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AN IMPLICIT FUNCTION IMPLIES SEVERAL CONTRACTION CONDITIONS

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Abstract. In this paper, we define a new implicit function which includes a majority of contractions of the existing literature of metric fixed point theory and then utilize the same to prove a general common fixed point theorem for two pairs of weakly compatible mappings satisfying the common property $(E.A)$. In the process, a host of previously known results are improved and generalized. Some related results are derived besides furnishing illustrative examples.

1. Introduction

A common fixed point theorem in metric spaces generally involves conditions on commutativity, continuity, contraction along with a condition on suitable containment of range of one mapping into the range of other. Hence, in order to prove a new metrical common fixed point theorem one is always required to improve one or more of these conditions.

The first ever attempt to improve commutativity conditions in common fixed point theorems is due to Sessa [29] wherein he introduced the notion of weakly commuting mappings. Inspired by the definition due to Sessa [29], researchers of this domain introduced several definitions of weak commutativity such as: Compatible mappings, Compatible mappings of type $(A), (B), (C)$ and (P), and some others whose systematic comparison and illustration (up to 2001) is available in Murthy [25]. But in our subsequent work, we use the most minimal but natural of these conditions namely 'weak compatibility' which is due to Jungck [19].

With a view to improve the continuity requirement in fixed point theorems, Kannan [21] proved a result for self mappings (without continuity) and shown that there do exist mappings which are discontinuous at their fixed points. However, common fixed point theorems without any continuity

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requirement were established by Singh and Mishra [31] and also Pant [26]. Here, we opt a method which is essentially inspired by Singh and Mishra [31].

The tradition of improving contraction conditions in fixed and common fixed point theorems is still in fashion and continues to be most effective tool to improve such results. For an extensive collection of contraction conditions one can be referred to Rhoades [28] and references cited therein. Most recently, with a view to accommodate many contraction conditions in one go, Popa [27] introduced implicit functions which are proving fruitful due to their unifying power besides admitting new contraction conditions. In this paper, we also introduce an implicit function to prove our results because of their versatility of deducing several contraction conditions in one go. This fact will be substantiated by furnishing several examples in Section 2.

Definition 1.1. [18] *A pair (A, S) of self mappings of a metric space (X, d) is said to be compatible if* $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, *whenever $\{x_n\}$ is a sequence in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t, \text{ for some } t \in X.$$

Definition 1.2. *A pair (A, S) of self mappings of a metric space (X, d) is said to be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t, \text{ for some } t \in X$$

but $\lim_{n\to\infty} d(ASx_n, SAx_n)$ *is either nonzero or nonexistent.*

Motivated by the notions of compatibility and noncompatibility, Aamri and Moutawakil [1] defined the generalization of these notions as follows.

Definition 1.3. [1] *A pair (A, S) of self mappings of a metric space (X, d) is said to satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t, \text{ for some } t \in X.$$

Clearly a pair of noncompatible mappings satisfies the property $(E.A)$.

Definition 1.4. [23] *Two pairs (A, S) and (B, T) of self mappings of a metric space (X, d) are said to satisfy the common property $(E.A)$ if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t, \text{ for some } t \in X.$$

Definition 1.5. [19] *A pair (A, S) of self mappings of a nonempty set X is said to be weakly compatible if $Ax = Sx$ for some $x \in X$ implies $ASx = SAx$.*

The main purpose of this paper is to define a new implicit function to enhance the domain of applicability which includes several known contraction conditions such as: Ćirić quasi-contraction, generalized contraction, ϕ-type contraction, rational inequality and others besides admitting new unknown contraction conditions which is used to prove a general common fixed point theorem for two pairs of weakly compatible self mappings satisfying the common property $(E.A)$. In the process, many known results are enriched and improved. Some related results are also derived besides furnishing illustrative examples.

2. Implicit functions

Now we define an implicit function and furnish a variety of examples which include most of the well known contractions of the existing literature besides admitting several new ones. Here it is fascinating to note that some of the presented examples are of nonexpansive type (e.g. Examples 2.16 and 2.19) and Lipschitzian type (e.g. Examples 2.12, 2.14 and 2.15). Here, it may be pointed out that most of the following examples do not meet the requirements of implicit function due to Popa [27]. In order to describe our implicit function, let Ψ be the family of lower semi-continuous functions $F : \Re_+^6 \to \Re$ satisfying the following conditions.

$(F_1) : F(t,0,t,0,0,t) > 0$, for all $t > 0$,

$(F_2) : F(t,0,0,t,t,0) > 0$, for all $t > 0$,

$(F_3) : F(t,t,0,0,t,t) > 0$, for all $t > 0$.

Example 2.1. Define $F(t_1, t_2, \cdots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}, \quad \text{where } k \in [0,1).$$

(F_1): $F(t,0,t,0,0,t) = t(1-k) > 0$, for all $t > 0$,

(F_2): $F(t,0,0,t,t,0) = t(1-k) > 0$, for all $t > 0$,

(F_3): $F(t,t,0,0,t,t) = t(1-k) > 0$, for all $t > 0$.

Example 2.2. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - k \max\{t_2, t_3, t_3 t_5, t_4 t_6\}, \quad \text{where } k \in [0,1).$$

(F_1): $F(t,0,t,0,0,t) = t(1-k) > 0$, for all $t > 0$,

(F_2): $F(t,0,0,t,t,0) = t > 0$, for all $t > 0$,

(F_3): $F(t,t,0,0,t,t) = t(1-k) > 0$, for all $t > 0$.

