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ON (σ,τ)-DERIVATIONS OF PRIME NEAR-RINGS-II

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Abstract. Let N be a left near-ring and let σ,τ be automorphisms of N. An additive mapping $d : N \longrightarrow N$ is called a (σ,τ)-derivation on N if $d(xy) = \sigma(x)d(y) + d(x)\tau(y)$ for all $x, y \in N$. In this paper, we obtain Leibniz' formula for (σ,τ)-derivations on near-rings which facilitates the proof of the following result: Let $n \geq 1$ be an integer, N be n-torsion free, and d a (σ,τ)-derivation on N with $d^n(N) = \{0\}$. If both σ and τ commute with d^n for all $n \geq 1$, then $d(Z) = \{0\}$. Further, besides proving some more related results, we investigate commutativity of N satisfying either of the properties: $d([x, y]) = 0$, or $d(xoy) = 0$, for all $x, y \in N$.

1. Introduction

Throughout the paper N will denote a zero-symmetric left near-ring with multiplicative centre Z. A near-ring N is said to be prime if $aNb = \{0\}$ implies that $a = 0$ or $b = 0$. An element x of N is said to be distributive if $(y + z)x = yx + zx$ for all $x, y, z \in N$. A near-ring N is called zero-symmetric if $0x = 0$, for all $x \in N$ (recall that left distributivity yields $x0 = 0$). For any $x, y \in R$, as usual $[x, y] = xy - yx$ and $xoy = xy + yx$ will denote the well-known Lie and Jordan products respectively, while the symbol (x, y) will denote the additive commutator $x + y - x - y$. An additive endomorphism d of N is called a derivation on N if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$ or equivalently, as noted in [8, Proposition 1], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. An element $x \in N$ for which $d(x) = 0$ is called a constant. Let σ,τ be automorphisms on N. Following [2], an additive endomorphism $d : N \longrightarrow N$ is called a (σ,τ)-derivation if there exist automorphisms $\sigma,\tau : N \longrightarrow N$ such that $d(xy) = \sigma(x)d(y) + d(x)\tau(y)$ for all $x, y \in N$. In case $\sigma = 1$, the identity mapping, d is called a τ-derivation. Similarly if $\tau = 1, d$ is called a σ-derivation.

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There are several results asserting that prime near-rings with certain constrained derivations have ring-like behavior. Recently many authors have studied commutativity of prime and semi-prime rings with derivations. In view of these results it is natural to look for comparable results on near-rings and this has been done in [1], [2], [3], [4], [5] and [8] etc. In order to facilitate our discussion we need to extend Leibniz' theorem for derivations in near-rings to (σ, τ)-derivations in near-rings. Though it has been obtained in [8] for derivations in near-rings, there appears to be a torsion restriction that has not be used in the proof. By simple calculations, it can be easily seen that Leibniz' rule holds even in the case of (σ, τ)-derivations in near-rings. Besides proving Leibniz' formula for (σ, τ)-derivations in near-rings, we extend some results due to Wang [8] and Bell [4] for (σ, τ)-derivations on near-rings. Some new results have also been obtained for prime near-rings. Finally, it is shown that under appropriate additional hypothesis a near-ring must be a commutative ring.

2. Preliminary results

We begin with the following known results. The proofs of the first three can be found in [2] while the fourth is essentially proved in [1].

Lemma 2.1. ([2, Lemma 2.1]). *An additive endomorphism d on a near-ring N is a (σ, τ)-derivation if and only if $d(xy) = d(x)\tau(y) + \sigma(x)d(y)$, for all $x, y \in N$.*

Lemma 2.2. ([2, Lemma 2.2]). *Let d be a (σ, τ)-derivation on a near-ring N. Then N satisfies the following partial distributive laws:*

(i) $(\sigma(x)d(y) + d(x)\tau(y))z = \sigma(x)d(y)z + d(x)\tau(y)z$, *for all* $x, y, z \in N$.

(ii) $(d(x)\tau(y) + \sigma(x)d(y))z = d(x)\tau(y)z + \sigma(x)d(y)z$, *for all* $x, y, z \in N$.

Lemma 2.3. ([2, Theorem 3.1]). *Let N be a prime near-ring admitting a non-trivial (σ, τ)-derivation d for which $d(N) \subseteq Z$. Then $(N, +)$ is abelian. Moreover, if N is 2-torsion free and σ,τ commute with d, then N is a commutative ring.*

Lemma 2.4. ([1, Proposition 2.1]). *Let N be a prime near-ring. If d is a (σ, σ)-derivation on N, then $d(Z) \subseteq Z$.*

3. Main results

The following theorem has its independent interest in the study of (σ, τ)-derivations in near-rings. In fact Leibniz' formula has already been obtained by Wang [8] for derivations in near-rings. Now, we shall extend this result for (σ, τ)-derivations in near-rings.

