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DUALITY BETWEEN GLOBAL SECTIONS AND SUSPENSIONS

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Abstract. We establish existence of global sections for non-weakly-mixing compact minimal flows whose acting group is SK. We also establish a duality between the notions of global sections and suspensions of compact flows.

1. Introduction

A *flow* is a triple $\mathcal{X} = (T, X, \pi)$, where T is a Hausdorff topological group, X a Hausdorff topological space and $\pi : T \times X \to X$ a continuous group action of T on X. It is *minimal* if every orbit is dense.

All the notions that we use, but not define in this paper, can be found in the monograph [5]. We say that a flow $\mathcal{X}$ is *compact* if the phase space X is compact and *abelian* if the acting group T is abelian. All the topological groups we deal with in this paper are abelian and all the flows are compact abelian. (Some of the constructions of this paper can be extended to non-abelian acting groups, but that will be the subject of another paper.) $\mathbb{T}$ will denote the topological group of complex numbers of module 1. A continuous map $f : X \to \mathbb{T}$ is called an *eigenfunction* of $\mathcal{X} = (T, X, \pi)$ if there is a continuous character $\chi \in \widehat{T}$ such that $f(t.x) = \chi(t)f(x)$ for all $t \in T$, $x \in X$. In this case χ is the *eigenvalue* which corresponds to f. The group of eigenvalues of $\mathcal{X}$ is denoted by $EV(\mathcal{X})$. A subset A of a topological group T is *syndetic* if there is a compact subset K of T such that $T = K + A$.

A compact minimal abelian flow $\mathcal{X}$ is *weakly mixing* if for any four nonempty open sets U_1, U_2, V_1, V_2 there exists a $t \in T$ such that $tU_1 \cap U_2 \neq \emptyset$ and $tV_1 \cap V_2 \neq \emptyset$. Equivalently, $EV(\mathcal{X}) = \{1\}$ ([5]).

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A topological group T is an *SK group* (see [3]) if the kernel of every continuous character of T is a syndetic subgroup of T.

For example, $\mathbb{R}$ is an SK group and $\mathbb{Z}$ is not an SK group. Every minimally almost periodic group (in particular every extremely amenable group) is an SK group. A finite product of SK groups is an SK group. In particular, every connected locally compact abelian group is SK.

Our Proposition 2.9 can probably be proved for other classes of groups (see [4]).

A cocycle of a flow $\mathcal{X} = (G, X)$ to a topological group T is a map $\sigma : G \times X \to T$ such that $\sigma(g_1 + g_2, x) = \sigma(g_1, g_2 x) + \sigma(g_2, x)$ for all $g_1, g_2 \in G,\ x \in X$. All cocycles are assumed to be continuous.

2. Duality between global sections and suspensions

Definition 2.1. Let $\mathcal{X} = (G, X)$, $\mathcal{Y} = (T, Y)$ be two flows, $\varphi : X \to Y$ a continuous map and $\sigma : G \times X \to T$ a cocycle of $\mathcal{X}$ to T. We say that the pair (φ, σ) is an *orbit-morphism* of $\mathcal{X}$ to $\mathcal{Y}$ if

$$\varphi(gx) = \sigma(g, x)\varphi(x)$$

for all $g \in G$, $x \in X$. We also say that φ is an *orbit-morphism* or an *orbit-morphism with respect to σ*. An orbit-morphism is *injective* (resp. *surjective*) if φ is injective (resp. surjective).

Example 2.2. Let $\mathcal{X} = (T, X)$, $\mathcal{Y} = (T, Y)$ be two flows with the same acting group. Then any morphism of flows $\varphi : \mathcal{X} \to \mathcal{Y}$ is an orbit-morphism with respect to the constant cocycle $\sigma : T \times X \to T,\ \sigma(t, x) = t,\ t \in T, x \in X$.

Example 2.3. Let $\mathcal{X} = (G, X)$, $\mathcal{Y} = (T, Y)$ be two flows, $h : G \to T$ a continuous group homomorphism and $\varphi : X \to Y$ a continuous map. Suppose that $\varphi(gx) = h(g)\varphi(x)$ for all $g \in G$, $x \in X$. Then φ is an orbit-morphism with respect to the cocycle $\sigma : G \times X \to T,\ \sigma(g, x) = h(g)$.