Example 2.3. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - k[\max\{t_2^2, t_3 t_4, t_5 t_6, t_3 t_5, t_4 t_6\}]^{\frac{1}{2}}, \quad \text{where } k \in [0,1).$$

Example 2.4. Define $F(t_1, t_2, \dots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \dots, t_6) = t_1 - \alpha[\beta \max\{t_2, t_3, t_4, t_5, t_6\} + (1-\beta) \\ (\max\{t_2^2, t_3t_4, t_5t_6, t_3t_6, t_4t_5\})^{\frac{1}{2}}]$$

where $\alpha \in [0, 1)$ and $\beta \geq 0$.

Example 2.5. Define $F(t_1, t_2, \dots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \dots, t_6) = t_1^2 - \alpha \max\{t_2^2, t_3^2, t_4^2\} - \beta \max\{t_3t_5, t_4t_6\} - \gamma t_5t_6$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \gamma < 1$.

Example 2.6. Define $F(t_1, t_2, \dots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \dots, t_6) = (1 + \alpha t_2)t_1 - \alpha \max\{t_3t_4, t_5t_6\} - \beta \max\{t_2, t_3, t_4, t_5, t_6\}$$

where $\alpha \geq 0$ and $\beta \in [0, 1)$.

Example 2.7. Define $F(t_1, t_2, \dots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \dots, t_6) = t_1 - \alpha t_2 - \beta \max\{t_3, t_4\} - \gamma \max\{t_3 + t_4, t_5 + t_6\}$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + 2\gamma < 1$.

Example 2.8. Define $F(t_1, t_2, \dots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \dots, t_6) = t_1 - \phi(\max\{t_2, t_3, t_4, t_5, t_6\})$$

where $\phi : \Re^+ \to \Re$ is an upper semi-continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$.

Example 2.9. Define $F(t_1, t_2, \dots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \dots, t_6) = t_1 - \phi(t_2, t_3, t_4, t_5, t_6)$$

where $\phi : \Re_+^5 \to \Re$ is an upper semi-continuous function such that $\max\{\phi(0, t, 0, 0, t), \phi(0, 0, t, t, 0), \phi(t, 0, 0, t, t)\} < t$ for each $t > 0$.

Example 2.10. Define $F(t_1, t_2, \dots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \dots, t_6) = t_1^2 - \phi(t_2^2, t_3t_4, t_5t_6, t_3t_6, t_4t_5)$$

where $\phi : \Re_+^5 \to \Re$ is an upper semi-continuous function such that $\max\{\phi(0, 0, 0, t, 0), \phi(0, 0, 0, 0, t), \phi(t, 0, t, 0, 0)\} < t$ for each $t > 0$.

Example 2.11. Define $F(t_1, t_2, \dots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \dots, t_6) = \begin{cases} t_1 - \alpha t_2 - \beta \frac{t_3^2 + t_4^2}{t_3 + t_4} - \gamma(t_5 + t_6), & \text{if } t_3 + t_4 \neq 0 \\ t_1, & \text{if } t_3 + t_4 = 0 \end{cases}$$

where $\alpha, \beta, \gamma \geq 0$ and $\beta + \gamma < 1$.

Example 2.12. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = \begin{cases} t_1^p - kt_2^p - \frac{t_3 t_4^p + t_5 t_6^p}{t_3 + t_4}, & \text{if } t_3 + t_4 \neq 0 \\ t_1, & \text{if } t_3 + t_4 = 0 \end{cases}$$

where $p \geq 1$ and $0 \leq k < \infty$.

Example 2.13. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = \begin{cases} t_1 - \alpha t_2 - \beta \frac{t_5^2 + t_6^2}{t_5 + t_6} - \gamma(t_3 + t_4), & \text{if } t_5 + t_6 \neq 0 \\ t_1, & \text{if } t_5 + t_6 = 0 \end{cases}$$

where $\alpha, \beta, \gamma \geq 0$ and $\beta + \gamma < 1$.

Example 2.14. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = \begin{cases} t_1 - kt_2 - \frac{t_3 t_4 + t_5 t_6}{t_5 + t_6}, & \text{if } t_5 + t_6 \neq 0 \\ t_1, & \text{if } t_5 + t_6 = 0 \end{cases}$$

where $0 \leq k < \infty$.

Example 2.15. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = \begin{cases} t_1 - kt_2 - \frac{t_3 t_4 + t_5 t_6}{t_3 + t_4} - \frac{t_3 t_5 + t_4 t_6}{t_5 + t_6}, & \text{if } t_3 + t_4 \neq 0 \\ & \text{and } t_5 + t_6 \neq 0 \\ t_1, & \text{if } t_3 + t_4 = 0 \\ & \text{or } t_5 + t_6 = 0 \end{cases}$$

where $0 \leq k < \infty$.

Example 2.16. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - \frac{t_3 t_4 + t_5 t_6}{1 + t_2}.$$

Example 2.17. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - \alpha t_2 - \beta \frac{t_3 + t_4}{1 + t_5 t_6}, \quad \text{where } \alpha, \beta \in [0, 1).$$

Example 2.18. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^2 - \alpha t_2^2 - \beta \frac{t_5 t_6}{1 + t_3^2 + t_4^2}$$

where $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$.

