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THREE POSITIVE PERIODIC SOLUTIONS OF NONLINEAR FUNCTIONAL DIFFERENCE EQUATIONS

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Abstract. Sufficient conditions for the existence of at least three positive T-periodic solutions of the nonlinear functional difference equations are established. An example is presented to illustrate the main results.

1. Introduction

Let R denote the set of real numbers, Z the set of integers and R^+ the set of positive numbers. In this paper, we investigate nonlinear functional difference equations

$$\Delta x(n) = -a(n)x(n) + f(n, x(n - \tau(n))), \quad n \in Z, \tag{1}$$

and

$$\Delta x(n) = -a(n)x(n) - f(n, x(n - \tau(n))), \quad n \in Z, \tag{2}$$

where $a(n)$ and $\tau(n)$ are T-periodic sequences with $T \geq 1$, $f(n, x)$ nonnegative and continuous in x and T-periodic in n.

The motivation of this paper is mainly due to papers [1,2,3,5]. In paper [1], Raffoul investigated the existence of at least one or two positive periodic solutions of the following functional difference equations

$$x(n + 1) = b(n)x(n) + \lambda h(n)f(x(n - \tau(n))), \tag{3}$$

and

$$x(n + 1) = b(n)x(n) - \lambda h(n)f(x(n - \tau(n))), \tag{4}$$

where $b(n)$, $h(n)$ and $\tau(n)$ are nonnegative with period $T \geq 1$, $b(n) > 0$, $\lambda > 0$ a constant.

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The methods to obtain positive periodic solutions of a difference equation are based on the theorems in cones in Banach spaces [1-7,14], lower and upper solutions methods and monotone iterative technique [15], critical point theorems [16,17].

The purpose of this paper is to study the existence of at least three positive T-periodic solutions of the equation (1). Comparing to the known results, equations (1) may be regarded as delay difference equation or forward difference equation or fixed type difference equation. We impose growth conditions on f, instead of requiring the existence of the limits $\lim_{x\to 0} f(x)/x$ and $\lim_{x\to+\infty} f(x)/x$ see [1,2,3,5], to obtain three positive T-periodic solutions by applying fixed point theorems in cones in Banach spaces.

Furthermore, Raffoul proposed the following open problem:

Open problem 3[1]. What can be said about the existence of positive T-periodic solutions of equations (3) and (4) when $b(n) > 1$ in (3) and $0 < b(n) < 1$ in (4) for all $n \in [0, T-1]$?

When applying our results to equation (3) or (4), above mentioned open problem is solved. See Remarks 3.1-3.3 in Section 3.

This paper is organized as follows. In Section 2, we give some background definitions and fixed-point theorems in cones in Banach spaces due to Leggett-Williams, Avery and Henderson [12,13,14]. The main results are presented in Section 3. The example to illustrate the main theorems is given in Section 4.

2. Preliminary results

In this section, we present some lemmas and some preliminary results.

Definition 2.1. *Let X be a Banach space and P be a cone in X. A map $\psi : P \to [0,+\infty)$ is a nonnegative continuous concave or convex functional map provided ψ is nonnegative, continuous and satisfies*

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y),$$

or

$$\psi(tx + (1-t)y) \leq t\psi(x) + (1-t)\psi(y),$$

for all $x, y \in P$ and $t \in [0,1]$.

Definition 2.2. *An operator $H; X \to X$ is completely continuous if it is continuous and maps bounded sets into pre-compact sets.*

Let $0 < a < b$ and r be given and let ψ be a nonnegative continuous concave functional on the cone P. Define the convex sets P_r, $\overline{P_r}$ and $P(\psi; a, b)$

by

$$P_r = \{y \in P | \; ||y|| < r\},$$
$$\overline{P_r} = \{y \in P | \; ||y|| \le r\},$$
$$P(\psi; a, b) = \{y \in P | \; a \le \psi(y), \; ||y|| < b\}.$$

Now, we state the fixed-point theorems due to Leggett-Williams [13] and Avery and Henderson [12] which lay the foundation needed to prove our main results.

Theorem 2.1. (Leggett-Williams Fixed-Point Theorem, [13]). *Suppose that X is a Banach spaces, P a cone of X. Let $H : \overline{P}_c \to \overline{P}_c$, $y \to Hy$ for $y \in P$ be a completely continuous operator, and let ψ be a nonnegative continuous concave functional on P such that $\psi(y) \le ||y||$ for all $y \in \overline{P}_c$. Suppose that there exist $0 < a < b < d \le c$ such that*

- (C_1) *$\{y \in P(\psi; b, d) | \psi(y) > b\} \neq \emptyset$ and $\psi(Hy) > b$ for $y \in P(\psi; b, d)$;*
- (C_2) *$||Hy|| < a$ for $||y|| \le a$;*
- (C_3) *$\psi(Hy) > b$ for $y \in P(\psi; b, c)$ with $||Hy|| > d$.*

Then H has at least three fixed points y_1, y_2 and y_3 such that $||y_1|| < a$, $b < \psi(y_2)$ and $||y_3|| > a$ with $\psi(y_3) < b$.

