}_n^+). \tag{4}$$

ii) *If* $p=1$, *then there is a constant* $C_q>0$ *such that for any* $t>0$ *and for any* Q

$$\frac{m\{x\in Q:\ |I_{\alpha,v}f|>t\}}{w(Q)^q}\leq\frac{C_q}{t^q}\,\|f\|_{1,w,v}^q\ \ \textit{for}\ f\in L_{1,w,v}(\mathbb{R}_n^+). \tag{5}$$

Proof. **i)** For $f \in L_{p,w,v}(\mathbb{R}_n^+)$ and for Q, let $f = f_1 + f_2,\ f_1 = f\chi_{2Q}$. Since $I_{\alpha,v}$ is bounded from $L_{p,v}$ to $L_{q,v}$ in [8],

$$\int_Q |I_{\alpha,v} f_1(x)|^q \left(\prod_{i=1}^n x_i^{2v_i}\right) dx \le \|I_{\alpha,v} f_1\|_{p,v}^q \le C_{p,q} \|f_1\|_{p,v}^q$$

$$\le C_{p,q} \left\{ \int_{2Q} |f(x)|^p \left(\prod_{i=1}^n x_i^{2v_i}\right) dx \right\}^{\frac{q}{p}}.$$

Therefore from Lemma 2 we have

$$\left\{ \frac{1}{w(Q)^{\frac{q}{p}}} \int_Q T^y |I_{\alpha,v} f_1(x)|^q \left(\prod_{i=1}^n y_i^{2v_i}\right) dy \right\}^{\frac{1}{p}}$$

$$\le \left\{ \frac{1}{w(Q)^{\frac{q}{p}}} \int_Q |I_{\alpha,v} f_1(x)|^q \left(\prod_{i=1}^n y_i^{2v_i}\right) dy \right\}^{\frac{1}{p}}$$

$$\le C_{p,q} \left\{ \frac{1}{w(Q)} \int_{2Q} T^y |f(x)|^p \left(\prod_{i=1}^n y_i^{2v_i}\right) dy \right\}^{\frac{1}{p}}$$

$$= C_{p,q} \|f\|_{p,w,v}. \tag{6}$$

For $x \in Q$ and for $y \in (2Q)^c$, $\frac{1}{|y|^{n+2|v|-\alpha}}$ is comparable to $\left(\frac{M\chi_Q(y)}{|Q|}\right)^{1-\frac{\alpha}{n+2|v|}}$. Then we have

$$|I_{\alpha,v} f_2(x)| \le \int \frac{T^y |f_2(x)|}{|y|^{n+2|v|-\alpha}} \left(\prod_{i=1}^n y_i^{2v_i}\right) dy$$

$$\le \left(\frac{1}{|Q|}\right)^{1-\frac{\alpha}{n+2|v|}} \int T^y |f(x)| \left(M\chi_Q(y)\right)^{1-\frac{\alpha}{n+2|v|}} \left(\prod_{i=1}^n y_i^{2v_i}\right) dy. \tag{7}$$

By Lemma 3 we have

$$\int_\rho^\infty \frac{w(0,t)}{t^{n+2|v|-\alpha p-\varepsilon+1}}\, dt \le C \frac{w(0,\rho)}{\rho^{n+2|v|-\alpha p-\varepsilon}} \text{ for } \varepsilon > 0.$$

Let $\delta = \frac{n+2|v|-\alpha p-\varepsilon}{n+2|v|}$. By Hölder's inequality, we have

$$|I_{\alpha,v} f_2(x)| \le \frac{C}{|Q|^{1-\frac{\alpha}{n+2|v|}}} \int T^y |f(x)| \left(M\chi_Q(y)\right)^{\frac{\delta}{p}}$$

$$\times \left(M\chi_Q(y)\right)^{1-\frac{\alpha}{n+2|v|}-\frac{\delta}{p}} \left(\prod_{i=1}^n y_i^{2v_i}\right) dy$$

$$\leq \frac{C}{|Q|^{1-\frac{\alpha}{n+2|v|}}}\left(\int T^y\,|f(x)|^p\,(M\chi_Q(y))^\delta\left(\prod_{i=1}^n y_i^{2v_i}\right)dy\right)^{\frac{1}{p}}$$
$$\times\left(\int (M\chi_Q(y))^{\frac{(1-\frac{\alpha}{n+2|v|}-\frac{\delta}{p})}{p-1}}\left(\prod_{i=1}^n y_i^{2v_i}\right)dy\right)^{\frac{1}{q}}.$$