Example 2.19. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^3 - \frac{t_3^2 t_4^2 + t_5^2 t_6^2}{1 + t_2}.$$

Example 2.20. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^3 - \alpha t_1^2 t_2 - \beta t_1 t_3 t_4 - \gamma t_5^2 t_6 - \eta t_5 t_6^2$$

where $\alpha, \beta, \gamma, \eta \geq 0$ and $\alpha + \gamma + \eta < 1$.

Since verification of requirements (F_1, F_2 and F_3) for Examples 2.3-2.20 are easy, details are not included.

3. Main results

We begin with the following observation.

Lemma 3.1. *Let A, B, S and T be self mappings of a metric space (X, d) such that*

(a) *the pair (A, S) (or (B, T)) satisfies the property $(E.A)$,*

(b) *$A(X) \subset T(X)$ (or $B(X) \subset S(X)$), and*

(c) *for all $x, y \in X$ and $F \in \Psi$*

$$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)) \leq 0. \tag{3.1.1}$$

Then the pairs (A, S) and (B, T) satisfy the common property $(E.A)$.

Proof. If the pair (A, S) enjoys property $(E.A)$, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t, \quad \text{for some } t \in X.$$

Since $A(X) \subset T(X)$, hence for each $\{x_n\}$ there exists $\{y_n\}$ in X such that $Ax_n = Ty_n$. Therefore, $\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = t$. Thus, in all we have $Ax_n \to t, Sx_n \to t$ and $Ty_n \to t$. Now, we assert that $By_n \to t$. If not, then using (3.1.1), we have

$$F(d(Ax_n, By_n), d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ty_n), \\ d(Sx_n, By_n), d(Ty_n, Ax_n)) \leq 0$$

which on making $n \to \infty$, reduces to

$$F(d(t, By_n), 0, 0, d(By_n, t), d(t, By_n), 0) \leq 0$$

a contradiction to (F_2). Hence $By_n \to t$ which shows that the pairs (A, S) and (B, T) satisfy the common property $(E.A)$. □

Remark 3.1. The converse of Lemma 3.1 is not true in general. For a counter example, one can see Example 4.1.

Now, we state and prove our main result for two pairs of weakly compatible mappings satisfying an implicit function.

Theorem 3.1. *Let A, B, S and T be self mappings of a metric space (X, d) which satisfy inequality* (3.1.1)*. Suppose that*

(a) *pairs (A, S) and (B, T) enjoy the common property $(E.A)$,*

(b) *$S(X)$ and $T(X)$ are closed subsets of X.*

Then the pair (A, S) as well as (B, T) have a coincidence point. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Since the pairs (A, S) and (B, T) enjoy common property $(E.A)$, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t, \quad \text{for some } t \in X.$$

If $S(X)$ is a closed subset of X, then $\lim_{n\to\infty} Sx_n = t \in S(X)$. Therefore, there exists a point $u \in X$ such that $Su = t$. Now we assert that $Au = Su$. If not, then using (3.1.1), we have

$$F(d(Au, By_n), d(Su, Ty_n), d(Au, Su), d(By_n, Ty_n), \\ d(Su, By_n), d(Ty_n, Au)) \leq 0$$

which on making $n \to \infty$, reduces to

$$F(d(Au, t), d(Su, t), d(Au, Su), d(t, t), d(Su, t), d(t, Au)) \leq 0$$

or

$$F(d(Au, Su), 0, d(Au, Su), 0, 0, d(Su, Au)) \leq 0$$

a contradiction to (F_1). Hence $Au = Su$. Therefore, u is a coincidence point of the pair (A, S).

If $T(X)$ is a closed subset of X, then $\lim_{n\to\infty} Ty_n = t \in T(X)$. Therefore, there exists a point $w \in X$ such that $Tw = t$. Now we assert that $Bw = Tw$. If not, then again using (3.1.1), we have

$$F(d(Ax_n, Bw), d(Sx_n, Tw), d(Ax_n, Sx_n), d(Bw, Tw), \\ d(Sx_n, Bw), d(Tw, Ax_n)) \leq 0$$

which on making $n \to \infty$, reduces to

$$F(d(t, Bw), d(t, Tw), d(t, t), d(Bw, Tw), d(t, Bw), d(Tw, t)) \leq 0$$

or

$$F(d(Tw, Bw), 0, 0, d(Bw, Tw), d(Tw, Bw), 0) \leq 0$$

a contradiction to (F_2). Hence $Bw = Tw$, which shows that w is a coincidence point of the pair (B, T).

Since the pair (A, S) is weakly compatible and $Au = Su$, hence $At = ASu = SAu = St$. Now we assert that t is a common fixed point of the pair (A, S). Suppose that $At \neq t$, then using (3.1.1), we have

$$F(d(At, Bw), d(St, Tw), d(At, St), d(Bw, Tw), d(St, Bw), d(Tw, At)) \leq 0$$

or

$$F(d(At, t), d(At, t), 0, 0, d(At, t), d(t, At)) \leq 0$$

a contradiction to (F_3).

