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ON THE DIFFERENCE EQUATION $x_{n+1} = \frac{ax_n^2+bx_{n-1}x_{n-k}}{cx_n^2+dx_{n-1}x_{n-k}}$

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Abstract. In this paper we investigate the global convergence result, boundedness, and periodicity of solutions of the recursive sequence

$$x_{n+1} = \frac{ax_n^2 + bx_{n-1}x_{n-k}}{cx_n^2 + dx_{n-1}x_{n-k}}, \quad n = 0, 1, \ldots$$

where the parameters a, b, c and d are positive real numbers and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}$ and x_0 are arbitrary positive numbers.

1. Introduction

Our goal in this paper is to investigate the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1} = \frac{ax_n^2 + bx_{n-1}x_{n-k}}{cx_n^2 + dx_{n-1}x_{n-k}}, \tag{1}$$

where a, b, c and $d \in (0, \infty)$ with the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}$ and $x_0 \in (0, \infty)$.

Recently there has been a lot of interest in studying the global attractivity, boundedness character and the periodic nature of nonlinear difference equations. For some results in this area, see for example [5-8]. See also [1-4].

Here, we recall some notations and results which will be useful in our investigation.

Definition 1. *The difference equation*

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{2}$$

is said to be persistence if there exist numbers m and M with $0 < m \leq M < \infty$ such that for any initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in (0, \infty)$

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there exists a positive integer N which depends on the initial conditions such that

$$m \leq x_n \leq M \quad \text{for all} \quad n \geq N.$$

Definition 2. (Stability)

(i) *The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ for some interval I of real numbers with*

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) *The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with*

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iii) *The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all x_{-k}, $x_{-k+1}, \ldots, x_{-1}, x_0 \in I$, we have*

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iv) *The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).*

(v) *The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.*

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, \ldots, \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A. *Assume that $p, q \in \mathbb{R}$ and $k \in \{0, 1, 2, \ldots\}$. Then*

$$|p| + |q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + p x_n + q x_{n-k} = 0, \quad n = 0, 1, \ldots .$$

Remark 1. Theorem A can be easily extended to a general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, \quad n = 0, 1, \ldots \tag{4}$$

where $p_1, p_2, \ldots, p_k \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$. Then Eq.(4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

2. Local stability of the equilibrium point

In this section we study the local stability character of the solutions of Eq.(1). Eq.(1) has a unique positive equilibrium point and is given by

$$\overline{x} = \frac{a+b}{c+d}. \tag{5}$$

Let $f : (0, \infty)^3 \longrightarrow (0, \infty)$ be a continuous function defined by

$$f(u, v, w) = \frac{au^2 + bvw}{cu^2 + dvw}. \tag{6}$$

Therefore it follows that

$$\begin{aligned}
\frac{\partial f(u,v,w)}{\partial u} &= \frac{2(ad-bc)uvw}{(cu^2+dvw)^2} \\
\frac{\partial f(u,v,w)}{\partial v} &= \frac{-(ad-bc)u^2 w}{(cu^2+dvw)^2} \\
\frac{\partial f(u,v,w)}{\partial w} &= \frac{-(ad-bc)u^2 v}{(cu^2+dvw)^2}.
\end{aligned}$$

Then we see that

$$\begin{aligned}
\frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial u} &= \frac{2(ad-bc)}{(a+b)(c+d)} = -c_2 \\
\frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial v} &= \frac{-(ad-bc)}{(a+b)(c+d)} = -c_1 \\
\frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial w} &= \frac{-(ad-bc)}{(a+b)(c+d)} = -c_0.
\end{aligned}$$

Then the linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1} + c_2 y_n + c_1 y_{n-1} + c_0 y_{n-k} = 0 \tag{7}$$

and its characteristic equation is

$$\lambda^{k+1} + c_2 \lambda^k + c_1 \lambda^{k-1} + c_0 = 0. \tag{8}$$

Theorem 2.1. *Assume that*

$$4\,|ad - bc| < (a+b)(c+d).$$

Then the positive equilibrium point of Eq.(1) *is locally asymptotically stable.*