Let γ, β, θ be nonnegative continuous convex functionals on the cone P and α, ψ be nonnegative continuous concave functionals on the cone P. Then for nonnegative numbers h, a, b, d, and c we define the following sets:

$$P(\gamma, c) = \{x \in P : \; \gamma(x) < c\},$$
$$P(\gamma, \alpha, a, c) = \{x \in P : a \le \alpha(x), \; \gamma(x) \le c\},$$
$$Q(\gamma, \beta, d, c) = \{x \in P : \; \beta(x) \le d, \; \gamma(x) \le c\},$$
$$P(\gamma, \theta, \alpha, a, b, c) = \{x \in P : \; a \le \alpha(x), \; \theta(x) \le b, \; \gamma(x) \le c\},$$
$$Q(\gamma, \beta, \psi, h, d, c) = \{x \in P : \; h \le \psi(x), \; \beta(x) \le d, \; \gamma(x) \le c\}.$$

Theorem 2.2. (Avery and Henderson [12]). *Suppose that X is a Banach spaces, P a cone of X, and there exists positive number c such that*

- (i) *γ, β, θ are three nonnegative continuous convex functionals, α, ψ are a nonnegative continuous concave functionals;*
- (ii) *there exists positive number M such that $\alpha(x) \le \beta(x), ||x|| \le M\gamma(x)$ for $x \in \overline{P(\gamma, c)}$;*
- (iii) *$H : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is completely continuous;*
- (iv) *there are positive numbers h, d, a, b with $0 < d < a$ such that*
- (D_1) *$\{x \in P(\gamma, \theta, \alpha, a, b, c) : \; \alpha(x) > a\} \neq \emptyset$ and $x \in P(\gamma, \theta, \alpha, a, b, c)$ implies $\alpha(Hx) > a$;*

(D_2) *$\{x \in Q(\gamma, \beta, \psi, h, d, c): \ \beta(x) < d\} \neq \emptyset$ and $x \in Q(\gamma, \beta, \psi, h, d, c)$ implies $\beta(Hx) < d$;*

(D_3) *$x \in P(\gamma, \alpha, a, c)$ with $\theta(Hx) > b$ implies $\alpha(Hx) > a$;*

(D_4) *$x \in Q(\gamma, \beta, d, c)$ with $\psi(Hx) < h$ implies $\beta(Hx) < d$.*

Then H has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ that satisfying

$$\beta(x_1) < d, \ a < \alpha(x_2), \ d < \beta(x_3), \ \alpha(x_3) < a.$$

Let X be the set of all real T-periodic sequences $\{x(n)\}_{n=-\infty}^{+\infty}$. This set is endowed with the norm $||x|| = \max_{n \in [0, T-1]} |x(n)|$. Then X is a Banach space. Suppose $a(n) < 1$ for all $n \in Z$. Let us define a cone in the Banach space X as

$$P = \{x \in X: \quad x(n) \geq \sigma ||x|| \text{ on } [0, T-1]\}.$$

Then P is a cone of space X, where σ is defined by

$$\sigma = \frac{\prod_{s=0}^{T-1} a^-(s)}{\prod_{s=0}^{T-1} a^+(s)},$$

where

$$a^+(n) = \max\{1, \ 1 - a(n)\}, \ \ a^-(n) = \min\{1, \ 1 - a(n)\}.$$

We call the sequence $\{x(n)\}_{n=-\infty}^{+\infty}$ a positive T-periodic solution of (1) if it is positive, T-periodic and satisfies (1) for all $n \in Z$.

Let a, b be nonnegative integers, denote $\prod_{i=a}^{b} x(i) = x(a)x(a+1)\cdots x(b)$ if $a \leq b$ and $\prod_{i=1}^{b} x(i) = 1$ if $a > b$. Let us list the assumptions.

(A_1) $f : Z \times R \to R^+$ is continuous in x and T-periodic in n.

(A_2) $a : Z \to R$ and $\tau : Z \to R$ are T-periodic sequences and $a(n) < 1$ for all $n \in Z$ with $\prod_{j=n}^{n+T-1}(1 - a(s)) \equiv$ *constant* < 1 for all $n \in Z$.