Also the pair (B, T) is weakly compatible and $Bw = Tw$, then $Bt = BTw = TBw = Tt$. Suppose that $Bt \neq t$, then using (3.1.1), we get

$$F(d(Au, Bt), d(Su, Tt), d(Au, Su), d(Bt, Tt), d(Su, Bt), d(Tt, Au)) \leq 0$$

or

$$F(d(Bt, t), d(Bt, t), 0, 0, d(Bt, t), d(t, Bt)) \leq 0$$

a contradiction to (F_3). Therefore, $Bt = t$ which shows that t is a common fixed point of the pair (B, T). Hence t is a common fixed point of both the pairs (A, S) and (B, T). Uniqueness of common fixed point is an easy consequence of inequality (3.1.1) in view of condition (F_3). This completes the proof. □

Theorem 3.2. *The conclusions of Theorem* 3.1 *remain true if the condition* (b) *of Theorem* 3.1 *is replaced by following.*

(b′) $\overline{A(X)} \subset T(X)$ *and* $\overline{B(X)} \subset S(X)$.

As a corollary of Theorem 3.2, we can have the following result which is also a variant of Theorem 3.1.

Corollary 3.1. *The conclusions of Theorems* 3.1 *and* 3.2 *remain true if the conditions* (b) *and* (b′) *are replaced by following.*

(b″) $A(X)$ *and* $B(X)$ *are closed subsets of* X *provided* $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$.

Theorem 3.3. *Let* A, B, S *and* T *be self mappings of a metric space* (X, d) *satisfying inequality* (3.1.1)*. Suppose that*

(a) *the pair* (A, S) *(or* (B, T)*) has property* $(E.A)$,

(b) $A(X) \subset T(X)$ *(or* $B(X) \subset S(X)$*), and*

(c) $S(X)$ *(or* $T(X)$*) is closed subset of* X.

Then the pairs (A, S) *and* (B, T) *have coincidence point. If the pairs* (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Proof. In view of Lemma 3.1, the pairs (A, S) and (B, T) satisfy the common property $(E.A)$, i.e. there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t \in X.$$

If $S(X)$ is a closed subset of X, then on the lines of Theorem 3.1, the pair (A, S) has coincidence point, say u, i.e. $Au = Su$. Since $A(X) \subset T(X)$ and $Au \in A(X)$, there exists $w \in X$ such that $Au = Tw$. Now we assert that $Bw = Tw$. If not, then using (3.1.1), we have

$$F(d(Ax_n, Bw), d(Sx_n, Tw), d(Ax_n, Sx_n), d(Bw, Tw),\\ d(Sx_n, Bw), d(Tw, Ax_n)) \leq 0$$

which on making $n \to \infty$, reduces to

$$F(d(t, Bw), d(t, Tw), d(t, t), d(Bw, Tw), d(t, Bw), d(Tw, t)) \leq 0$$

or

$$F(d(Tw, Bw), 0, 0, d(Bw, Tw), d(Tw, Bw), 0) \leq 0$$

a contradiction to (F_2). Hence $Bw = Tw$, which shows that w is a coincidence point of the pair (B, T). The rest of the proof can be completed on the lines of Theorem 3.1. □

By choosing A, B, S and T suitably, one can deduce corollaries for a pair as well as for a triod of mappings. The details of two possible corollaries for a triod of mappings are not included. As a sample, we outline the following natural result for a pair of self mappings.

Corollary 3.2. *Let A and S be self mappings of a metric space (X, d). Suppose that*

(a) *A and S have property $(E.A)$,*

(b) *for all $x, y \in X$ and $F \in \Psi$*

$$F(d(Ax, Ay), d(Sx, Sy), d(Ax, Sx), d(Ay, Sy), d(Sx, Ay), d(Sy, Ax)) \leq 0 \tag{3.1.2}$$

(c) *$S(X)$ is a closed subset of X.*

Then A and S have a coincidence point. Moreover, if the pair (A, S) is weakly compatible, then A and S have a unique common fixed point.

Corollary 3.3. *The conclusions of Theorem 3.1 remain true if inequality (3.1.1) is replaced by one of the following contraction conditions. For all $x, y \in X$,*

(a_1)

$$d(Ax, By) \leq k \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)\}$$

where $k \in [0, 1)$.

(a_2)

$$d(Ax, By) \leq k \max\{d(Sx, Ty), d(Ax, Sx), d(Ax, Sx)d(Sx, By), d(By, Ty)d(Ty, Ax)\}$$

where $k \in [0, 1)$.

(a_3)

$$d(Ax, By) \leq k[\max\{d^2(Sx, Ty), d(Ax, Sx)d(By, Ty), d(Sx, By)d(Ty, Ax), d(Ax, Sx)d(Sx, By), d(By, Ty)d(Ty, Ax)\}]^{\frac{1}{2}}$$

where $k \in [0, 1)$.

(a_4)

$$d(Ax, By) \leq \alpha[\beta \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)\} + (1-\beta)(\max\{d^2(Sx, Ty), d(Ax, Sx)d(By, Ty), d(Sx, By)d(Ty, Ax), d(Ax, Sx)d(Ty, Ax), d(By, Ty)d(Sx, By)\})^{\frac{1}{2}}]$$

where $\alpha \in [0, 1)$ *and* $\beta \geq 0$.

(a_5)

$$d^2(Ax, By) \leq \alpha \max\{d^2(Sx, Ty), d^2(Ax, Sx), d^2(By, Ty)\} + \beta \max\{d(Ax, Sx)d(Sx, By), d(By, Ty)d(Ty, Ax)\} + \gamma d(Sx, By)d(Ty, Ax)$$

where $\alpha, \beta, \gamma \geq 0$ *and* $\alpha + \gamma < 1$.

