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ON THE COMPOSITION OF THE DISTRIBUTIONS $x_+^\lambda \ln^m x_+$ AND $x_+^{-1/\lambda}$

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ABSTRACT. Let F be a distribution and let f be a locally summable function. The neutrix composition $F(f)$, of F and f, is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The neutrix composition of the distributions $x_+^\lambda \ln^m x_+$ and $x_+^{-1/\lambda}$ is evaluated for $-1 < \lambda < 0$ and $m = 1, 2, \ldots$.

1. INTRODUCTION

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support, let $\mathcal{D}[a,b]$ be the space of infinitely differentiable functions with support contained in the interval $[a,b]$ and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

We define the locally summable function $x_+^\lambda \ln^m x_+$ for $\lambda > -1$ and $m = 0, 1, 2, \ldots$ by

$$x_+^\lambda \ln^m x_+ = \begin{cases} x^\lambda \ln^m x, & x > 0, \\ 0, & x < 0 \end{cases}$$

and we define the distribution $x_+^{-1} \ln^m x_+$ $m = 0, 1, 2, \ldots$ by

$$x_+^{-1} \ln^m x_+ = \frac{\ln^{m+1} x_+}{m+1}.$$

It follows that if φ is an arbitrary function in $\mathcal{D}[-1,1]$, then

$$\langle x_+^{-1} \ln^m x_+, \varphi(x) \rangle = \int_0^1 x^{-1} \ln^m x[\varphi(x) - \varphi(0)]\, dx. \tag{1}$$

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We now let N be the neutrix, see [1], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^{\lambda}\ln^{r-1} n,\ \ln^r n: \qquad \lambda > 0,\ r = 1, 2, \dots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Now let $\rho(x)$ be an infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

If now f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t)\rangle$$

for $n = 1, 2, \dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition was given in [2]. and was originally called the composition of distributions.

Definition 1. *Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a, b), with $-\infty < a < b < \infty$, if*

$$\mathop{\mathrm{N-lim}}_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)\, dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a, b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \dots$.

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)\, dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a, b]$.

The following theorems were proved in [2] and [3] respectively:

Theorem 1. *The neutrix compositions $(x_-^{\mu})_-^{\lambda}$ and $(x_+^{\mu})_-^{\lambda}$ exist and*

$$(x_-^{\mu})_-^{\lambda} = (x_+^{\mu})_-^{\lambda} = 0$$

for $\mu > 0$ and $\lambda\mu \neq -1, -2, \ldots$ and

$$(x_-^\mu)_-^\lambda = (-1)^{\lambda\mu}(x_+^\mu)_-^\lambda = \frac{\pi \operatorname{cosec}(\pi\lambda)}{2\mu(-\lambda\mu - 1)!}\delta^{(-\lambda\mu-1)}(x)$$

for $\mu > 0$, $\lambda \neq -1, -2, \ldots$ and $\lambda\mu = -1, -2, \ldots$.

Theorem 2. *The neutrix composition $(x_+^r)_-^{-s}$ exists and*

$$(x_+^r)_-^{-s} = \frac{(-1)^{rs+s}c(\rho)}{r(rs-1)!}\delta^{(rs-1)}(x)$$

for $r, s = 1, 2, \ldots$, where $c(\rho) = \int_0^1 \ln t\, dt$.

In the previous theorem, the distribution x_-^{-s} is defined by

$$x_-^{-s} = -\frac{(\ln x_-)^{(s)}}{(s-1)!}$$

for $s = 1, 2, \ldots$ and not as in Gel'fand and Shilov [7].

The next three theorems were proved in [4], [6] and [5] respectively.

Theorem 3. *The neutrix composition $(x_+^r)^{-1}$ exists and*

$$(x_+^r)^{-1} = x_+^{-r} + (-1)^r \frac{2c(\rho) - r\phi(r-1)}{r!}\delta^{(r-1)}(x),$$

for $r = 1, 2, \ldots$, where

$$\phi(r) = \begin{cases} \sum_{i=1}^r 1/i, & i \geq 1, \\ 0, & i = 0. \end{cases}$$

Theorem 4. *The neutrix composition $(x_+^\mu)_+^\lambda$ exists and*

$$(x_+^\mu)_+^\lambda = x_+^{\lambda\mu}$$

for $\lambda < 0$, $\mu > 0$ and $\lambda, \lambda\mu \neq -1, -2, \ldots$.