Lemma 2.1. *Assume* (A_2) *holds. Then $x \in X$ is a solution of equation* (1) *if and only if*

$$\begin{aligned} x(n) = & \sum_{s=n}^{n+T-2} G(n,s) f(s, x(s - \tau(s))) \\ & + \frac{1}{1 - \prod_{s=n}^{n+T-1}(1 - a(s))} f(n-1, x(n-1-\tau(n-1))), \quad (5) \end{aligned}$$

where

$$G(n,s) = \frac{\prod_{j=s+1}^{n+T-1}(1 - a(j))}{1 - \prod_{j=0}^{T-1}(1 - a(j))}, \ \ s \in [n, n+T-1].$$

Proof. If $x \in X$ satisfies equation (1), then

$$x(n+1) - (1-a(n))x(n) = f(n, x(n-\tau(n))), \quad n \in Z.$$

It is easy to see that

$$x(n) = \frac{1}{a(n)} f(n, x(n-\tau(n))), \ n \in Z$$

for $T = 1$. If $T \geq 2$, then we get

$$\begin{aligned}
&x(n+1) - (1-a(n))x(n) = f(n, x(n-\tau(n))), \ n \in Z,\\
&x(n+2) - (1-a(n+1))x(n+1) = f(n+1, x(n+1-\tau(n+1))), \ n \in Z,\\
&\cdots\cdots,\\
&x(n+T) - (1-a(n+T-1))x(n+T-1)\\
&\qquad\qquad = f(n+T-1, x(n+T-1-\tau(n+T-1))).
\end{aligned}$$

Summing the above equalities, we get

$$\begin{aligned}
x(n+T) - x(n) \prod_{j=n}^{n+T-1} (1-a(j)) &= \sum_{s=n}^{n+T-2} \prod_{j=s+1}^{n+T-1} (1-a(j)) f(j, x(j-\tau(j)))\\
&\quad + f(n+T-1, x(n+T-1-\tau(n+T-1))).
\end{aligned}$$

Since $x \in X$ implies $x(T+n) = x(n)$, in view of (A_2), we get (5).

On the other hand, it is easy to show that if $x \in X$ and satisfies (5), then x satisfies (1), and so x is a T-periodic solution of (1). □

Remark 2.1. It follows from Lemma 2.1 and (5) that if $(A_1), (A_2)$ hold, and $a(n) < 1$ with $\prod_{j=n}^{n+T-1}(1-a(j)) \equiv \prod_{j=0}^{T-1}(1-a(j)) > 1$, then equation (1) has no positive periodic solution. In fact, suppose x is a positive periodic solution of equation (1). Then

$$\begin{aligned}
0 < x(n) = &\sum_{s=n}^{n+T-2} G(n,s) f(s, x(s-\tau(s)))\\
&+ \frac{1}{1-\prod_{j=n}^{n+T-1}(1-a(j))} f(n-1, x(n-1-\tau(n-1))) \leq 0
\end{aligned}$$

is a contradiction.

If $a(n) < 1$ for all $n \in Z$, (A_2) holds, and $\prod_{j=n}^{n+T-1}(1-a(j)) \equiv \prod_{j=0}^{T-1}(1-a(j)) < 1$, then we get, for $T \geq 2$, $n \leq s \leq n+T-2$, that

$$G(n,s) = \frac{\prod_{j=s}^{n+T-1}(1-a(j))}{1-\prod_{j=0}^{T-1}(1-a(j))} \geq \frac{\prod_{j=0}^{T-1} a^-(j)}{1-\prod_{j=0}^{T-1}(1-a(j))},$$

and

$$G(n,s) = \frac{\prod_{j=s}^{n+T-1}(1-a(j))}{1-\prod_{j=0}^{T-1}(1-a(j))} \le \frac{\prod_{j=0}^{T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))}.$$

Let

$$\sigma = \begin{cases} 1, & T=1, \\ \prod_{j=0}^{T-1} a^-(j)[a^+(s)]^{-1}, & T \ge 2. \end{cases}$$

Define an operator $H: X \to X$ by

$$\begin{aligned} H(x)(n) = & \sum_{s=n}^{n+T-2} G(n,s) f(s, x(s-\tau(s))) \\ & + \frac{1}{1-\prod_{j=n}^{n+T-1}(1-a(j))} f(n-1, x(n-1-\tau(n-1))) \end{aligned}$$

for $x \in X$.

Lemma 2.2. *Assume $a(n) < 1$ for all $n \in Z$ and $\prod_{j=0}^{T-1}(1-a(j)) < 1$, (A_1), (A_2) hold. If $x \in P$, then $H(x) \in P$.*

Proof. For $x \in P$, we get

$$\begin{aligned} H(x)(n) = & \sum_{s=n}^{n+T-2} G(n,s) f(s, x(s-\tau(s))) \\ & + \frac{1}{1-\prod_{j=n}^{n+T-1}(1-a(j))} f(n-1, x(n-1-\tau(n-1))),\ n \in Z. \end{aligned}$$

If $n \le -1$, then

$$\begin{aligned} & \sum_{s=n}^{n+T-1} f(s, x(s-\tau(s))) \\ & = \sum_{s=n}^{-1} f(s, x(s-\tau(s))) + \sum_{s=0}^{T-1} f(s, x(s-\tau(s))) - \sum_{s=n+T}^{T-1} f(s, x(s-\tau(s))) \\ & = \sum_{s=n}^{-1} f(s, x(s-\tau(s))) + \sum_{s=0}^{T-1} f(s, u(s)) - \sum_{s=n}^{1} f(s, x(s-\tau(s))) \\ & = \sum_{s=0}^{T-1} f(s, x(s-\tau(s))). \end{aligned}$$