(a_6)

$$(1 + \alpha d(Sx, Ty))d(Ax, By) \leq \alpha \max\{d(Ax, Sx)d(By, Ty), d(Sx, By) d(Ty, Ax)\} + \beta \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)\}$$

where $\alpha \geq 0$ *and* $\beta \in [0, 1)$.

(a_7)

$$d(Ax, By) \leq \alpha d(Sx, Ty) + \beta \max\{d(Ax, Sx), d(By, Ty)\} + \gamma \max\{d(Ax, Sx) + d(By, Ty), d(Sx, By) + d(Ty, Ax)\}$$

where $\alpha, \beta, \gamma \geq 0$ *and* $\alpha + \beta + 2\gamma < 1$.

(a_8)

$$d(Ax, By) \leq \phi(\max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)\})$$

where $\phi : \Re^+ \to \Re$ *is an upper semi-continuous function such that* $\phi(0) = 0$ *and* $\phi(t) < t$ *for all* $t > 0$.

(a_9)

$$d(Ax, By) \leq \phi(d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax))$$

where $\phi : \Re^5_+ \to \Re$ *is an upper semi-continuous function such that* $\max\{\phi(0, t, 0, 0, t), \phi(0, 0, t, t, 0), \phi(t, 0, 0, t, t)\} < t$ *for each* $t > 0$.

(a_{10})

$$d^2(Ax, By) \leq \phi(d^2(Sx, Ty), d(Ax, Sx)d(By, Ty), d(Sx, By)d(Ty, Ax), d(Ax, Sx)d(Ty, Ax), d(By, Ty)d(Sx, By))$$

where $\phi : \Re^5_+ \to \Re$ *is an upper semi-continuous function such that* $\max\{\phi(0, 0, 0, t, 0), \phi(0, 0, 0, 0, t), \phi(t, 0, t, 0, 0)\} < t$ *for each* $t > 0$.

In the following contraction conditions, we denote $D = d(Ax, Sx) + d(By, Ty)$ *and* $D_1 = d(Sx, By) + d(Ty, Ax)$.

(a_{11})

$$d(Ax, By) \leq \begin{cases} \alpha d(Sx, Ty) + \beta \frac{d^2(Ax,Sx)+d^2(By,Ty)}{d(Ax,Sx)+d(By,Ty)} & \\ +\gamma(d(Sx, By) + d(Ty, Ax)), & \text{if } D \neq 0 \\ 0, & \text{if } D = 0 \end{cases}$$

where $\alpha, \beta, \gamma \geq 0$ *and* $\beta + \gamma < 1$.

(a_{12})

$$d^p(Ax, By) \leq \begin{cases} kd^p(Sx, Ty) + \frac{d(Ax,Sx)d^p(By,Ty)+d(Sx,By)d^p(Ty,Ax)}{d(Ax,Sx)+d(By,Ty)} & \text{if } D \neq 0 \\ 0, & \text{if } D = 0 \end{cases}$$

where $p \geq 1$ *and* $0 \leq k < \infty$.

(a_{13})

$$d(Ax, By) \leq \begin{cases} \alpha d(Sx, Ty) + \beta \frac{d^2(Sx,By)+d^2(Ty,Ax)}{d(Sx,By)+d(Ty,Ax)} & \\ +\gamma(d(Ax, Sx) + d(By, Ty)), & \text{if } D_1 \neq 0 \\ 0, & \text{if } D_1 = 0 \end{cases}$$

where $\alpha, \beta, \gamma \geq 0$ *and* $\beta + \gamma < 1$.

(a_{14})

$$d(Ax, By) \leq \begin{cases} kd(Sx, Ty) + \frac{d(Ax,Sx)d(By,Ty)+d(Sx,By)d(Ty,Ax)}{d(Sx,By)+d(Ty,Ax)} & \text{if } D_1 \neq 0 \\ 0, & \text{if } D_1 = 0 \end{cases}$$

where $0 \leq k < \infty$.

(a_{15})

$$d(Ax,By) \leq \begin{cases} kd(Sx,Ty) \\ +\frac{d(Ax,Sx)d(By,Ty)+d(Sx,By)d(Ty,Ax)}{d(Ax,Sx)+d(By,Ty)} \\ +\frac{d(Ax,Sx)d(Sx,By)+d(By,Ty)d(Ty,Ax)}{d(Sx,By)+d(Ty,Ax)} & \textit{if } D \neq 0, D_1 \neq 0 \\ 0, & \textit{if } D = 0 \textit{ or } D_1 = 0 \end{cases}$$

where $0 \leq k < \infty$.

(a_{16})

$$d(Ax,By) \leq \frac{d(Ax,Sx)d(By,Ty)+d(Sx,By)d(Ty,Ax)}{1+d(Sx,Ty)}$$

(a_{17})

$$d(Ax,By) \leq \alpha d(Sx,Ty) + \beta \frac{d(Ax,Sx)+d(By,Ty)}{1+d(Sx,By)d(Ty,Ax)}$$

where $\alpha, \beta \in [0,1)$.

(a_{18})

$$d^2(Ax,By) \leq \alpha d^2(Sx,Ty) + \beta \frac{d(Sx,By)d(Ty,Ax)}{1+d^2(Ax,Sx)+d^2(By,Ty)}$$

where $\alpha, \beta \geq 0$ *and* $\alpha + \beta < 1$.

