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EXTREMAL VECTORS FOR A CLASS OF OPERATORS

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Abstract. In this note, we study extremal vectors for a class of operators with a special application to the the Volterra operator. The study requires the description of the asymptotic behavior of a class of infinite series involving a real parameter.

1. Introduction

In 1996 Per Enflo introduced the concept of extremal vectors in connection with the Invariant Subspace Problem [1]. In particular, backward minimal vectors can be used to give constructive proofs of the existence of invariant subspaces for certain classes of operators on a Hilbert Space. In this paper we investigate the behavior of backward minimal vectors for a class of operators, including the Volterra operator. In particular, our main interest will be to give estimates on the norms of these vectors. Necessary for this study is the determination of the asymptotic behavior of a class of infinite series involving a real parameter. This work is contained in Section 2.

2. The sum of a class of series

The series we investigate here are of the form $\sum_{n\in J}\frac{n^c}{(n^d+K)^2}$ where $J=\{1,3,\ldots\}$ is the set of odd positive integers. The method of summation used involves the calculus of residues in Complex Analysis. Before stating the main result of this section (Theorem 2.3), we establish some terminology and state two technical lemmas. The proofs of these lemmas are given at the end of this section.

Let

$$T(z)=-\frac{\pi}{2}\tan\left(\frac{\pi z}{2}\right).$$

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Let c and d be positive even integers with $c \leq 2d-2$ and for $j = 1, \ldots, d$, let

$$\alpha_j = exp\left(\mathbf{i}\frac{2j-1}{d}\pi\right) = \cos\frac{2j-1}{d}\pi + \mathbf{i}\sin\frac{2j-1}{d}\pi$$

be the d-th roots of -1 and let $\beta_j = \alpha_j \sqrt[d]{K}$.

Lemma 2.1. *Let*

$$M = \sum_{j=1}^{d} T(\beta_j)\,\alpha_j^{c+1} \quad \textit{and} \quad N = \sum_{j=1}^{d} T'(\beta_j)\,\alpha_j^{c+2}.$$

Then both M and N are real numbers.

Lemma 2.2. *For each $j = 1, \ldots, d$, let $S_j = \{1, 2, \ldots, d\} \setminus \{j\}$, and let*

$$A_j = \sum_{\ell \in S_j} \frac{1}{\alpha_j - \alpha_\ell}, \quad \textit{and} \quad B_j = \prod_{\ell \in S_j} \frac{1}{\alpha_j - \alpha_\ell}.$$

Then

$$A_j = \frac{d-1}{2\alpha_j} \quad \textit{and} \quad B_j = -\frac{\alpha_j}{d}.$$

We now state the main theorem of this section :

Theorem 2.3. *Let $K \geq 1$ be real, c and d be positive even integers with $c \leq 2d-2$, and J be the set of odd, positive integers. Then*

$$\sum_{n \in J} \frac{n^c}{(n^d + K)^2} = \frac{1}{K^p}\left(\frac{(d-c-1)M - N\sqrt[d]{K}}{2d^2}\right)$$

where $p = 2 - \frac{c+1}{d}$ and M and N are as in Lemma 2.1.

Proof. Let $f(z) = \frac{z^c}{(z^d+K)^2}$ and let γ_k be a Mittag-Leffler rectangle with corners $\pm 2k \pm (2k)\mathbf{i}$. If we assume $k > K$, then by the residue theorem,

$$\frac{1}{2\pi\mathbf{i}} \int_{\gamma_k} T(z) f(z)\,dz = \sum_{p} \operatorname{Res}\left(T(z)f(z), z = p\right)$$

where the sum is taken over all the poles p of the function Tf, lying inside the rectangle γ_k. Since $\lim_{z\to\infty} zf(z) = 0$ and $|T(z)|$ is bounded on γ_k, $\int_{\gamma_k} T(z)f(z)\,dz \to 0$ as $k \to \infty$ [2]. Moreover, Tf has as poles the set of odd integers, each a simple pole, and the d complex numbers $\beta_1, \ldots \beta_d$, each of order 2.