Similarly $n > -1$ implies $\sum_{s=n}^{n+T-1} f(s, u(s)) = \sum_{s=0}^{T-1} f(s, u(s))$. We note that

$$G(n,s) = \frac{\prod_{j=s}^{n+T-1}(1-a(j))}{1-\prod_{j=0}^{T-1}(1-a(j))} \leq \frac{\prod_{j=0}^{T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))}.$$

Thus (A_1), $a(n) < 1$, $\prod_{s=n}^{n+T-1}(1-a(s)) < 1$ for all $n \in Z$ and the definition of $G(n,s)$ imply that

$$\begin{aligned}
H(x)(n) = &\sum_{s=n}^{n+T-2} G(n,s) f(s, x(s-\tau(s))) \\
&+ \frac{1}{1-\prod_{j=n}^{n+T-1}(1-a(j))} f(n-1, x(n-1-\tau(n-1))) \\
\leq &\frac{\prod_{j=n+1}^{n+T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))} \sum_{s=n}^{n+T-2} f(s, x(s-\tau(s))) \\
&+ \frac{1}{1-\prod_{j=0}^{T-1}(1-a(j))} f(n-1, x(n-1-\tau(n-1))) \\
\leq &\frac{\prod_{j=0}^{T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))} \sum_{s=n}^{n+T-1} f(s, x(s-\tau(s))) \\
= &\frac{\prod_{j=0}^{T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))} \sum_{s=0}^{T-1} f(s, x(s-\tau(s)))
\end{aligned}$$

and similarly we get

$$\begin{aligned}
H(x)(n) \geq &\frac{\prod_{j=n+1}^{n+T-1} a^-(j)}{1-\prod_{j=0}^{T-1}(1-a(j))} \sum_{s=n}^{n+T-2} f(s, x(s-\tau(s))) \\
&+ \frac{1}{1-\prod_{j=0}^{T-1}(1-a(j))} f(n-1, x(n-1-\tau(n-1))) \\
\geq &\frac{\prod_{j=0}^{T-1} a^-(j)}{1-\prod_{j=0}^{T-1}(1-a(j))} \sum_{s=0}^{T-1} f(s, x(s-\tau(s))).
\end{aligned}$$

It follows that

$$||H(x)|| \leq \frac{\prod_{j=0}^{T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))} \sum_{s=0}^{T-1} f(s, x(s-\tau(s))) \leq \sigma^{-1} H(x)(n)$$

and so $H(x)(n) \geq \sigma ||H(x)||$ for all $n \in Z$. Thus $H(x) \in P$. □

Lemma 2.3. *Suppose* (A_1) *and* (A_2) *hold. Then*

(i) H *is completely continuous.*

(ii) *x is a positive T-periodic solution of* (1) *if and only if x is a fixed point of the operator H on P.*

The proof of Lemma 2.3 is standard and is omitted, one may see [12].

3. Main results

In this section, we present the main results of this paper. In relation to equation (1), we suppose

$$a(n) < 1 \text{ for all } n \in Z, \quad \prod_{s=0}^{T-1} a(s) < 1.$$

Let

$$\psi(x) = \min_{n \in [0,T-1]} x(n) \text{ for } x \in P.$$

Then ψ is a nonnegative continuous concave functional on P and $\psi(x) \le ||x||$ for all $x \in P$.

Theorem 3.1. *Assume $(A_1)-(A_2)$. Furthermore, suppose that there are constants $0 < d < a$ such that*

(A_3) *f satisfies*

$$\overline{\lim_{x \to +\infty}} \frac{f(n,x)}{|x|} < \frac{1 - \prod_{j=0}^{T-1}(1-a(j))}{T \prod_{j=0}^{T-1} a^+(j)} = c$$

uniformly for $n \in [0, T-1]$;

(A_4) *it holds that*

$$f(n,x) < d \frac{1 - \prod_{j=0}^{T-1}(1-a(j))}{T \prod_{j=0}^{T-1} a^+(j)}$$

for $n \in [0, T-1]$ and $x \in [0, d]$;

(A_5) *$f(n,x) > La$ for $t \in [0, T-1]$ and $x \in [a, a/\sigma]$ and $L = \frac{1-\prod_{j=0}^{T-1}(1-a(j))}{T\prod_{j=0}^{T-1} a^-(j)}$.*

Then equation (1) *has at least three positive T-periodic solutions x_1, x_2 and x_3 satisfying*

$$||x_1|| < d, \quad \min_{t \in [0,T-1]} x_2(t) > a, \ ||x_2|| \ge d \text{ and } \min_{t \in [0,T-1]} x_3(t) \le a.$$