Theorem 5. *If $F_{m,\lambda}(x)$ denotes the distibution $x_+^\lambda \ln^m x_+$, then the neutrix composition $F_{m,\lambda}(x_+^\mu)$ exists and*

$$F_{m,\lambda}(x_+^\mu) = \mu^m x_+^{\lambda\mu} \ln^m x_+ \tag{2}$$

for $-1 < \lambda < 0$, $\mu > 0$, $m = 0, 1, 2, \ldots$ and $\lambda\mu \neq -1, -2, \ldots$.

To prove the next theorem, we need the following lemma which can easily proved by induction.

Lemma.

$$\int_1^n v^\alpha \ln^r v\, dv = \frac{(-1)^r r!(n^{\alpha+1} - 1)}{(\alpha+1)^{r+1}} + \sum_{i=0}^{r-1} \frac{(-1)^i r! n^{\alpha+1} \ln^{r-i} n}{(r-i)!(\alpha+1)^{i+1}}$$

for $\alpha \neq -1$ and $r = 1, 2, \ldots$.

We now prove

Theorem 6. *If $F_{m,\lambda}(x)$ denotes the distibution $x_+^\lambda \ln^m x_+$, then the neutrix composition $F_{m,\lambda}(x_+^{-1/\lambda})$ exists and*

$$F_{m,\lambda}(x_+^{-1/\lambda}) = (-\lambda)^{-m} x_+^{-1} \ln^m x_+ + c_{\lambda,m}(\rho)\delta(x) \tag{3}$$

for $-1 < \lambda < 0$ and $m = 0, 1, 2, \ldots$, where

$$c_{\lambda,m}(\rho) = -\lambda \int_0^1 v^{-\lambda-1} \int_{-1}^v (v-u)^\lambda \ln^m(v-u)\rho(u)\,du\,dv.$$

Proof. We put

$$[F_{m,\lambda}(x)]_n = (x_+^\lambda \ln^m x_+) * \delta_n(x).$$

Then

$$[F_{m,\lambda}(x)]_n = \begin{cases} \int_{-1/n}^{1/n} (x-t)^\lambda \ln^m(x-t)\delta_n(t)\,dt, & 1/n < x, \\ \int_{-1/n}^{x} (x-t)^\lambda \ln^m(x-t)\delta_n(t)\,dt, & -1/n \le x \le 1/n, \\ 0, & x < -1/n \end{cases}$$

and so

$$[F_{m,\lambda}(x_+^\mu)]_n = \begin{cases} \int_{-1/n}^{1/n} (x^\mu-t)^\lambda \ln^m(x^\mu-t)\delta_n(t)\,dt, & 1/n < x^\mu, \\ \int_{-1/n}^{x^\mu} (x^\mu-t)^\lambda \ln^m(x^\mu-t)\delta_n(t)\,dt, & 0 \le x^\mu \le 1/n, \\ \int_{-1/n}^{0} (-t)^\lambda \ln^m(-t)\delta_n(t)\,dt, & x < 0. \end{cases} \tag{4}$$

It follows that

$$\begin{aligned} \int_{-1}^1 x^k [F_{m,\lambda}(x_+^\mu)]_n\,dx &= \int_0^{n^\lambda} x^k \int_{-1/n}^{x^\mu} (x^\mu-t)^\lambda \ln^m(x^\mu-t)\delta_n(t)\,dt\,dx \\ &\quad + \int_{n^\lambda}^1 \int_{-1/n}^{1/n} x^k (x^\mu-t)^\lambda \ln^m(x^\mu-t)\delta_n(t)\,dt\,dx \\ &\quad + \int_{-1}^0 \int_{-1/n}^0 x^k (-t)^\lambda \ln^m(-t)\delta_n(t)\,dt\,dx \\ &= -\lambda n^{k\lambda} \int_0^1 v^{-(k+1)\lambda-1} \int_{-1}^v (v-u)^\lambda [\ln(v-u) - \ln n]^m \rho(u)\,du\,dv \\ &\quad - \lambda n^{k\lambda} \int_{-1}^1 \rho(u) \int_1^n v^{-(k+1)\lambda-1} (v-u)^\lambda [\ln(v-u) - \ln n]^m\,dv\,du \\ &\quad + n^{-\lambda} \int_{-1}^0 x^k \int_{-1}^0 (-u)^\lambda \ln^m(-u/n)\rho(u)\,du\,dx \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{5}$$

where we have put $\mu = -\lambda^{-1}$ and the substitutions $u = nt$ and $v = nx^{\mu}$ have been made.