Consequently, we have

$$\sum_{m \in \mathbb{Z}\setminus 2\mathbb{Z}} \operatorname{Res}\left(T(z)f(z), z = m\right) + \sum_{j=1}^{d} \operatorname{Res}\left(T(z)f(z), z = \beta_j\right) = 0. \qquad (1)$$

Since T has simple poles at all the odd integers with residues equal 1,

$$\begin{aligned}\sum_{m\in\mathbb{Z}\setminus 2\mathbb{Z}} \operatorname{Res}(T(z)f(z), z=m) &= \sum_{m\in\mathbb{Z}\setminus 2\mathbb{Z}} f(m)\operatorname{Res}(T(z), z=m)\\ &= \sum_{m\in\mathbb{Z}\setminus 2\mathbb{Z}} f(m)\\ &= 2\sum_{n\in J} f(n)\\ &= 2\sum_{n\in J}\frac{n^c}{(n^d+K)^2}.\end{aligned}$$

Now, for each $j=1,\dots,d$, Tf has a double pole at $z=\beta_j$. Thus, letting

$$G_j(z) = \prod_{\ell\in S_j}\frac{1}{(z-\beta_\ell)^2},$$

we have that

$$\begin{aligned}\operatorname{Res}(T(z)f(z), z=\beta_j) &= \lim_{z\to\beta_j}\frac{d}{dz}(z-\beta_j)^2T(z)f(z) = \lim_{z\to\beta_j}\frac{d}{dz}z^cT(z)G_j(z)\\ &= c\beta_j^{c-1}T(\beta_j)G_j(\beta_j) + \beta_j^cT(\beta_j)G_j'(\beta_j) + \beta_j^cT'(\beta_j)G_j(\beta_j).\end{aligned}$$

But

$$G_j(\beta_j) = \frac{1}{\left(K^{\frac{2}{d}}\right)^{d-1}}\prod_{\ell\in S_j}\frac{1}{\alpha_j-\alpha_\ell} = K^{\frac{2-2d}{d}}B_j^2 = K^{\frac{2-2d}{d}}\frac{\alpha_j^2}{d^2}.$$

Using a logarithmic derivative we have

$$\begin{aligned}G_j'(\beta_j) &= -2G_j(\beta_j)\left(\frac{1}{(\alpha_j\sqrt[d]{K}-\alpha_1\sqrt[d]{K})}+\cdots+\frac{1}{(\alpha_j\sqrt[d]{K}-\alpha_d\sqrt[d]{K})}\right)\\ &= -2K^{\frac{2-2d}{d}}B_j^2K^{-\frac{1}{d}}\left(\frac{1}{(\alpha_j-\alpha_1)}+\cdots+\frac{1}{(\alpha_j-\alpha_d)}\right)\\ &= -2K^{\frac{2-2d}{d}}B_j^2K^{-\frac{1}{d}}A_j\\ &= -2K^{\frac{1-2d}{d}}B_j^2A_j\end{aligned}$$