Proof. It follows from Remark 1 that Eq.(7) is asymptotically stable if all roots of Eq.(8) lie in the open disc $|\lambda| < 1$, which results from inequality

$$|c_2| + |c_1| + |c_0| < 1.$$

$$\left|\frac{2(ad-bc)}{(a+b)(c+d)}\right| + \left|\frac{-(ad-bc)}{(a+b)(c+d)}\right| + \left|\frac{-(ad-bc)}{(a+b)(c+d)}\right| < 1,$$

or

$$4\,|bc-ad| < (c+d)(a+b).$$

The proof is complete. □

3. Boundedness of solutions

Here we study the permanence of Eq.(1).

Theorem 3.1. *Every solution of Eq.*(1) *is bounded and persists.*

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1). It follows from Eq.(1) that

$$\begin{aligned} x_{n+1} &= \frac{ax_n^2 + bx_{n-1}x_{n-k}}{cx_n^2 + dx_{n-1}x_{n-k}} = \frac{ax_n^2}{cx_n^2 + dx_{n-1}x_{n-k}} + \frac{bx_{n-1}x_{n-k}}{cx_n^2 + dx_{n-1}x_{n-k}} \\ &\le \frac{ax_n^2}{cx_n^2} + \frac{bx_{n-1}x_{n-k}}{dx_{n-1}x_{n-k}}. \end{aligned}$$

Then

$$x_n \le \frac{a}{c} + \frac{b}{d} = M \quad \text{for all} \quad n \ge 1. \tag{9}$$

Now we wish to show that there exists $m > 0$ such that

$$x_n \ge m \quad \text{for all} \quad n \ge 1.$$

The transformation

$$x_n = \frac{1}{y_n},$$

will reduce Eq.(1) to the equivalent form

$$\begin{aligned} y_{n+1} &= \frac{dy_n^2 + cy_{n-1}y_{n-k}}{by_n^2 + ay_{n-1}y_{n-k}} \\ &= \frac{dy_n^2}{ay_{n-1}y_{n-k} + by_n^2} + \frac{cy_{n-1}y_{n-k}}{ay_{n-1}y_{n-k} + by_n^2}. \end{aligned}$$

It follows that

$$y_{n+1} \le \frac{d}{b} + \frac{c}{a} = \frac{bc+ad}{ab} = H \quad \text{for all} \quad n \ge 1.$$

Thus we obtain

$$x_n = \frac{1}{y_n} \ge \frac{1}{H} = \frac{ab}{bc+ad} = m \quad \text{for all} \quad n \ge 1. \tag{10}$$

From (9) and (10) we see that

$$m \leq x_n \leq M \quad \text{for all} \quad n \geq 1.$$

Therefore every solution of Eq.(1) is bounded and persists. □

4. Periodicity of solutions

In this section we study the existence of prime period two solutions of Eq.(1).

Theorem 4.1. *Let k be even. Eq.*(1) *has a prime period two solution if and only if*

$$(i) \quad 4da < (c-d)(b-a).$$

Proof. First suppose that there exists a prime period two solution

$$\dots, p, q, p, q, \dots$$

of Eq.(1). We will prove that Condition (i) holds.

When $k - even$, we see from Eq.(1) that

$$p = \frac{aq^2 + bpq}{cq^2 + dpq} = \frac{aq + bp}{cq + dp},$$

and

$$q = \frac{ap^2 + bpq}{cp^2 + dpq} = \frac{ap + bq}{cp + dq}.$$

Then

$$cpq + dp^2 = aq + bp, \tag{11}$$

and

$$cpq + dq^2 = ap + bq. \tag{12}$$

Subtracting (11) from (12) gives

$$d(p^2 - q^2) = (b-a)(p-q).$$

Since $p \neq q$, it follows that

$$p + q = \frac{(b-a)}{d}. \tag{13}$$

Also, since p and q are positive, $(b-a)$ should be positive.