Proof. By (A_3), we see there is $0 < \mu < c$ and $M > 0$ such that

$$f(n,x) \le \mu |x|$$

for all $n \in [0, T-1]$ and $|x| \ge M$. Let

$$\beta = \max_{|x| \in [0,M]\ n \in [0,T-1]} f(n,x).$$

Then

$$0 \le f(n,x) \le \mu|x| + \beta$$

for $n \in [0, T-1],\ x \in [0,+\infty)$. Choose $e > \max\{\beta/(c-\mu), a/\sigma\}$. Then, for $x \in \overline{P_e}$, we have

$$\begin{aligned}||H(x)|| &= \max_{0\le n\le T-1}\Bigg(\sum_{s=n}^{n+T-2} G(n,s) f(s, x(s-\tau(s))) \\ &\qquad + \frac{1}{1-\prod_{j=0}^{T-1}(1-a(j))} f(n-1, x(n-1))\Bigg) \\ &\le \frac{\prod_{j=0}^{T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))}\sum_{s=0}^{T-1} f(s, x(s-\tau(s))) \\ &\le \frac{(\mu e+\beta)T\prod_{j=0}^{T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))} = \frac{\mu e+\beta}{c} < e.\end{aligned}$$

Then $H(x) \in \overline{P_e}$, Hence $H(\overline{P_e}) \subset P_e$. On the other hand, it is easy to prove that H is completely continuous by (A_1). Now, we prove that (C_1), (C_2) and (C_3) of Theorem 2.1 hold.

First, choose $\theta \in (1, 1/\sigma)$ and let $\phi_0(n) = \theta a$ for $n \in Z$, it follows from $e > a/\sigma$ that $\phi_0 \in \{x \in P(\psi, a, a/\sigma),\ \psi(\phi_0) > a$. Hence $\{x \in P(\psi, a, a/\sigma), \psi(x) > a\} \ne \emptyset$. For $x \in P(\psi, a, a/\sigma)$, $\psi(x) \ge a$ and $a \le x(t) \le a/\sigma$, it follows from (A_5) that

$$\begin{aligned}\psi(H(x)) &= \min_{n\in[0,T-1]}\min_{n\in[0,T-1]}\Bigg(\sum_{s=n}^{n+T-2} G(n,s) f(s, x(s-\tau(s))) \\ &\qquad + \frac{1}{1-\prod_{j=0}^{T-1}(1-a(j))} f(n-1, x(n-1))\Bigg) \\ &\ge \frac{\prod_{j=0}^{T-1} a^-(j)}{1-\prod_{j=0}^{T-1}(1-a(j))}\sum_{s=0}^{T-1} f(s, x(s-\tau(s))) \\ &> \frac{T\prod_{j=0}^{T-1} a^-(j)}{1-\prod_{j=0}^{T-1}(1-a(j))} La = a.\end{aligned}$$

Secondly, for $x \in \overline{P_d}$, it follows from (A_4) that

$$\begin{aligned}||H(x)|| &= \max_{0\le n\le T-1}\Bigg(\sum_{s=n}^{n+T-2} G(n,s) f(s, x(s-\tau(s))) \\ &\qquad + \frac{1}{1-\prod_{j=0}^{T-1}(1-a(j))} f(n-1, x(n-1))\Bigg)\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\prod_{j=0}^{T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))}\sum_{s=0}^{T-1} f(s,x(s-\tau(s)))\\
&\leq \frac{dT\prod_{j=0}^{T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))}\\
&= \frac{\prod_{j=0}^{T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))}\sum_{s=0}^{T-1} f(s,x(s-\tau(s)))\\
&< \frac{\prod_{j=0}^{T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))}Td\frac{1-\prod_{j=0}^{T-1}(1-a(j))}{T\prod_{j=0}^{T-1} a^+(j)} = d.
\end{aligned}$$

Finally, if $x \in P(\psi, a, e)$ and $||H(x)|| > a/\sigma$, then $\min_{t\in[0,T-1]} x(n) \geq a$ and $||x|| \leq e$. Hence we have

$$\psi(H(x)) = \min_{t\in[0,T-1]} H(x)(n) \geq \sigma||H(x)|| > \sigma\frac{a}{\sigma} = a.$$

From the steps above, (C_1), (C_2) and (C_3) of Theorem 2.1 are satisfied. Then, by Theorem 2.1, H has three fixed points x_1, x_2 and $x_3 \in \overline{P_e}$ such that

$$||x_1|| < d,\ \psi(x_2) > a,\ ||x_2|| \geq d,\ \psi(x_3) \leq a,\ ||x_i|| \leq e \text{ for } i = 1,2,3,$$

i.e. equation (1) has three positive T-periodic solutions x_1, x_2 and x_3 such that

$$||x_1|| < d,\quad \min_{n\in[0,T-1]} x_2(t) > a,\ ||x_2|| \geq d$$
$$\text{and } \min_{n\in[0,T-1]} x_3(n) \leq a,\ ||x_i|| \leq e \text{ for } i = 1,2,3.$$