(a_{19})

$$d^3(Ax,By) \leq \frac{d^2(Ax,Sx)d^2(By,Ty)+d^2(Sx,By)d^2(Ty,Ax)}{1+d(Sx,Ty)}$$

(a_{20})

$$\begin{aligned} d^3(Ax,By) \leq{}& \alpha d^2(Ax,By)d(Sx,Ty) + \beta d(Ax,By)d(Ax,Sx)d(By,Ty) \\ &+ \gamma d^2(Sx,By)d(Ty,Ax) + \eta d(Sx,By)d^2(Ty,Ax) \end{aligned}$$

where $\alpha, \gamma, \eta, \beta \geq 0$ *and* $\alpha + \gamma + \eta < 1$.

Proof. The proof follows from Theorem 3.1 and Examples 2.1-2.20. □

Remark 3.2. Corollaries corresponding to contraction conditions (a_1) to (a_{20}) are new results as these never require any conditions on the containment of ranges. Some contraction conditions (e.g. $a_1, a_4, a_6 - a_{15}$) in the above corollary are well known and generalize relevant results from [2-15,17,18,20,22-24,26,30,32] while some others are new ones (e.g. a_2, a_3, a_5, $a_{16} - a_{20}$).

As an application of Theorem 3.1, we have the following result for four finite families of self mappings.

Theorem 3.4. *Let* $\{A_1, A_2, \ldots, A_m\}$, $\{B_1, B_2, \ldots, B_p\}$, $\{S_1, S_2, \ldots, S_n\}$ *and* $\{T_1,\ T_2, \ldots, T_q\}$ *be four finite families of self mappings of a metric space* (X, d) *with* $A = A_1A_2 \ldots A_m$, $B = B_1B_2 \ldots B_p$, $S = S_1S_2, \ldots S_n$ *and* $T = T_1T_2 \ldots T_q$ *satisfying condition (3.1.1) and the pairs* (A, S) *and* (B, T) *share common property* $(E.A)$. *If* $S(X)$ *and* $T(X)$ *are closed subsets of* X, *then*

(a) *the pair* (A, S) *has a coincidence point,*

(b) *the pair* (B, T) *has a coincidence point.*

Moreover, if $A_iA_j = A_jA_i, B_kB_l = B_lB_k, S_rS_s = S_sS_r, T_tT_u = T_uT_t$, $A_iB_k = B_kA_i$ *and* $S_rT_t = T_tS_r$ *for all* $i, j \in I_1 = \{1, 2, \ldots, m\}, k, l \in I_2 = \{1, 2, \ldots, p\}, r, s \in I_3 = \{1, 2, \ldots, n\}$ *and* $t, u \in I_4 = \{1, 2, \ldots, q\}$, *then (for all* $i \in I_1, k \in I_2, r \in I_3$ *and* $t \in I_4$*)* A_i, B_k, S_r *and* T_t *have a common fixed point.*

Proof. Proof follows on the lines of result due to Imdad et al. [16, Theorem 2.2]. □

By setting $A_1 = A_2 = \cdots = A_m = G, B_1 = B_2 = \cdots = B_p = H, S_1 = S_2 = \cdots = S_n = I$ and $T_1 = T_2 = \cdots = T_q = J$ in Theorem 3.4, we deduce the following:

Corollary 3.4. *Let* G, H, I *and* J *be self mappings of a metric space* (X, d), *pairs* (G^m, I^n) *and* (H^p, J^q) *have common property* $(E.A)$ *and satisfying the condition*

$$F(d(G^m x, H^p y), d(I^n x, J^q y), d(G^m x, I^n x), d(H^p y, J^q y), d(I^n x, H^p y), d(J^q y, G^m x)) \leq 0$$

for all $x, y \in X$ *and* $F \in \Psi$ *where* m, n, p *and* q *are fixed positive integers. If* $I^n(X)$ *and* $J^q(X)$ *are closed subsets of* X, *then* G, H, I *and* J *have a unique common fixed point provided* $GI = IG$ *and* $HJ = JH$.

Remark 3.3. By restricting the four families as $\{A_1, A_2\}, \{B_1, B_2\}, \{S_1\}$ and $\{T_1\}$ in Theorem 3.4, we deduce a substantial but partial generalization of the main results of Imdad and Khan [12,13] as such a result will deduce stronger commutativity condition besides relaxing continuity requirements and weakening completeness requirement of the space to the closedness of subspaces.

Remark 3.4. Corollary 3.4 is a slight but partial generalization of Theorem 3.1 as the commutativity requirements (i.e. $GI = IG$ and $HJ = JH$) in this corollary are stronger as compared to weak compatibility in Theorem 3.1.

Remark 3.5. Results similar to Corollary 3.3 can be derived from Theorems 3.2-3.3 and Corollaries 3.1, 3.2 and 3.4. For the sake of brevity, we have not included the details.

4. Illustrative examples

Now we furnish examples demonstrating the validity of the hypotheses and degree of generality of our results proved in this paper over the majority of earlier results proved till date with rare possible exceptions.

Example 4.1. Consider $X = [-1, 1]$ equipped with the usual metric. Define self mappings A, B, S and T on X as

$A(-1) = A1 = 3/5,\ Ax = x/4,\ -1 < x < 1,$
$B(-1) = B1 = 3/5,\ Bx = -x/4,\ -1 < x < 1,$
$S(-1) = 1/2,\ Sx = x/2,\ -1 < x < 1,$ and $S1 = -1/2$, and
$T(-1) = -1/2,\ Tx = -x/2,\ -1 < x < 1,$ and $T1 = 1/2$.