Theorem 3.1. *Let N be a near-ring and d a (σ,τ)-derivation on N. If both σ and τ commute with d^n, for all positive integer $n \geq 1$, then for all $x, y \in N$*

$$d^n(xy) = \sum_{r=0}^{n} \binom{n}{r} d^{n-r}(\sigma^r(x)) d^r(\tau^{n-r}(y)).$$

Proof. In view of Lemma 2.1, we have

$$d(x)\tau(y) + n\sigma(x)d(y) = n\sigma(x)d(y) + d(x)\tau(y), \text{ for all } x, y \in N. \tag{3.1}$$

This implies that

$$nd(x)\tau(y) + n\sigma(x)d(y) = n(d(x)\tau(y) + \sigma(x)d(y)), \text{ for all } x, y \in N. \tag{3.2}$$

Now, we apply induction on n. When $n = 2$, we get

$$\begin{aligned} d^2(xy) &= d(d(xy)) \\ &= d(d(x)\tau(y) + \sigma(x)d(y)) \\ &= d^2(x)\tau^2(y) + \sigma(d(x))d(\tau(y)) \\ &\qquad + d(\sigma(x))\tau(d(y)) + \sigma^2(x)d^2(y), \text{ for all } x, y \in N. \end{aligned} \tag{3.3}$$

Since σ and τ commute with d, equation (3.3) reduces to

$$d^2(xy) = d^2(x)\tau^2(y) + 2d(\sigma(x))d(\tau(y)) + \sigma^2(x)d^2(y), \text{ for all } x, y \in N.$$

This implies that,

$$d^2(xy) = \sum_{r=0}^{2} \binom{2}{r} d^{2-r}(\sigma^r(x)) d^r(\tau^{2-r}(y)), \text{ for all } x, y \in N. \tag{3.4}$$

Assume that Leibniz' rule holds for $n-1$, then

$$d^{n-1}(xy) = \sum_{r=0}^{n-1} \binom{n-1}{r} d^{n-r-1}(\sigma^r(x)) d^r(\tau^{n-r-1}(y)), \text{ for all } x, y \in N. \tag{3.5}$$

That is,

$$\begin{aligned} d^{n-1}(xy) &= d^{n-1}(x)\tau^{n-1}(y) + \cdots + \binom{n-1}{i-1} d^{n-i}(\sigma^{i-1}(x)) d^{i-1}(\tau^{n-i}(y)) \\ &\quad + \binom{n-1}{i} d^{n-i-1}(\sigma^i(x)) d^i(\tau^{n-i-1}(y)) + \cdots + \sigma^{n-1}(x)d^{n-1}(y) \\ &\qquad \text{for all } x, y \in N. \end{aligned} \tag{3.6}$$

By application of (3.2), the above expression yields that

$$\begin{aligned}
d^n(xy) &= d(d^{n-1}(xy)) \\
&= d(d^{n-1}(x)\tau^{n-1}(y) + \cdots + \binom{n-1}{i-1} d^{n-i}(\sigma^{i-1}(x))d^{i-1}(\tau^{n-i}(y)) \\
&\quad + \binom{n-1}{i} d^{n-i-1}(\sigma^{i}(x))d^{i}(\tau^{n-i-1}(y)) + \cdots + \sigma^{n-1}(x)d^{n-1}(y)) \\
&= d((d^{n-1}(x)\tau^{n-1}(y)) + \cdots + \binom{n-1}{i-1} d(d^{n-i}(\sigma^{i-1}(x))d^{i-1}(\tau^{n-i}(y)) \\
&+ \binom{n-1}{i} d(d^{n-i-1}(\sigma^{i}(x))d^{i}(\tau^{n-i-1}(y))) + \cdots + d(\sigma^{n-1}(x)d^{n-1}(y))
\end{aligned}$$