where A_j and B_j are as in lemma 2.2. Hence

$$\begin{aligned}
&c\beta_j^{c-1}T(\beta_j)G_j(\beta_j) + \beta_j^c T(\beta_j)G_j'(\beta_j)\\
&= T(\beta_j)\left(c\alpha_j^{c-1}\sqrt[d]{K}^{c-1}G_j(\beta_j) + \alpha_j^c\sqrt[d]{K}^c G_j'(\beta_j)\right)\\
&= T(\beta_j)\left(c\alpha_j^{c-1}K^{\frac{c-1}{d}}K^{\frac{2-2d}{d}}B_j^2 - 2\alpha_j^c K^{\frac{c}{d}}K^{\frac{1-2d}{d}}B_j^2A_j\right)\\
&= T(\beta_j)K^{\frac{c+1-2d}{d}}\left(c\alpha_j^{c-1}B_j^2 - 2\alpha_j^c B_j^2 A_j\right)\\
&= T(\beta_j)K^{\frac{c+1-2d}{d}}\left(c\alpha_j^{c-1}\frac{\alpha_j^2}{d^2} - 2\alpha_j^c\frac{\alpha_j^2}{d^2}\cdot\frac{(d-1)}{2\alpha_j}\right)\\
&= T(\beta_j)K^{\frac{c+1-2d}{d}}\alpha_j^{c+1}\left(\frac{c-d+1}{d^2}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathrm{Res}\,(T(z)f(z), z=\beta_j)\\
&= c\beta_j^{c-1}T(\beta_j)G_j(\beta_j) + \beta_j^c T(\beta_j)G_j'(\beta_j) + \beta_j^c T'(\beta_j)G_j(\beta_j)\\
&= K^{\frac{c+1-2d}{d}}T(\beta_j)\,\alpha_j^{c+1}\left(\frac{c-d+1}{d^2}\right) + K^{\frac{c}{d}}\alpha_j^c T'(\beta_j)K^{\frac{2-2d}{d}}\frac{\alpha_j^2}{d^2}\\
&= K^{\frac{c+1-2d}{d}}\left(T(\beta_j)\,\alpha_j^{c+1}\left(\frac{c-d+1}{d^2}\right) + K^{\frac{1}{d}}\alpha_j^{c+2}T'(\beta_j)\frac{1}{d^2}\right).
\end{aligned}$$

Finally, recalling the definitions of M and N, and that

$$\sum_{m\in\mathbb{Z}\setminus 2\mathbb{Z}} \mathrm{Res}\,(T(z)f(z), z=m) = -\sum_{j=1}^{d}\mathrm{Res}\,(T(z)f(z), z=\beta_j)\,,$$

we now have that

$$\begin{aligned}
2\sum_{n\in J}\frac{n^c}{(n^d+K)^2} &= -\sum_{j=1}^{d}K^{\frac{c+1-2d}{d}}\Bigg(T(\beta_j)\,\alpha_j^{c+1}\left(\frac{c-d+1}{d^2}\right)\\
&\qquad\qquad + K^{\frac{1}{d}}\alpha_j^{c+2}T'(\beta_j)\frac{1}{d^2}\Bigg)\\
&= \frac{1}{K^{\frac{2d-c-1}{d}}}\left(\frac{(d-c-1)M - N\sqrt[d]{K}}{d^2}\right).
\end{aligned}$$

Dividing by 2 completes the proof of Theorem 2.3. □

In what follows, we will be comparing functions of a real variable according to their asymptotic behavior. The real variable will be the parameter K from the series above (and also the $K = K_\varepsilon$ from Theorem 3.2). We make the following definition :

Definition 2.4. *Let f, $g : [1, \infty) \longrightarrow \mathbb{R}$ be two positive real valued functions (of the parameter K). Then f and g are* ***asymptotically equivalent****, denoted $f \backsim g$, if there exist positive constants A and B independent of K, so that*

$$A \cdot g(K) < f(K) < B \cdot g(K).$$

Note here that $\backsim$ is an equivalence relation. Moreover, we have the following :

Proposition 2.5. *Let*

$$W(K) = (d - c - 1)M - N\sqrt[d]{K}$$

be as in Theorem 2.3*. Then*

$$W(K) \backsim 1.$$

Proof. Since the sum in Theorem 2.3 is clearly positive, it is sufficient to show that $W(K) \to C$ for some non-zero constant C as $K \to \infty$. Since $T(z) = \frac{\pi \mathbf{i}(e^{\mathbf{i}z} - e^{-\mathbf{i}z})}{2(e^{\mathbf{i}z} + e^{-\mathbf{i}z})}$ and since $\sin\left(\frac{2j-1}{d}\pi\right) > 0$ for $1 \le j \le \frac{d}{2}$ and $\sin\left(\frac{2j-1}{d}\pi\right) < 0$ for $\frac{d}{2} < j \le d$, one readily obtains

$$\lim_{K\to\infty} T(\beta_j) = \begin{cases} -\frac{\pi}{2}\mathbf{i}, & \text{for } 1 \le j \le \frac{d}{2} \\ \frac{\pi}{2}\mathbf{i}, & \text{for } \frac{d}{2} < j \le d. \end{cases}$$