Example 2.4. Let $\mathcal{Y} = (T, Y)$ be a flow, S a subgroup of T, $X \subset Y$ stable under the action of S. Then the canonical injection $i : X \to Y$ is an orbit-morphism from $\mathcal{X} = (S, X)$ to $\mathcal{Y}$ with respect to the cocycle $\sigma : S \times X \to T,\ \sigma(s, x) = s$. (Under some additional conditions this orbit-morphism is an example of a global section, see Definition 2.6.)

Example 2.5. Let $\mathcal{X} = (G, X)$ be a flow, T a topological group, $\sigma : G \times X \to T$ a cocycle of $\mathcal{X}$ to T. Define a (G, T)-biflow on $T \times X$ in the following way:

$$t'.(t, x) = (t' + t, x)$$
$$g.(t, x) = (t - \sigma(g, x), gx),$$

for $t', t \in T$, $g \in G$, $x \in X$. Let $\widetilde{X} = (T \times X)/G$ be the quotient of this biflow with respect to G and let $\widetilde{\mathcal{X}} = (T, \widetilde{X})$. Denote by $q : T \times X \to \widetilde{X}$ the canonical map and let $[t, x] = q(t, x)$. Then the action of T on $\widetilde{X}$, the quotient of the action of T on $T \times X$, is given by

$$t'.[t, x] = [t' + t, x],$$

for $t', t \in T$, $x \in X$. We have

$$[t, x] = [t', x'] \Leftrightarrow x' = gx \;\&\; t' = t - \sigma(g, x) \text{ for some } g \in G.$$

Let $\varphi : \mathcal{X} \to \widetilde{\mathcal{X}}$ be defined by $\varphi(x) = q(0, x)$. Then $(\varphi, \sigma) : \mathcal{X} \to \widetilde{\mathcal{X}}$ is an orbit-morphism of $\mathcal{X}$ to $\widetilde{\mathcal{X}}$. (This orbit-morphism is (an example of) a suspension of $\mathcal{X}$ by the cocycle σ, see Definition 2.10.)

Definition 2.6. Let $\mathcal{X} = (G, X)$, $\mathcal{Y} = (T, Y)$ be two flows. We say that $\mathcal{X}$ is a *global section* of $\mathcal{Y}$ (*with respect to a cocycle* σ) if there is an orbit-morphism $(\varphi, \sigma) : \mathcal{X} \to \mathcal{Y}$ such that the following conditions hold:

(i) $T.\varphi(X) = Y$;
(ii) If $t.\varphi(x) \in \varphi(X)$ for some $x \in X$, then $t = \sigma(g, x)$ for some $g \in G$;
(iii) For every neighborhood N of 0 in T and every $x_0 \in X$ there is a neighborhood V of $\varphi(x_0)$ in Y such that, if $t\varphi(x) \in V$ for some $t \in T$ and $x \in X$, then $t = \sigma(g, x) + \varepsilon$ for some $g \in G$, $\varepsilon \in N$.

Definition 2.7. Let $\mathcal{Y} = (T, Y)$ be a flow, S a subgroup of T, X a closed subset of Y stable under the action of S. We say that $\mathcal{X} = (S, X)$ is a *canonical global section* of $\mathcal{Y}$ (*with respect to* S) if $(i, \sigma) : \mathcal{X} \to \mathcal{Y}$ is a global section of $\mathcal{Y}$, where $i : X \to Y$ is the canonical injection and $\sigma : S \times X \to T$ the cocycle $\sigma(s, x) = s$.

Proposition 2.8 ([1])**.** *Let $\mathcal{Y} = (T, Y)$ be a flow, X a closed subset of Y, S a closed syndetic subgroup of T. Suppose that:*

(i) *$T.X = Y$;*
(ii) *$S.X \subset X$;*
(iii) *If $tX \cap X \neq \emptyset$ for some $t \in T$, then $t \in S$.*

Then $\mathcal{X} = (S, X)$ is a canonical global section of $\mathcal{Y}$.