Consider sequences $\{x_n = \frac{1}{n}\}$ and $\{y_n = \frac{-1}{n}\}$ in X. Clearly,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 0$$

which shows that pairs (A, S) and (B, T) satisfy the common property $(E.A)$. Define a continuous implicit function $F : \Re_+^6 \to \Re$ such that $F(t_1, t_2, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}$ where $k \in [0, 1)$ and $F \in \Psi$. By a routine calculation, one can verify the inequality (3.1.1) with $k = \frac{1}{2}$. Also, $A(X) = B(X) = \{\frac{3}{5}\} \cup (\frac{-1}{4}, \frac{1}{4}) \not\subset S(X) = T(X) = [\frac{-1}{2}, \frac{1}{2}]$. Therefore, all the conditions of Theorem 3.1 are satisfied and 0 is a unique common fixed point of the pairs (A, S) and (B, T) which is their coincidence point as well.

Here it is worth noting that none of the theorems (with rare possible exceptions) can be used in the context of this example as Theorem 3.1 never requires any condition on the containment of ranges of mappings while completeness condition is replaced by closedness of subspaces. Moreover, the continuity requirements of involved mappings are completely relaxed whereas all earlier theorems (prior to 1997) require the continuity of at least one involved mapping.

Now, we furnish an example which presents a situation applicable to Theorems 3.1, 3.2 and 3.3.

Example 4.2. Consider $X = [2, 20]$ equipped with the usual metric. Define self mappings A, B, S and T on X as

$Ax = 2,\ x \in \{2\} \cup (5, 20],\ Ax = 4,\ 2 < x \leq 5,$
$Bx = 2,\ x \in \{2\} \cup (5, 20],\ Bx = 3,\ 2 < x \leq 5,$

$S2 = 2,\ Sx = 8,\ 2 < x \leq 5,\ Sx = (x+1)/3,\ x > 5$ and
$T2 = 2,\ Tx = 12 + x,\ 2 < x \leq 5,\ Tx = x - 3,\ x > 5.$

Clearly, both the pairs (A, S) and (B, T) satisfy the common property $(E.A)$ as there exist two sequences $\{x_n = 5 + \frac{1}{n}\}, \{y_n = 5 + \frac{1}{n}\} \in X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 2.$$

Also $A(X) = \{2, 4\} \subset [2, 17] = T(X)$ and $B(X) = \{2, 3\} \subset [2, 7] \cup \{8\} = S(X)$. Define $F(t_1, t_2, \ldots, t_6) : \Re^6_+ \to \Re$ with $t_3 + t_4 \neq 0$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - \alpha t_2 - \beta \left[\frac{t_3^2 + t_4^2}{t_3 + t_4}\right] - \gamma[t_5 + t_6]$$

where $\alpha, \beta, \gamma \geq 0$ with at least one is nonzero and $\beta + \gamma < 1$.

By a routine calculation one can verify that contraction condition (3.1.1) is satisfied for $\alpha = \gamma = \frac{1}{4}$ and $\beta = \frac{1}{5}$. If $x, y \in \{2\} \cup (5, 20]$, then $d(Ax, By) = 0$ and verification is trivial. If $x \in (2, 5]$ and $y > 5$, then

$$\begin{aligned}
&\alpha d(Sx, Ty) + \beta \left[\frac{d^2(Sx, Ax) + d^2(Ty, By)}{d(Sx, Ax) + d(Ty, By)}\right] + \gamma[d(Sx, By) + d(Ty, Ax)] \\
&\quad = \frac{1}{4}|y - 11| + \frac{1}{5}\left[\frac{4^2 + |y-5|^2}{4 + |y-5|}\right] + \frac{1}{4}[6 + |y - 7|] \\
&\quad \geq \begin{cases} \frac{4}{5} + \frac{1}{4}(24 - 2y) > 2 = d(Ax, By), & \text{if } y \in (5, 7] \\ \frac{4}{5} + \frac{10}{4} = \frac{33}{10} > 2 = d(Ax, By), & \text{if } y \in (7, 11] \\ \frac{4}{5} + \frac{1}{4}(2y - 12) > 2 = d(Ax, By), & \text{if } y > 11. \end{cases}
\end{aligned}$$

Similarly, one can verify the other cases. One may note that the pairs (A, S) and (B, T) commute at 2 which is their common coincidence point. All the needed pairwise commutativity at coincidence point 2 are immediate. Thus all the conditions of Theorems 3.1, 3.2 and 3.3 are satisfied and 2 is the unique common fixed point of A, B, S and T. Here one may notice that all the mappings in this example are even discontinuous at their unique common fixed point 2.

Example 4.2 may create an impression that Theorems 3.1, 3.2 and 3.3 are not different results. In what follows, we show that these results can be situationally useful, i.e. there do exist situations when one theorem is applicable whereas others are not. In order to substantiate this view point, we furnish the following examples.

Example 4.3. In the setting of Example 4.2 retain the same A, B, T and implicit function F and modify S as follows.

$$S2 = 2,\ S20 = 2,\ Sx = 8,\ 2 < x \leq 5,\ Sx = (x+1)/3,\ 5 < x < 20.$$

Clearly, $S(X) = [2, 7) \cup \{8\}$ which is not a closed subset of X. Here, Theorems 3.2 and 3.3 are applicable but not Theorem 3.1.