The proof is complete. □

To establish the second result, define nonnegative continuous concave functionals α, ψ and nonnegative convex functionals β, θ, γ on P by

$$\gamma(x) = \beta(x) = \theta(x) = \max_{n\in[0,T-1]} x(n),\quad x \in P,$$

and

$$\psi(x) = \alpha(x) = \min_{n\in[0,T-1]} x(n),\quad x \in P.$$

We observe here that for each $x \in P$,

$$\alpha(x) = \psi(x) \leq \beta(x) = \gamma(x) = \theta(x),\quad ||x|| = \gamma(x).$$

Then

$$\begin{aligned}
P(\gamma, c) &= \{x \in P : \max_{n\in[0,T-1]} x(n) < c\}, \\
P(\gamma, \alpha, a, c) &= \{x \in P : a \le \min_{n\in[0,T-1]} x(n),\ \max_{n\in[0,T-1]} x(n) \le c\}, \\
Q(\gamma, \beta, d, c) &= \{x \in P : \max_{n\in[0,T-1]} x(n) \le d,\ \max_{n\in[0,T-1]} x(n) \le c\}, \\
P(\gamma, \theta, \alpha, a, b, c) &= \{x \in P : a \le \min_{n\in[0,T-1]} x(n), \\
&\qquad \max_{n\in[0,T-1]} x(n) \le b,\ \max_{n\in[0,T-1]} x(n) \le c\}, \\
Q(\gamma, \beta, \psi, h, d, c) &= \{x \in P : h \le \min_{n\in[0,T-1]} x(n), \\
&\qquad \max_{n\in[0,T-1]} x(n) \le d,\ \max_{n\in[0,T-1]} x(n) \le c\}.
\end{aligned}$$

Theorem 3.2. *Suppose that* (A_1) *and* (A_2) *hold and there exist positive numbers* $0 < a < b < c$ *such that*

$$0 < a < b < \frac{b}{\sigma} < \sigma c < c,$$

and $f(n,x)$ *satisfies conditions:*

$$f(n,x) \le a\frac{1 - \prod_{j=0}^{T-1}(1-a(j))}{T\prod_{j=0}^{T-1} a^+(j)},\quad n \in [0, T-1],\ \ \sigma a \le x \le a. \tag{6}$$

$$f(n,x) \ge b\frac{1 - \prod_{j=0}^{T-1}(1-a(j))}{T\prod_{j=0}^{T-1} a^-(j)},\quad n \in [0, T-1],\ \ b \le x \le \frac{b}{\sigma}, \tag{7}$$

$$f(n,x) \le c\frac{1 - \prod_{j=0}^{T-1}(1-a(j))}{T\prod_{j=0}^{T-1} a^+(j)},\quad n \in [0, T-1],\ \ \sigma c \le x \le c. \tag{8}$$

Then equation (1) *has at least three positive* T*-periodic solutions* $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ *and*

$$\begin{aligned}
&\max_{n\in[0,T-1]} x_1(n) < a,\ b < \min_{n\in[0,T-1]} x_2(n), \\
&\qquad a < \max_{n\in[0,T-1]} x_3(n),\ \min_{n\in[0,T-1]} x_3(n) < b.
\end{aligned}$$

Proof. Define the completely continuous operator $H : X \to X$ by

$$H(x)(n) = \sum_{s=n}^{n+T-1} G(n,s) f(s, x(s-\tau(s)))$$

for $x \in X$. It is easy to prove that $HP \subset P$ and x is a positive solution of equation (1) if and only if x is a fixed point of H on cone P.

First, if $x \in \overline{P(\gamma, c)}$, then $\sigma c \le x(n) \le ||x|| = \gamma(x) \le c$. It follows from condition (8) that

$$\begin{aligned}\gamma(H(x)) &= \max_{n\in[0,T-1]} H(x)(n)\\ &= \max_{n\in[0,T-1]} \Bigg(\sum_{s=n}^{n+T-2} G(n,s) f(s, x(s-\tau(s)))\\ &\qquad + \frac{1}{1-\prod_{j=0}^{T-1}(1-a(j))} f(n-1, x(n-1-\tau(n-1))\Bigg)\\ &\le \frac{\prod_{j=0}^{T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))} \sum_{s=0}^{T-1} f(s, x(s-\tau(s)))\\ &= Tc \frac{1-\prod_{j=0}^{T-1}(1-a(j))}{T\prod_{j=0}^{T-1} a^+(j)} \frac{\prod_{j=0}^{T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))} = c.\end{aligned}$$

Therefore $H: \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$.

Secondly, it is immediate that

$$\{x \in P(\gamma, \theta, \alpha, b, \frac{b}{\sigma}, c) : \alpha(x) > b\} \ne \emptyset,$$
$$\{x \in Q(\gamma, \beta, \psi, \sigma a, a, c) : \beta(x) < a\} \ne \emptyset.$$

In the following steps, we verify the remaining conditions of Theorem 2.2.