It follows immediately that

$$\operatorname*{N-lim}_{n\to\infty} I_3 = 0 \tag{6}$$

for $k = 0, 1, 2, \dots$ and

$$\lim_{n\to\infty} I_1 = 0 \tag{7}$$

for $k = 1, 2, \dots$.

When $k = 0$, we have

$$I_1 = -\lambda \int_0^1 v^{-\lambda-1} \int_{-1}^{v} (v-u)^{\lambda}[\ln(v-u) - \ln n]^m \rho(u)\, du\, dv$$

so that

$$\operatorname*{N-lim}_{n\to\infty} I_1 = -\lambda \int_0^1 v^{-\lambda-1} \int_{-1}^{v} (v-u)^{\lambda} \ln^m(v-u)\rho(u)\, du\, dv = c_{\lambda,m}(\rho). \tag{8}$$

Further,

$$\begin{aligned}
&\int_1^n v^{-(k+1)\lambda-1}(v-u)^{\lambda}[\ln(v-u) - \ln n]^m\, dv = \\
&\quad = \sum_{s=0}^{m} \binom{m}{s} (-1)^{m-s} \ln^{m-s} n \int_1^n v^{-(k+1)\lambda-1}(v-u)^{\lambda} \ln^s(v-u)\, dv \\
&\quad = \sum_{s=0}^{m-1} \sum_{i=1}^{s} \binom{m}{s}\binom{s}{i} (-1)^{m-s} \ln^{m-s} n \int_1^n v^{-k\lambda-1}(1-u/v)^{\lambda} \\
&\qquad\qquad \times \ln^i(1-u/v) \ln^{s-i} v\, dv \\
&\qquad + \sum_{s=0}^{m-1} \binom{m}{s} (-1)^{m-s} \ln^{m-s} n \int_1^n v^{-k\lambda-1}(1-u/v)^{\lambda} \ln^s v\, dv \\
&\qquad + \sum_{i=1}^{m} \binom{m}{i} \int_1^n v^{-k\lambda-1}(1-u/v)^{\lambda} \ln^i(1-u/v) \ln^{m-i} v\, dv \\
&\qquad + \int_1^n v^{-k\lambda-1}(1-u/v)^{\lambda} \ln^m v\, dv \\
&\quad = \sum_{s=0}^{m-1} \sum_{i=1}^{s} (-1)^{m-s+i} \binom{m}{s}\binom{s}{i} \ln^{m-s} n \int_1^n v^{-k\lambda-1} \\
&\qquad\qquad \times \Big[\frac{u^i}{v^i} + \frac{(i/2-\lambda)u^{i+1}}{v^{i+1}} + O(v^{-i-2})\Big] \ln^{s-i} v\, dv
\end{aligned}$$

$$+\sum_{s=0}^{m-1}\binom{m}{s}(-1)^{m-s}\ln^{m-s}n\int_1^n v^{-k\lambda-1}$$
$$\times\Big[1-\frac{\lambda u}{v}+O(v^{-2})\Big]\ln^s v\,dv$$
$$+\sum_{i=1}^{m}(-1)^i\binom{m}{i}\int_1^n v^{-k\lambda-1}\Big[\frac{u^i}{v^i}+\frac{(i/2-\lambda)u^{i+1}}{v^{i+1}}+O(v^{-i-2})\Big]\ln^{m-i}v\,dv$$
$$+\int_1^n v^{-k\lambda-1}\Big[1-\frac{\lambda u}{v}+O(v^{-2})\Big]\ln^m v\,dv. \qquad (9)$$