So

$$\begin{aligned}
\lim_{K\to\infty} M &= \lim_{K\to\infty} \sum_{j=1}^{d} T(\beta_j)\,\alpha_j^{c+1} \\
&= \lim_{K\to\infty} \sum_{j=1}^{d/2} T(\beta_j)\,\alpha_j^{c+1} + T(\beta_{j+\frac{d}{2}})\,\alpha_{j+\frac{d}{2}}^{c+1} \\
&= -\pi\mathbf{i} \sum_{j=1}^{d/2} \alpha_j^{c+1} \qquad \left(\text{since } \alpha_{j+d/2} = -\alpha_j,\ T(\beta_{j+\frac{d}{2}}) = -T(\beta_j)\right) \\
&= -\pi\mathbf{i}\left(\alpha_1^{c+1} + \left(\alpha_1^3\right)^{c+1} + \left(\alpha_1^5\right)^{c+1} + \cdots + \left(\alpha_1^{d-1}\right)^{c+1}\right) \\
&= \frac{-\pi\mathbf{i}\,\alpha_1^{c+1}\left(1 - \left(\alpha_1^{c+1}\right)^d\right)}{\left(1 - \left(\alpha_1^{c+1}\right)^2\right)} = \frac{-2\pi\mathbf{i}\,\alpha_1^{c+1}}{\left(1 - \left(\alpha_1^{c+1}\right)^2\right)} \\
&= \frac{-2\pi\mathbf{i}\,\alpha_1^{c+1}\overline{\alpha_1}^{c+1}}{\overline{\alpha}_1^{c+1}\left(1 - \left(\alpha_1^{c+1}\right)^2\right)} = \frac{2\pi\mathbf{i}}{\alpha_1^{c+1} - \overline{\alpha_1}^{c+1}} = \frac{\pi}{\sin\frac{(c+1)\pi}{d}}.
\end{aligned}$$

Also, since

$$T'(\beta_j) = -\pi^2 \left(e^{\mathrm{i}\frac{\pi}{2}\beta_j} + e^{-\mathrm{i}\frac{\pi}{2}\beta_j}\right)^{-1} = \mathbf{O}\Big(\frac{1}{e^{\sqrt[d]{K}}}\Big),$$

the limit

$$\lim_{K\to\infty} N\sqrt[d]{K} = \lim_{K\to\infty} T'(\beta_j)\,\alpha_j^{c+2}\sqrt[d]{K} = 0.$$

Hence

$$\lim_{K\to\infty} W(K) = \lim_{K\to\infty}(d-c-1)M - N\sqrt[d]{K} = \frac{\pi(d-c-1)}{\sin\frac{(c+1)\pi}{d}}$$

which is positive for $c \le 2d-2$. □

Proposition 2.6. *Let* $f(K) = \sum_{n\in J}\frac{n^c}{(n^d+K)^2}$ *be as in Theorem* 2.3. *Then*

$$f(K) \backsim \frac{1}{K^{2-\frac{c+1}{d}}}.$$

Proof. We have from Theorem 2.3 that $f(K) = \frac{W(K)}{2d^2K^p}$ where $p = 2-\frac{c+1}{d}$. But by Proposition 2.5, we have $W(K) \backsim 1$, so $f(K) = \frac{W(K)}{2d^2K^p} \backsim \frac{1}{K^p}$. □