Proposition 2.9. *Let $\mathcal{Y} = (T, Y)$ be a compact minimal flow whose acting group is SK. Suppose that $\mathcal{Y}$ is not weakly mixing. Then for every $x \in Y$ there exists a proper closed subset X of Y, containing x, such that the inclusion $i : X \to Y$ is a canonical global section of $\mathcal{Y}$ with respect to a proper closed syndetic subgroup S of T.*

Proof. Since T is an SK group and $\mathcal{Y}$ is not weakly mixing, there is an eigenvalue γ of $\mathcal{Y}$ whose kernel $S = \ker\gamma$ is a proper closed syndetic subgroup of T ([3, Proposition 7.2]). Let $x \in Y$ and let $X = \overline{Sx}$. Then X is a proper closed subset of Y. (Otherwise let f be an eigenfunction of $\mathcal{Y}$ corresponding to γ and mapping x to 1. Then from $f(sx) = \gamma(s)f(x)$ we would get $f \equiv 1$, hence $\gamma \equiv 1$, a contradiction.) Let K be a compact subset of T such that $S + K = T$. Then $TX = T\overline{Sx} \supset K\overline{Sx} = \overline{(S+K)x} = \overline{Tx} = Y$, hence $TX = Y$. Also $S.X \subset X$ since $s'.\overline{Sx} = \overline{S.s'x} = \overline{Sx}$ since the orbit-closures under S form a partition of Y ([2, Theorem 2.32]). Note that $S = \{t \in T |\, tx \in \overline{Sx}\}$. Indeed, if f is an eigenfunction corresponding to γ, which maps x to 1, then from $f(tx) = \gamma(t)f(x)$ we get for every $tx \in \overline{Sx}$ that $\gamma(t) = 1$, hence $t \in S$. Suppose now that $tX \cap X \neq \emptyset$ for some $t \in T$, i.e., that $t\overline{Sx} \cap \overline{Sx} \neq \emptyset$. Let $y, z \in \overline{Sx}$ be such that $ty = z$. Then $\overline{Sty} = \overline{Sz}$, i.e., $t\overline{Sy} = \overline{Sz}$. Since the orbit-closures under S form a partition of Y, $\overline{Sx} = \overline{Sy} = \overline{Sz}$, hence $t\overline{Sx} = \overline{Sx}$, hence $tx \in \overline{Sx}$. Hence $t \in S$. Thus all the conditions of the previous proposition are satisfied and so $i : X \to Y$ is a canonical global section of $\mathcal{Y}$ with respect to the subgroup S of T. □

Definition 2.10. Let $\mathcal{X} = (G, X)$, $\mathcal{Y} = (T, Y)$ be two flows. Let $\varphi : X \to Y$ be a continuous map, such that $\varphi(X)$ is a closed subset of Y and $T.\varphi(X) = Y$. Let $\sigma : G \times X \to T$ be a cocycle. We say that $\mathcal{Y}$ is a *suspension* of $\mathcal{X}$ (*with respect to* (φ, σ)) if the map $p : T \times X \to Y$, defined by $p(t, x) = t.\varphi(x)$, $t \in T$, $x \in X$, is compatible with the action of G on $T \times X$ and induces an isomorphism $f : \widetilde{X} \to Y$. (The flow $\widetilde{\mathcal{X}} = (T, \widetilde{X})$ is constructed in Example 2.5.)

Theorem 2.11. *Let $\mathcal{X} = (G, X)$, $\mathcal{Y} = (T, Y)$ be two flows, $\varphi : X \to Y$ an injective continuous map. Suppose that $\varphi(X)$ is a closed subset of Y such that $T.\varphi(X) = Y$. Let $\sigma : G \times X \to T$ be a cocycle. Then the following are equivalent:*

(i) *$\mathcal{X}$ is a global section of $\mathcal{Y}$ (with respect to (φ, σ));*
(ii) *$\mathcal{Y}$ is a suspension of $\mathcal{X}$ (with respect to (φ, σ)).*

Proof. (i)⇒(ii): Suppose that $\mathcal{X}$ is a global section of $\mathcal{Y}$ (with respect to (φ, σ). Consider the map $p : T \times X \to Y$, $p(t, x) = t\varphi(x)$. It is surjective since $T.\varphi(X) = Y$. We show that it is compatible with the G-action on $T \times X$. Suppose $g.(t_1, x_1) = (t_2, x_2)$, i.e., $x_2 = g.x_1 \,\&\, t_2 = t_1 - \sigma(g, x_1)$. We want to show that then $t_1\varphi(x_1) = t_2\varphi(x_2)$. We have: $t_2\varphi(x_2) = (t_1 - \sigma(g, x_1)).\varphi(gx_1) = (t_1 - \sigma(g, x_1)).\sigma(g, x_1)\varphi(x_1) = t_1\varphi(x_1)$. (Here we used that (φ, σ) is an orbit-morphism.) Hence p induces a continuous bijective map $f : \widetilde{X} \to Y$, where the flow $\widetilde{\mathcal{X}} = (T, \widetilde{X})$ is constructed as in Example 2.5. We need to show that f^{-1} is continuous.