Using the lemma, it follows that

$$n^{k\lambda}\ln^{m-s}n\int_1^n v^{-k\lambda-1}\Big[\frac{u^i}{v^i}+\frac{(i/2-\lambda)u^{i+1}}{v^{i+1}}+\dots\Big]\ln^{s-i}v\,dv$$
$$=O(n^{-i}\ln^{m-i}n)+O(n^{k\lambda-i}\ln^{m-s}n), \qquad (10)$$

for $i=1,\dots,s$; $s=0,1,\dots,m-1$ and $k=0,1,2,\dots,$

$$n^{k\lambda}\ln^{m-s}n\int_1^n v^{-k\lambda-1}\Big[1-\frac{\lambda u}{v}+\dots\Big]\ln^s v\,dv$$
$$=\begin{cases}\dfrac{\ln^{m+1}n}{s+1}+O(n^{-1}\ln^m n)+O(n^{-(k+1)/\mu-\lambda}\ln^m n), & k=0,\\ -\dfrac{s!(1-n^{k\lambda})\ln^{m-s}n}{(k\lambda)^{s+1}}+O(n^{-1}\ln^m n), & k=1,2,\dots\end{cases} \qquad (11)$$

for $s=0,1,\dots,m-1$,

$$n^{k\lambda}\int_1^n v^{-k\lambda-1}\Big[\frac{u^i}{v^i}+\frac{(i/2-\lambda)u^{i+1}}{v^{i+1}}+\dots\Big]\ln^{m-i}v\,dv$$
$$=O(n^{-i}\ln^{m-i}n), \qquad (12)$$

for $i=1,\dots,m$ and $k=0,1,2,\dots$ and

$$n^{k\lambda}\int_1^n v^{-k\lambda-1}\Big[1-\frac{\lambda u}{v}+\dots\Big]\ln^m v\,dv$$
$$=\begin{cases}\dfrac{\ln^{m+1}n}{m+1}+O(n^{-1}\ln^m n)+O(n^{-(k+1)/\mu-\lambda}\ln^m n), & k=0,\\ -\dfrac{m!(1-n^{k\lambda})}{(k\lambda)^{s+1}}+O(n^{-1}\ln^m n), & k=1,2,\dots.\end{cases} \qquad (13)$$

It now follows from equations (5) and (9) to (13) that

$$\operatorname*{N-lim}_{n\to\infty} I_2=0 \qquad (14)$$

when $k = 0$ and

$$\operatorname*{N-lim}_{n\to\infty} I_2 = \frac{m!}{k^{m+1}\lambda^m}\int_{-1}^{1}\rho(u)\,du = \frac{m!}{k^{m+1}\lambda^m} \tag{15}$$

for $k = 1, 2, \ldots$.

It now follows from equations (5), (6), (7) and (15) that

$$\operatorname*{N-lim}_{n\to\infty}\int_{-1}^{1} x^k [F_{m,\lambda}(x_+^\mu)]_n\,dx = \frac{m!}{k^{m+1}\lambda^m}, \tag{16}$$

for $k = 1, 2, \ldots,$

We now consider the case $k = 2$ and let ψ be an arbitrary continuous function. Then

$$\int_0^{n^\lambda} x^2\psi(x)[F_{m,\lambda,}(x_+^\mu)]_n\,dx$$
$$= -\lambda n^{2\lambda}\int_0^1 v^{-3\lambda-1}\psi[(v/n)^{-\lambda}]\int_{-1}^{v}(v-u)^\lambda[\ln(v-u)-\ln n]^m\rho(u)\,du\,dv$$

and it follows that

$$\lim_{n\to\infty}\int_0^{n^\lambda} x^2\psi(x)[F_{m,\lambda}(x_+^\mu)]_n\,dx = 0. \tag{17}$$

When $x \le 0$, we have

$$\int_{-1}^{0} x^2\psi(x)[F_{m,\lambda}(x_+^\mu)_n]\,dx = n^{-\lambda}\int_{-1}^{0} x^2\psi(x)\int_{-1}^{0}(-u)^\lambda(\ln u - \ln n)^m\rho(u)\,du\,dx$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty}\int_{-1}^{0} x^2\psi(x)F_{m,\lambda}(x_+^\mu)_n\,dx = 0. \tag{18}$$