Example 4.4. In the setting of Example 4.2 retain the same A, B and implicit function F and modify S and T as follows.

$$S2 = 2, \quad S20 = 2, \quad Sx = 8, \quad 2 < x \leq 5, \quad Sx = (x+1)/3, \quad 5 < x < 20,$$
$$T2 = 2, \quad T20 = 2, \quad Tx = 12 + x, \quad 2 < x \leq 5, \quad Tx = x - 3, \ \ 5 < x < 20.$$

Clearly, $S(X) = [2, 7) \cup \{8\}$ and $T(X) = [2, 17)$ which are not closed subsets of X. Here Theorem 3.2 is applicable but not Theorems 3.1 and 3.3.

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References

[1] M. Aamri and D. El Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl., 270 (2002), 181–188.
[2] A. Ahmad and M. Imdad, *A common fixed point theorem for four mappings satisfying a rational inequality*, Publ. Math. Debrecen, 41 (3–4) (1992), 181–187.
[3] R. Chugh and S. Kumar, *Common fixed points for weakly compatible maps*, Proc. Indian Acad. Sci. (Math. Sci.), 111 (2) (2001), 241–247.
[4] Lj. B. Ćirić, *Generalized contractions and fixed point theorems*, Publ. Inst. Math., 12 (26) (1971), 19–26.
[5] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., 45 (1974), 267–273.
[6] J. Danes, *Two fixed point theorems in topological and metric spaces*, Bull. Austral. Math. Soc., 14 (2) (1976), 259–265.
[7] B. Fisher, *Common fixed point and constant mappings satisfying a rational inequality*, Math. Sem. Notes, 6 (1978), 29–35.
[8] B. Fisher, *Mappings satisfying rational inequalities*, Nanta Math., 12 (2) (1979), 193–199.
[9] G. E. Hardy and T. D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull., 16 (1973), 201–206.
[10] S. A. Husain and V. M. Sehgal, *On common fixed points for a family of mappings*, Bull. Austral. Math. Soc., 13 (2) (1975), 261–267.
[11] M. Imdad and Javid Ali, *Pairwise coincidentally commuting mappings satisfying a rational inequality*, Italian J. Pure Appl. Math., 20 (2006), 87–96.
[12] M. Imdad and Q. H. Khan, *Six mappings satisfying a rational inequality*, Rad. Mat., 9 (1999), 251–260.
[13] M. Imdad and Q. H. Khan, *A common fixed point theorem for six mappings satisfying a rational inequality*, Indian J. Math., 44 (2002), 47–57.
[14] M. Imdad and T. I. Khan, *On common fixed points of pairwise coincidentally commuting noncontinuous mappings satisfying a rational inequality*, Bull. Calcutta Math. Soc., 93 (4) (2001), 263–268.

[15] M. Imdad, T. I. Khan and Javid Ali, *Noncontinuous noncommuting mappings satisfying a rational inequality*, Nonlinear Anal. Forum, 11 (2006), 175–183.
[16] M. Imdad, S. Kumar and M. S. Khan, *Remarks on some fixed point theorems satisfying implicit relations*, Rad. Mat., 11 (2002), 135–143.
[17] G. S. Jeong and B. E. Rhoades, *Some remarks for improving fixed point theorems for more than two maps*, Indian J. Pure Appl. Math., 28 (9) (1997), 1177–1196.
[18] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci., 9 (1986), 771–779.
[19] G. Jungck, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. Sci., 4 (2) (1996), 199–215.
[20] S. M. Kang and Y. P. Kim, *Common fixed points theorems*, Math. Japonica, 37 (6) (1992), 1031–1039.
[21] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., 60 (1968), 71–76.
[22] M. S. Khan and M. Imdad, *A common fixed point theorem for a class of mappings*, Indian J. Pure Appl. Math., 14 (1983), 1220–1227.
[23] Y. Liu, Jun Wu and Z. Li, *Common fixed points of single-valued and multi-valued maps*, Internat. J. Math. Math. Sci., 19 (2005), 3045–3055.
[24] S. Mudgal and R. K. Vats, *Common fixed points for intimate maps*, South East Asian J. Math. Math. Sci., 4 (1) (2005), 53–60.
[25] P. P. Murthy, *Important tools and possible applications of metric fixed point theory*, Nonlinear Anal.(TMA), 47 (2001), 3479–3490.
[26] R. P. Pant, *Common fixed points of four mappings*, Bull. Calcutta Math. Soc., 90 (1998), 281–286.
[27] V. Popa, *Some fixed point theorems for weakly compatible mappings*, Rad. Mat., 10 (2001), 245–252.
[28] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., 226 (1977), 257–290.
[29] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math., 32 (46) (1982), 149–153.
[30] N. Shahzad, *Invariant approximations, generalized I-contractions and R-subweakly commuting maps*, Fixed Point Theory Appl., 2 (1) (2005), 79–86.
[31] S. L. Singh and S. N. Mishra, *Remarks on Jachymski's fixed point theorems for compatible maps*, Indian J. Pure Appl. Math., 28 (5) (1997), 611–615.
[32] S. P. Singh and B. A. Meade, *On common fixed point theorems*, Bull. Austral. Math. Soc., 16 (1977), 49–53.

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