2.1. **Proofs of Lemmas 2.1 and 2.2.**

Proof. (Lemma 2.1) : Note that for $j = 1,\ldots,\frac{d}{2}$, $\overline{\alpha_j} = \alpha_{d-j+1}$, and consequently $\overline{\beta_j} = \beta_{d-j+1}$. Also, for any complex number z,

$$T(\overline{z}) = \overline{T(z)} \quad \text{and} \quad T'(\overline{z}) = \overline{T'(z)}.$$

Therefore

$$\begin{aligned} M = \sum_{j=1}^{d} T(\beta_j)\,\alpha_j^{c+1} &= \sum_{j=1}^{d/2} T(\beta_j)\,\alpha_j^{c+1} + T(\beta_{d-j+1})\,\alpha_{d-j+1}^{c+1} \\ &= \sum_{j=1}^{d/2} T(\beta_j)\,\alpha_j^{c+1} + T(\overline{\beta_j})\,\overline{\alpha_j}^{c+1} = \sum_{j=1}^{d/2} T(\beta_j)\,\alpha_j^{c+1} + \overline{T(\beta_j)\,\alpha_j^{c+1}} \\ &= 2\sum_{j=1}^{d/2} \Re e\left(T(\beta_j)\,\alpha_j^{c+1}\right). \end{aligned}$$

From Theorem 2.3, since the sum is real and M is real, N must be real. □

Proof. (Lemma 2.2) : Note that for $j = 1,\ldots,d$, if we let $H_j(z) = \frac{z-\alpha_j}{z^d+1}$, then

$$\lim_{z\to\alpha_j} H_j(z) = B_j$$

and

$$\lim_{z\to\alpha_j} (\ln H_j(z))' = -A_j.$$

Hence

$$\begin{aligned}
A_j &= -\lim_{z\to\alpha_j}\left(\frac{H_j'(z)}{H_j(z)}\right)\\
&= \lim_{z\to\alpha_j}\frac{dz^{d-1}(z-\alpha_j)-(z^d+1)}{(z^d+1)(z-\alpha_j)}\\
&= \lim_{z\to\alpha_j}\frac{d(d-1)z^{d-2}(z-\alpha_j)+dz^{d-1}-dz^{d-1}}{(z^d+1)+dz^{d-1}(z-\alpha_j)}\\
&= \lim_{z\to\alpha_j}\frac{d(d-1)z^{d-2}(z-\alpha_j)}{(z^d+1)+dz^{d-1}(z-\alpha_j)}\\
&= \lim_{z\to\alpha_j}\frac{d(d-1)(d-2)z^{d-3}(z-\alpha_j)+d(d-1)z^{d-2}}{dz^{d-1}+dz^{d-1}+d(d-1)z^{d-2}(z-\alpha_j)}\\
&= \frac{d-1}{2\alpha_j}.
\end{aligned}$$

Also,

$$\begin{aligned}
B_j &= \lim_{z\to\alpha_j} H_j(z) = \lim_{z\to\alpha_j}\frac{z-\alpha_j}{z^d+1}\\
&= \lim_{z\to\alpha_j}\frac{1}{dz^{d-1}} = \frac{1}{d\alpha_j^{d-1}} = -\frac{\alpha_j}{d}.
\end{aligned}$$

□

3. Extremal vectors for a class of operators

We begin by giving the the definition of a backward minimal vector and previously known results (Theorem 3.2) [1] which are pertinent to our discussion.

Definition 3.1. *Let $T : H \longrightarrow H$ be a bounded linear operator on a Hilbert Space with dense range. Let $x_0 \in H$ with $x_0 \notin \mathcal{R}(T)$. Then, for $\varepsilon > 0$ with $\varepsilon < \|x_0\|$, there is a unique vector y_{ε,x_0} so that $\|y_{\varepsilon,x_0}\| = \inf\{\|y\| : \|Ty - x_0\| \leq \varepsilon\}$. The vectors y_{ε,x_0} are called* **backward minimal vectors**.