Step 1. We prove that

$$x \in P(\gamma, \theta, \alpha, b, \frac{b}{\sigma}, c) \text{ implies } \alpha(H(x)) > b. \tag{9}$$

In fact, we have

$$b \le \min_{n\in[0,T-1]} x(n), \quad \max_{n\in[0,T-1]} x(n) \le \frac{b}{\sigma}, \quad \max_{n\in[0,T-1]} x(n) \le c.$$

Thus using (7), one gets

$$\begin{aligned}\alpha(H(x)) &= \min_{n\in[0,T-1]} H(x)(n)\\ &= \min_{n\in[0,T-1]} \Bigg(\sum_{s=n}^{n+T-2} G(n,s) f(s, x(s-\tau(s)))\\ &\qquad + \frac{1}{1-\prod_{j=0}^{T-1}(1-a(j))} f(n-1, x(n-1-\tau(n-1)))\Bigg)\\ &\ge \frac{\prod_{j=0}^{T-1} a^-(j)}{1-\prod_{j=0}^{T-1}(1-a(j))} \sum_{s=0}^{T-1} f(s, x(s-\tau(s)))\end{aligned}$$

$$= \frac{\prod_{j=0}^{T-1} a^-(j)}{1-\prod_{j=0}^{T-1}(1-a(j))} Tb \frac{1-\prod_{j=0}^{T-1}(1-a(j))}{T\prod_{j=0}^{T-1} a^-(j)} = b.$$

Step 2. We show that

$$x \in Q(\gamma, \beta, \psi, \sigma a, a, c) \text{ implies } \beta(H(x)) < a. \tag{10}$$

In fact, we have

$$\sigma a \leq \min_{n\in[0,T-1]} x(n), \quad \max_{n\in[0,T-1]} x(n) \leq a, \quad \max_{n\in[0,T-1]} x(n) \leq c.$$

Thus (5) implies that

$$\begin{aligned}
\beta(H(x)) &= \max_{n\in[0,T-1]} H(x)(n) \\
&= \max_{n\in[0,T-1]} \Bigg(\sum_{s=n}^{n+T-2} G(n,s) f(s, x(s-\tau(s))) \\
&\qquad + \frac{1}{1-\prod_{j=0}^{T-1}(1-a(j))} f(n-1, x(n-1-\tau(n-1))) \Bigg) \\
&\leq \frac{\prod_{j=0}^{T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))} \sum_{s=0}^{T-1} f(s, x(s-\tau(s))) \\
&= Ta \frac{1-\prod_{j=0}^{T-1}(1-a(j))}{T\prod_{j=0}^{T-1} a^+(j)} \frac{\prod_{j=0}^{T-1} a^+(j)}{1-\prod_{j=0}^{T-1}(1-a(j))} = a.
\end{aligned}$$

Step 3. We verify that

$$x \in Q(\gamma, \beta, a, c) \text{ with } \psi(H(x)) < \sigma a \text{ implies } \beta(H(x)) < a. \tag{11}$$

It follows from $\beta(H(x)) \leq \frac{1}{\sigma}\psi(H(x))$ that

$$\begin{aligned}
\beta(H(x)) &= \max_{n\in[0,T-1]} \Bigg(\sum_{s=n}^{n+T-2} G(n,s) f(s, x(s-\tau(s))) \\
&\qquad + \frac{1}{1-\prod_{j=0}^{T-1}(1-a(j))} f(n-1, x(n-1-\tau(n-1))) \Bigg) \\
&\leq \frac{1}{\sigma}\psi(H(x)) < a.
\end{aligned}$$

Step 4. We prove that

$$x \in P(\gamma, \alpha, b, c) \text{ with } \theta(H(x)) > \frac{1}{\sigma} b \text{ implies } \alpha(H(x)) > b. \tag{12}$$

It follows from $\alpha(H(x)) \geq \sigma\theta(H(x))$ that

$$\begin{aligned}\alpha(H(x)) = \min_{n\in[0,T-1]} \Bigg(& \sum_{s=n}^{n+T-2} G(n,s)f(s,x(s-\tau(s))) \\ & + \frac{1}{1-\prod_{j=0}^{T-1}(1-a(j))} f(n-1,x(n-1-\tau(n-1)))\Bigg) \\ \geq \sigma\theta(H(x)) > b. &\end{aligned}$$

Therefore, the hypotheses of the Theorem 2.2 are satisfied and there exist three positive solutions x_1, x_2, x_3 for equation (1) satisfying

$$\max_{n\in[0,T-1]} x_i(n) \leq c, \ i = 1,2,3;$$

and

$$\max_{n\in[0,T-1]} x_1(n) < a, \quad b < \min_{n\in[0,T-1]} x_2(n),$$
$$a < \max_{n\in[0,T-1]} x_3(n), \quad \max_{n\in[0,T-1]} x_3(n) < b.$$

□

Remark 3.1. Similar to Theorems 3.1 and 3.2, we can establish existence results for at least three positive periodic solutions of equation (2).