Let $y_\alpha \to y$ in Y. Since $T.\varphi(X) = Y$, we have $y_\alpha = t_\alpha\varphi(x_\alpha)$, where (t_α) is a net in T and (x_α) a net in X. Also $y = t\varphi(x)$ for some $t \in T$, $x \in X$. Now we have $t_\alpha\varphi(x_\alpha) \to t\varphi(x)$ in Y and need to show that $[t_\alpha, x_\alpha] \to [t, x]$ in $\widetilde{X}$. We have $(-t + t_\alpha)\varphi(x_\alpha) \to \varphi(x)$. Now we use the third condition from the definition of global sections. We choose smaller and smaller neighborhoods N_α of 0 in G and corresponding neighborhoods V_α of $\varphi(x)$ so that $-t+t_\alpha = \sigma(g_\alpha, x_\alpha) + \varepsilon_\alpha$, with $\varepsilon \to 0$, i.e., $t - t_\alpha + \sigma(g_\alpha, x_\alpha) \to 0$. Using the second condition from the definition of global sections we conclude that $g_\alpha x_\alpha \to x$. Hence $[t_\alpha, x_\alpha] = [g_\alpha.(t_\alpha, x_\alpha)] = [t_\alpha - \sigma(g_\alpha, x_\alpha), g_\alpha x_\alpha] \to [t, x]$.

(ii)⇒(i): Suppose that $\mathcal{Y}$ is a suspension of $\mathcal{X}$ (with respect to (φ, σ). First note that (φ, σ) is an orbit-morphism (Example 2.5 and Definition 2.10). The first condition from the definition of global sections is satisfied by the assumption. Next we show the second condition from the definition of global sections. Suppose that $t.\varphi(x) \in \varphi(X)$ for some $t \in T$, $x \in X$. Then $p(t, x) = p(0, x')$, hence $q(t, x) = q(0, x')$, i.e., $[t, x] = [o, x']$. Hence $t = \sigma(g, x)$ by the definition of $\widetilde{Y}$.

Now we show the third condition from the definition of global sections. We prove by contradiction that this condition is satisfied. Let N be a neighborhood of 0 in G and $y_0 \in \varphi(X)$ such that for every neighborhood V_α of y_0 in Y there are $\varphi(x_\alpha) \in \varphi(X)$ and $t_\alpha \in T$ such that $t_\alpha \varphi(x_\alpha) \in V_\alpha$ and $t_\alpha \notin \sigma(g, x_\alpha) + N$ for any $g \in G$. Then $t_\alpha \varphi(x_\alpha) \to y_0$, i.e., $p(t_\alpha, x_\alpha) \to p(0, x_0)$, where $x_0 \in X$ is such that $\varphi(x_0) = y_0$. Hence $q(t_\alpha, x_\alpha) \to q(0, x_0)$ in $\widetilde{X}$, i.e., $[t_\alpha, x_\alpha] \to [0, x_0]$. Consider a neighborhood $N \times V$ of $(0, x_0)$ in $T \times X$. Its image under q is a neighborhood of $[0, x_0]$ in $\widetilde{X}$ since q is an open map. Hence all $[t_\alpha, x_\alpha]$ are in $q(N \times V)$ for $\alpha \geq \alpha_0$. Hence $(t_\alpha, x_\alpha) \in q^{-1}(q(N \times V))$ for $\alpha \geq \alpha_0$. Hence $g_\alpha.(t_\alpha, x_\alpha) \in N \times V$ for some $g_\alpha \in G\,(\alpha \geq \alpha_0)$. This implies $t_\alpha - \sigma(g_\alpha, x_\alpha) \in N$ for $\alpha \geq \alpha_0$, a contradiction. □

References

[1] R. Ellis, *Global sections of transformation groups*, Illinois J. Math., 8 (1964), 380–394.

[2] W. Gottschalk and G. Hedlund, *Topological Dynamics*, Colloquium Publications, 36, American Mathematical Society, R.I., 1955.

[3] A. Miller, *A criterion for minimality of restrictions of compact minimal abelian flows*, Top. Appl., 119 (2002), 95–111.

[4] V. Pestov, *Topological groups: where to from here?*, Top. Proc., 24 (1999), 421–502.

[5] J. de Vries, *Elements of topological dynamics*, Mathematics and its Applications, 257, Kluwer Academic Publishers Group, Dordrecht, 1993.

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