Again, adding (11) and (12) yields

$$2cpq + d(p^2 + q^2) = (p+q)(a+b). \tag{14}$$

It follows by (13), (14) and the relation

$$p^2 + q^2 = (p+q)^2 - 2pq \quad \text{for all} \ \ p, q \in R,$$

that

$$2(c-d)pq = \frac{2a(b-a)}{d}.$$

Again, since p and q are positive and $b > a$, we see that $c > d$. Thus

$$pq = \frac{a(b-a)}{d(c-d)}. \tag{15}$$

Now it is clear from Eq.(13) and Eq.(15) that p and q are the two positive distinct roots of the quadratic equation

$$t^2 - \frac{(b-a)}{d}t + \frac{a(b-a)}{d(c-d)} = 0, \tag{16}$$

and so

$$\left[\frac{b-a}{d}\right]^2 - \frac{4a(b-a)}{d(c-d)} > 0.$$

Since $c-d$ and $b-a$ have the same sign,

$$\frac{b-a}{d} > \frac{4a}{(c-d)},$$

which is equivalent to

$$4da < (c-d)(b-a).$$

Therefore Inequality (i) holds.

Second suppose that Inequality (i) is true. We will show that Eq.(1) has a prime period two solution.

Assume that

$$p = \frac{\frac{b-a}{d} - \sqrt{\left[\frac{b-a}{d}\right]^2 - \frac{4a(b-a)}{d(c-d)}}}{2},$$

and

$$q = \frac{\frac{b-a}{d} + \sqrt{\left[\frac{b-a}{d}\right]^2 - \frac{4a(b-a)}{d(c-d)}}}{2}.$$

We see from Inequality (i) that

$$(c-d)(b-a) > 4da,$$

or

$$[b-a]^2 > \frac{4da(b-a)}{(c-d)},$$

which equivalents to

$$\left[\frac{b-a}{d}\right]^2 > \frac{4a(b-a)}{d(c-d)}.$$

Therefore p and q are distinct positive real numbers.

Let k be even, $0 < p < q$,

$$x_{-k} = p, \;\; x_{-k+1} = q, \ldots, x_0 = p.$$

We wish to show that

$$x_1 = x_{-1} = q \quad \text{and} \quad x_2 = x_0 = p.$$

It follows from Eq.(1) that

$$\frac{ap^2+bpq}{cp^2+dpq}=q=x_1 \quad \text{and} \quad \frac{ax_1^2+bpq}{cx_1^2+dpq}=p,$$

$$\frac{ap+bq}{cp+dq}=q \quad \text{and} \quad \frac{aq+bp}{cq+dp}=p.$$

Then

$$cpq+dp^2=aq+bp \quad \text{and} \quad cpq+dq^2=ap+bq.$$

Adding and subtracting the above relations

$$p+q=\frac{(b-a)}{d} \quad \text{and} \quad pq=\frac{a(b-a)}{d(c-d)},$$

so p and q are the solutions of equation

$$t^2-\frac{(b-a)}{d}t+\frac{a(b-a)}{d(c-d)}=0.$$

So if k even and $4da<(c-d)(b-a)$ we have a prime period two solution and the proof is complete. □

5. Global stability of Eq.(1)

In this section we investigate the global asymptotic stability of Eq.(1).

Lemma 1. *For any values of the quotient $\frac{a}{c}$ and $\frac{b}{d}$, the function $f(u,v,w)$ defined by Eq.(6) is monotone in each of its three arguments.*

Proof. The proof follows from the calculations after formula (6). □

Theorem 5.1. *The equilibrium point $\overline{x}$ is a global attractor of Eq.(1) if one of the following statements holds*

$$(1) \quad ad\geq bc \ \ \textit{and} \ \ 4cb-2da>(a-b)d\left[\frac{ad}{bc}\right]^2. \tag{17}$$

$$(2) \quad ad\leq bc \ \ \textit{and} \ \ 4da-2cb>(b-a)c\left[\frac{bc}{ad}\right]^2. \tag{18}$$

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1) and again let f be a function defined by Eq.(6).