Now

$$\begin{aligned}
&|Q|^{-1}\int (M\chi_Q(y))^{\frac{(1-\frac{\alpha}{n+2|v|}-\frac{\delta}{p})}{p-1}}\left(\prod_{i=1}^n y_i^{2v_i}\right)dy\\
&\quad\leq |Q|^{-1}\left\{\int_{2Q}\left(\prod_{i=1}^n y_i^{2v_i}\right)dy\right.\\
&\qquad\left.+\sum_{k=1}^{\infty}\int_{2^{k+1}Q\backslash 2^kQ}(M\chi_Q(y))^{\frac{(1-\frac{\alpha}{n+2|v|}-\frac{\delta}{p})}{p-1}}\left(\prod_{i=1}^n y_i^{2v_i}\right)dy\right\}\\
&\quad\leq C\,|Q|^{-1}\left\{w(2Q)+\sum_{k=1}^{\infty}2^{-k\frac{(n+2|v|)}{p-1}(1-\frac{\alpha}{n+2|v|}-\frac{\delta}{p})}\left|2^{k+1}Q\right|\right\}\\
&\quad\leq C\sum_{k=0}^{\infty}2^{\frac{-k\varepsilon}{p-1}}\leq C_{p,q}.
\end{aligned}$$

Therefore by Lemma 3 we have

$$|I_{\alpha,v}f_2(x)|\leq C_{p,q}\,|Q|^{-\frac{1}{q}}\,w(Q)^{\frac{1}{q}}\,\|f\|_{p,w,v}\ \text{ for } x\in Q$$

and from Lemma 2

$$\left\{\frac{1}{w(Q)^{\frac{q}{p}}}\int_Q\left|T^y\left(I_{\alpha,v}f_2\right)(x)\right|^q\left(\prod_{i=1}^n y_i^{2v_i}\right)dy\right\}^{\frac{1}{p}}\leq C_{p,q}\,\|f\|_{p,w,v}\,. \tag{8}$$

By (6) and (8) we have (4).

ii) For $f\in L_{1,w,v}$ and for Q , let $f=f_1+f_2,\ f_1=f\chi_{2Q}$. Since $I_{\alpha,v}$ is bounded from $L_{1,v}$ to weak $L_{q,v}$ in [8],

$$\begin{aligned}
m\{x\in Q:\ |I_{\alpha,v}f(x)|>t\}&\leq C_q\left(\frac{\|f_1\|_{1,w,v}}{t}\right)^q\\
&\leq C_q\left(\frac{w(Q)\,\|f\|_{1,w,v}}{t}\right)^q.
\end{aligned}$$

It follows from (7) and Lemma 3 with $p = 1$, $\delta = 1 - \frac{\alpha}{n+2|v|} = \frac{1}{q}$ that

$$|I_{\alpha,v} f_2(x)| \leq C_q \, |Q|^{-\frac{1}{q}} \int T^y \, |f(x)| \, (M\chi_Q(y))^{\frac{1}{q}} \left(\prod_{i=1}^{n} y_i^{2v_i} \right) dy$$
$$\leq C_q \, |Q|^{-\frac{1}{q}} \, w(Q) \, \|f\|_{1,w,v} \quad \text{for } x \in Q.$$

Then from Lemma 2 we obtain

$$m\{x \in Q : \ |I_{\alpha,v} f_2(x)| > t\} \leq \int_Q \left(\frac{|I_{\alpha,v} f_2(x)|}{t} \right)^q \left(\prod_{i=1}^{n} y_i^{2v_i} \right) dy$$
$$\leq C_q \left(\frac{w(Q) \, \|f\|_{1,w,v}}{t} \right)^q .$$

Thus we have (5). This completed the theorem. □

References

[1] F. Chiarenza and M. Frasca, *Morrey spaces and Hardy-Littlewood Maximal Function*, Rend. Mat. Appl., 7 (3-4) (1987), 271–279.

[2] B. M. Levitan, B.M., *Expansion in Fourier Series and Integrals with Bessel Functions*, Uspehi Mat. Nauk (N.S), 6 no. 2(42) (1951), 102–143 (in Russian).

[3] B. M. Levitan, *Generelized Translation Operators and Some of Their Applications*, Moscova 1962 (Translation 1964).

[4] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans.Amer. Math. Soc., 43 (1938), 126–166.

[5] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces,* Math. Nachr., 166 (1994), 95–103.

[6] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Application*, Gordan and Breach Science USA 1993.

[7] E. M. Stein, *Singular Integrals and Differential Properties of Functions*, Princeton University Press, Princeton, New Jersey, (1970).

[8] H. Yıldırım, *Riesz Potentials Generated by a Generalized Shift Operator*, Ph.D. Thesis 1995.

(Received: October 18, 2006)
(Revised: December 25, 2007)

Department of Mathematics
Faculty of Science and Arts
Kocatepe University
Afyon–TURKEY
E–mail: sarikaya@aku.edu.tr
E–mail: hyildir@aku.edu.tr