Remark 3.2. From the results obtained in this paper, we see that $a(n)$ in equations (1) oscillates about 0, i.e., $b(n)$ in equations (3) oscillates about 1 when the results obtained are applied to equation (3). This fills a gap in the known papers [1-3].

Remark 3.3. Let $f(x) = g(x)$. Transform equation (3) into

$$\Delta x(n) = -(1-b(n))x(n) + \lambda h(n)g(x(n-\tau(n))). \tag{13}$$

With regard to equation (1), one sees that $a(n) = 1 - b(n)$ and $f(n, x(n - \tau(n))) = \lambda h(n)g(x(n-\tau(n)))$. When $b(n) > 1$ for $n \in Z$, we have that $a(n) < 0$. Then Theorem 3.1 and Theorem 3.2 can be applied. Hence we present an answer to Open problem 3 related to equation (3).

4. An example

Now, we present an example to illustrate the main results obtained in Section 3.

Example 4.1. Consider the following equation

$$\Delta x(n) = -\left(\frac{1}{8} + \frac{1}{4}\sin\frac{n\pi}{2}\right)x(n) + \frac{1}{4}\left(4+\cos\frac{n\pi}{2}\right)g\left(x\left(n-\cos\frac{n\pi}{2}\right)\right). \tag{14}$$

Regarding equation (1), one sees that $a(n) = \frac{1}{8} + \frac{1}{4}\sin\frac{n\pi}{2}$, $\tau(n) = \cos\frac{n\pi}{2}$ and $f(n,x) = h(n)g(x)$ with $h(n) = \frac{1}{4}\left(4 + \cos\frac{n\pi}{2}\right)$ and g being continuous, h is a 4-periodic positive sequence and a and τ 8-periodic sequences. If

$$g(x) = \begin{cases} 5 \times \frac{4099}{2048\times 9}, \quad x \in [0,10], \\ 200 \times \frac{4099\times 128}{4096\times 245} + (x-100)\frac{200\times\frac{4099\times128}{4096\times245} - 5\times\frac{4099}{2048\times9}}{100-10}, \quad x \in [10,100], \\ 200 \times \frac{4099\times 128}{4096\times 245}, \quad x \in [100, \frac{576}{245}\times 100], \\ 3000\times\frac{4099}{2048\times 9} + \left(x - \frac{245}{576}\times 6000\right)\frac{3000\times\frac{4099}{2048\times9} - 200\times\frac{4099\times128}{4096\times245}}{\frac{245}{576}\times 6000 - \frac{576}{245}\times 100}, \\ \qquad\qquad x \in [\frac{576}{245}\times 100, \frac{245}{576}\times 6000], \\ 3000\times\frac{4099}{2048\times9}, \quad x \geq \frac{245}{576}\times 6000, \end{cases}$$

it is easy to see that

$$\sigma = \frac{\prod_{s=0}^{T-1} a^-(s)}{\prod_{s=0}^{T-1} a^+(s)} = \frac{\prod_{s=0}^{T-1}\min\{1, 1-a(n)\}}{\prod_{s=0}^{T-1}\max\{1,1-a(n)\}} = \frac{245}{576};$$

(A_1) $f : Z \times R \to R^+$ is continuous in x and T-periodic in n;

(A_2) $a : Z \to R$ and $\tau : Z \to R$ are T-periodic sequences and $a(n) < 1$ for all $n \in Z$ with $\prod_{j=n}^{n+T-1}(1-a(s)) \equiv -\frac{3}{4096} < 1$ for all $n \in Z$;

Choose positive constants $a = 10, b = 100, c = 6000$. It is easy to check that

$$0 < a < b < \frac{b}{\sigma} < \sigma c < c$$

and

$$f(n,x) \leq a\frac{1-\prod_{j=0}^{T-1}(1-a(j))}{T\prod_{j=0}^{T-1}a^+(j)} = 10 \times \frac{4099}{2048\times 9},$$

$$n \in [0,3], \quad 10 \times \frac{245}{576} \leq x \leq 10,$$

$$f(n,x) \geq b\frac{1-\prod_{j=0}^{T-1}(1-a(j))}{T\prod_{j=0}^{T-1}a^-(j)} = 100 \times \frac{4099\times 128}{4096\times 245},$$

$$n \in [0,3], \quad 100 \leq x \leq \frac{576}{245}\times 100,$$

$$f(n,x) \leq c\frac{1-\prod_{j=0}^{T-1}(1-a(j))}{T\prod_{j=0}^{T-1}a^+(j)} = 6000 \times \frac{4099}{2048\times 9},$$

$$n \in [0,3], \quad \frac{245}{576}\times 6000 \leq x \leq 6000.$$

Therefore equation (1) has at least three positive 4-periodic solutions x_1, x_2 and x_3 satisfying

$$\max_{n\in[0,3]} x_1(n) < 10,\ 100 < \min_{n\in[0,3]} x_2(n),\ 10 < \max_{n\in[0,3]} x_3(n),\ \min_{n\in[0,3]} x_3(n) < 100.$$

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