We will prove the theorem when Case (1) is true and the proof of Case (2) is similar and is left to the reader.

Assume that (17) is true, then it results from the calculations after formula (6) that the function $f(u,v,w)$ is non-decreasing in u and non-increasing in v,w. Thus from Eq.(1), we see that

$$x_{n+1}=\frac{ax_n^2+bx_{n-1}x_{n-k}}{cx_n^2+dx_{n-1}x_{n-k}}\leq\frac{ax_n^2+b(0)}{cx_n^2+d(0)}=\frac{a}{c}.$$

Then

$$x_n \leq \frac{a}{c} = H \quad \text{for all} \quad n \geq 1. \tag{19}$$

$$\begin{aligned} x_{n+1} &= \frac{ax_n^2 + bx_{n-1}x_{n-k}}{cx_n^2 + dx_{n-1}x_{n-k}} \geq \frac{a(0) + bx_{n-1}x_{n-k}}{c(0) + dx_{n-1}x_{n-k}} \\ &\geq \frac{bx_{n-k}}{dx_{n-k}} \geq \frac{b}{d} = h \quad \text{for all} \quad n \geq 1. \end{aligned} \tag{20}$$

Then from Eq.(19) and Eq.(20), we see that

$$0 < h = \frac{b}{d} \leq x_n \leq \frac{a}{c} = H \quad \text{for all} \ \ n \geq 1.$$

Let $\{x_n\}_{n=0}^{\infty}$ solution of Eq.(1) with

$$I := \lim_{n\to\infty} \inf x_n \text{ and } S := \lim_{n\to\infty} \sup x_n.$$

It suffices to show that $I = S$.

Now it follows from Eq.(1) that

$$I \geq f(I, S, S),$$

or,

$$I \geq \frac{aI^2 + bS^2}{cI^2 + dS^2},$$

and so

$$aI^2 + bS^2 - cI^3 \leq dS^2 I. \tag{21}$$

Similarly, we see from Eq.(1) that

$$S \leq f(S, I, I),$$

or,

$$S \leq \frac{aS^2 + bI^2}{cS^2 + dI^2},$$

and so

$$aS^2 + bI^2 - cS^3 \geq dSI^2. \tag{22}$$

Therefore it follows from Eq.(21) and Eq.(22) that

$$aI^3 + bS^2 I - cI^4 \leq dS^2 I^2 \leq aS^3 + bSI^2 - cS^4$$
$$c(I^4 - S^4) + bSI(I - S) - a(I^3 - S^3) \geq 0,$$

if and only if

$$(I - S)\left[c(I^2 + S^2)(I + S) + bSI - a(I^2 + S^2 + IS)\right] \geq 0,$$

or

$$(I - S)\left[(I^2 + S^2)\{c(I + S) - a\} + SI\{b - a\}\right] \geq 0,$$

and so

$$I \geq S \quad \text{if} \ \ (I^2 + S^2)\{c(I + S) - a\} + SI\{b - a\} > 0.$$

Now, we know by (17) that

$$4cb - 2da > (a-b)d\left[\frac{ad}{bc}\right]^2$$
$$2\left[\frac{2cb}{d} - a\right] > (a-b)\left[\frac{ad}{bc}\right]^2$$

$$\left[I^2 + S^2\right](c(I+S) - a) \geq \left[\left[\frac{b}{d}\right]^2 + \left[\frac{b}{d}\right]^2\right]\left[c\left[\frac{b}{d} + \frac{b}{d}\right] - a\right]$$
$$> (a-b)\left[\frac{a}{c}\right]\left[\frac{a}{c}\right] > (a-b)IS$$
$$(I^2 + S^2)\{c(I+S) - a\} + SI\{b-a\} > 0,$$

and so it follows that

$$I \geq S.$$

Therefore

$$I = S.$$

This completes the proof. □

Remark 2. It follows from Eq.(1), when $\frac{a}{c} = \frac{b}{d}$, that $x_{n+1} = \lambda$ for all $n \geq -k$ and for some constant λ.

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