When $x \ge n^\lambda$, we have

$$\begin{aligned}[F_{m,\lambda}(x_+^\mu)]_n &= \int_{-1/n}^{1/n}(x^\mu - t)^\lambda \ln^m(x^\mu - t)\delta_n(t)\,dt\\ &= \int_{-1}^{1}(x^\mu - u/n)^\lambda \ln^m(x^\mu - u/n)\rho(u)\,du\\ &= x^{-1}\int_{-1}^{1}\Big[\ln^m x^\mu - \frac{\lambda u\ln^m x^\mu}{nx^\mu} - \frac{mu\ln^{m-1}x^\mu}{nx} + O(n^{-2})\Big]\rho(u)\,du\\ &= x^{-1}\ln^m x^\mu + O(n^{-1})\end{aligned} \tag{19}$$

and it follows from equations (17) and (19) that

$$\lim_{n\to\infty}\int_0^1 x^2\psi(x)[F_{m,\lambda}(x_+^\mu)]_n\,dx = \mu^m\int_0^1 x\ln^m x\psi(x)\,dx. \tag{20}$$

Now let $\varphi(x)$ be an arbitrary function in $\mathcal{D}[-1,1]$. By Taylor's Theorem we have

$$\varphi(x) = \varphi(0) + x\varphi'(0) + \frac{x^2}{2!}\varphi''(\xi x)$$

where $0 < \xi < 1$. Then

$$\begin{aligned}\langle [F_{m,\lambda}(x_+^\mu)]_n, \varphi(x)\rangle &= \int_{-1}^1 [F_{m,\lambda}(x_+^\mu)]_n\varphi(x)\,dx\\ &= \varphi(0)\int_{-1}^1 [F_{m,\lambda}(x_+^\mu)]_n\,dx + \varphi'(0)\int_{-1}^1 x[F_{m,\lambda}(x_+^\mu)]_n\,dx\\ &+ \int_0^1 \frac{x^2}{2!}[F_{m,\lambda}(x_+^\mu)]_n\varphi''(\xi x)\,dx + \int_{-1}^0 \frac{x^2}{2!}[F_{m,\lambda}(x_+^\mu)]_n\varphi^{(r)}(\xi x)\,dx.\end{aligned}$$

Using equations (8), (16) and (20), it follows that

$$\begin{aligned}\mathrm{N}-\lim_{n\to\infty}\langle [F_{m,\lambda}(x_+^\mu)]_n, \varphi(x)\rangle &= c_{\lambda,m}\varphi(0) + \frac{m!}{\lambda^m}\varphi'(0)\\ &\quad + \mu^m\int_0^1 \frac{x\ln^m x}{2!}\varphi''(\xi x)\,dx\\ &= \mu^m\int_0^1 x^{-1}\ln^m x[\varphi(x) - \varphi(0) - x\varphi'(0)]dx\\ &\quad + c_{\lambda,m}\varphi(0) + \frac{m!}{\lambda^m}\varphi'(0)\\ &= (-\lambda)^{-m}\int_0^1 x^{-1}\ln^m x[\varphi(x) - \varphi(0)]\,dx + c_{\lambda,m}\varphi(0)\\ &= \langle (-\lambda)^{-m}x_+^{-1}\ln^m x_+ + c_{\lambda,m}\delta(x), \varphi(x)\rangle,\end{aligned}$$

on using equation (1). This proves equation (3) on the interval $[-1,1]$. However, equation (3) clearly holds on any interval not containing the origin, and the proof is complete. □

Replacing x by $-x$ in Theorem 6, we get

Corollary. *If $G_{m,\lambda}(x)$ denotes the distibution $x_-^\lambda \ln^m x_-$, then the neutrix composition $G_{m,\lambda}(x_-^{-1/\lambda})$ exists and*

$$G_{m,\lambda}(x_-^{-1/\lambda}) = (-\lambda)^{-m}x_-^{-1}\ln^m x_- + c_{\lambda,m}(\rho)\delta(x) \tag{21}$$

for $-1 < \lambda < 0$ and $m = 0, 1, 2, \ldots$.

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