In what follows, we will assume that x_0 is known, and hence simply write y_ε for y_{ε,x_0} . We also note here that, in fact, $\|Ty_\varepsilon - x_0\| = \varepsilon$, [1].

Theorem 3.2. [1] *For $\varepsilon > 0$ there exists a constant $K = K_\varepsilon > 0$ so that*

$$y_\varepsilon = -K_\varepsilon T^*(Ty_\varepsilon - x_0).$$

We now give two important equations, also found in [1], which will be used in what follows. The first equation is obtained by solving for y_ε in the formula of Theorem 3.2. The second is obtained by first applying T to the same formula and then solving for Ty_ε.

$$y_\varepsilon = K_\varepsilon(I + K_\varepsilon T^*T)^{-1}T^*x_0. \tag{2}$$

$$Ty_\varepsilon = K_\varepsilon(I + K_\varepsilon TT^*)^{-1}TT^*x_0. \tag{3}$$

As mentioned above, our interest is in estimating the norms of backward minimal vectors. Because we are assuming that x_0 is not in the (dense) range of T, these norms necessarily grow unboundedly as the parameter ε tends to 0.

Theorem 3.3. *Let ε and $K = K_\varepsilon$ be as in Theorem 3.2. Let $T : H \longrightarrow H$ be a linear operator on a Hilbert space so that TT^* can be decomposed with a sequence of eigenvalues (λ_n) and corresponding complete orthonormal sequence (f_n) of eigenvectors. If $x_0 = \sum b_n f_n$, then*

$$\varepsilon^2 = \sum \frac{b_n^2}{(1 + K\lambda_n)^2} \quad \text{and} \tag{4}$$

$$\|y_\varepsilon\|^2 = K^2 \sum \frac{\lambda_n b_n^2}{(1 + K\lambda_n)^2}. \tag{5}$$

Proof. Let $V \equiv K(I + KTT^*)^{-1}TT^*$, so that by the spectral mapping theorem, V has the same eigenvectors (f_n) as TT^* but with eigenvalues $(\frac{K\lambda_n}{1+K\lambda_n})$. Thus, using equation (3), we have that

$$\begin{aligned}
\varepsilon^2 &= \|Ty_\varepsilon - x_0\|^2 = \|Vx_0 - x_0\|^2 \\
&= \|V(\sum b_n f_n) - \sum b_n f_n\|^2 \\
&= \left\| \sum (\frac{K\lambda_n b_n}{1 + K\lambda_n} - b_n) f_n \right\|^2 \\
&= \left\| \sum \frac{-b_n}{1 + K\lambda_n} f_n \right\|^2 = \sum \frac{b_n^2}{(1 + K\lambda_n)^2}.
\end{aligned}$$

which gives equation (4).

Now, noting that $T^*T(T^*(\frac{f_n}{\sqrt{\lambda_n}})) = T^*(\sqrt{\lambda_n} f_n) = \lambda_n T^*(\frac{f_n}{\sqrt{\lambda_n}})$, we see that T^*T has eigenvalues (λ_n) with (orthonormal) eigenvectors $(T^*(\frac{f_n}{\sqrt{\lambda_n}}))$. Letting $W \equiv (I + KT^*T)^{-1}$, then W has the same eigenvectors as T^*T with eigenvalues $(\frac{1}{1+K\lambda_n})$. Using equation (2) we have

$$\begin{aligned}\|y_\varepsilon\|^2 &= \|KWT^*(x_0)\|^2 = K^2\|WT^*(\sum b_n f_n)\|^2 \\ &= K^2\|W(\sum b_n T^* f_n)\|^2 = K^2 \left\| \sum \frac{b_n\sqrt{\lambda_n}}{1+K\lambda_n} T^*\left(\frac{f_n}{\sqrt{\lambda_n}}\right)\right\|^2 \\ &= K^2 \sum \frac{b_n^2\lambda_n}{(1+K\lambda_n)^2}.\end{aligned}$$

□

4. The Volterra operator

We now apply Theorem 3.3 to the Volterra operator, which is defined as:

Definition 4.1. *Let $T : L^2(0, \frac{\pi}{2}) \longrightarrow L^2(0, \frac{\pi}{2})$ by $Tf(x) \equiv \int_0^x f(t)dt$. Then T is called the* **Volterra** *operator.*

Fact : *If T is the Volterra operator, then the adjoint T^* of T is given by*

$$T^* f(x) = \int_x^{\frac{\pi}{2}} f(t)\, dt.$$

In what follows, we use the function x_0 which is constantly one a.e., as it is not in the range of the Volterra operator. For a more general x_0, one simply needs to expand x_0 with respect to the eigenvectors $(\sin(nx))$ as discussed below.

Theorem 4.2. *Let ε and $K = K_\varepsilon$ be as in Theorem 3.2. If T is the Volterra operator, x_0 is the function which is constantly one a.e. on $(0, \frac{\pi}{2})$, and y_ε is the corresponding backward minimal vector, then $\|y_\varepsilon\| \backsim \frac{1}{\varepsilon}$.*

Proof. Because T is compact, so is TT^*, and since TT^* is self adjoint, TT^* has a sequence of eigenvalues (λ_n) with a corresponding orthonormal sequence of eigenvectors (f_n) so that for every f in $L^2(0, \frac{\pi}{2})$, $Tf = \sum_n \lambda_n \langle f, f_n\rangle f_n$. Consider the equation $TT^* f = \lambda f$. Because T and T^* are integral operators, one can differentiate this equation twice, and then use standard differential equation techniques to show that TT^* has eigenvalues $\lambda_n = \frac{1}{n^2}$ $(n \in J = \{1, 3, 5, \ldots\})$ with corresponding eigenvectors $f_n(x) = \frac{4}{\pi}\sin nx$. Expanding x_0 with respect to (f_n) gives $x_0 = \frac{4}{\pi}\sum_{n\in J} \frac{\sin nx}{n}$, i.e. $(b_n) = (\frac{1}{n})$.

Thus, by equation (4),

$$\varepsilon^2 = \sum_{n\in J} \frac{n^2}{(n^2+K)^2}.$$

Now, using Proposition 2.6 with $c = d = 2,$ we see that

$$\varepsilon^2 \backsim \frac{1}{K^{\frac{1}{2}}} \text{ or } \varepsilon \backsim \frac{1}{K^{\frac{1}{4}}}.$$

Also, by equation (5),

$$\|y_\varepsilon\|^2 = K^2 \sum_{n\in J} \frac{1}{(n^2+K)^2}.$$

Using Proposition 2.6 again, this time with $c = 0$ and $d = 2,$ we have that

$$\|y_\varepsilon\|^2 \backsim \frac{K^2}{K^{\frac{3}{2}}} \text{ or } \|y_\varepsilon\| \backsim K^{\frac{1}{4}},$$

and thus we see that

$$\|y_\varepsilon\| \backsim \frac{1}{\varepsilon}.$$

□

5. Open problems

In [1], the authors consider extremal vectors of not only operators, but of powers of the operators. A study of the rate of growth of the norms of the backward minimal vectors of powers T^n of the Volterra operator similar to the above would require an analysis of the eigenvalues and eigenvectors of T^n. However, for $n \geq 2,$ the eigenvalues of T^n are solutions of complicated transcendental equations, and can only be estimated, and the eigenvectors can only be estimated as well. While an asymptotic estimate would do, even such estimates seem to be difficult, or at least very cumbersome. We do make the following conjecture :

Conjecture : *If T^n is the nth power of the Volterra operator and x_0 is one a.e., then $y_\varepsilon \backsim \frac{1}{\varepsilon^{2n-1}}$.*

References

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