for all $x, y \in N$. This implies that

$$\begin{aligned}
d^n(xy) &= d^n(x)\tau^n(y) + \cdots + \binom{n-1}{i-1} d^{n-i}(\sigma^{i}(x))d^{i}(\tau^{n-i}(y)) \\
&\quad + \binom{n-1}{i} d^{n-i}(\sigma^{i}(x))d^{i}(\tau^{n-i}(y)) + \cdots + \sigma^{n}(x)d^{n}(y) \\
&= d^n(x)\tau^n(y) + \cdots + \left[\binom{n-1}{i-1} + \binom{n-1}{i}\right] d^{n-i}(\sigma^{i}(x))d^{i}(\tau^{n-i}(y)) \\
&\quad + \cdots + \sigma^{n}(x)d^{n}(y) \\
&= d^n(x)\tau^n(y) + \cdots + \binom{n}{i} d^{n-i}(\sigma^{i}(x))d^{i}(\tau^{n-i}(y)) + \cdots + \sigma^{n}(x)d^{n}(y)
\end{aligned}$$

for all $x, y \in N$. Hence,

$$d^n(xy) = \sum_{r=0}^{n} \binom{n}{r} d^{n-r}(\sigma^r(x))d^r(\tau^{n-r}(y)),$$

for all $x, y \in N$. This completes the proof of the theorem. □

Corollary 3.1. ([8, Proposition 3]). *Let N be a near-ring. If N admits a derivation d, then for any integer $n \geq 1$ and for all $x, y \in N$, we have*

$$d^n(xy) = \sum_{r=0}^{n} \binom{n}{r} d^{n-r}(x)d^r(y)$$

where $0 \leq r \leq n$.

As an application of the above theorem we get the following results:

Theorem 3.2. *Let $n \geq 1$ be a fixed positive integer and let N be an n-torsion free near-ring. Suppose that σ, τ are automorphisms of N and d a (σ, τ)-derivation on N such that σ, τ commute with d^k, for all integers*

$k \geq 1$. If $d^n(N) = \{0\}$, then for each $x \in N$, either $d(x) = 0$ or there exists an integer i, $0 < i < n$ such that $d^i(x)$ is a non-zero divisor of zero.

Proof. The result is obvious for $n = 1$. By our hypothesis, we have $d^n(N) = \{0\}$. We may assume that $d^{n-1}(N) \neq \{0\}$ with $d^{n-1}(x_0) \neq 0$, for some $x_0 \in N$. Further, suppose that $d(x) \neq 0$. Then there exists i with $0 < i < n$ for which $d^i(x) \neq 0$ and $d^{i+1}(x) = 0$. By application of Theorem 3.1 and simple calculations for $d^n(x_0 d^{i-1}(x)) = 0$, for all $x \in N$, we find that $nd^{n-1}(\sigma(x_0))d(\tau^{n-1}(d^{i-1}(x))) = 0$, for all $x \in N$. This implies that

$$nd^{n-1}(\sigma(x_0))\tau^{n-1}(d^i(x)) = 0, \text{ for all } x \in N.$$

Since τ is an automorphism of N and N is n-torsion free, the above expression yields that $\sigma(d^{n-1}(x_0))\tau^{n-1}(d^i(x)) = 0$, for all $x \in N$. Thus, it follows that $(\tau^{n-1})^{-1}(\sigma(d^{n-1}(x_0)))d^i(x) = 0$, for all $x \in N$. Since σ and τ are automorphism of N and $d^{n-1}(x_0) \neq \{0\}$, it follows that $d^i(x)$ is a non-zero divisor of zero. Hence, we get the required result. □

Theorem 3.3. *Let $n \geq 1$ be a fixed positive integer and let N be an n-torsion free prime near-ring. Suppose that σ is an automorphism of N, and d a (σ,σ)-derivation on N such that σ commutes with d^k for all integers $k \geq 1$. If $d^n(N) = \{0\}$, then $d(Z) = \{0\}$.*

Proof. The result is obvious for $n = 1$. Now let $n \geq 2$ and $d(Z) \neq \{0\}$. Choose $z \in Z$ such that $d(z) \neq 0$. By our Lemma 2.4 and Theorem 3.2, there exists a positive integer i, $0 < i < n$ such that $d^i(z)$ is a non-zero divisor of zero contained in the center Z. Since N is prime near-ring, $d^i(z)$ cannot be a zero-divisor. This contradiction shows that $d(Z) = \{0\}$. □

The following example shows that the conclusion of the above result need not be true even for arbitrary rings with $\sigma = 1$, the identity mapping on N.

Example 3.1. Let S be any ring. Next, let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in S \right\}$. Define a map $d : R \longrightarrow R$ such that $d\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then, it can be easily seen that d is a derivation on R such that $d^2(R) = \{0\}$, but $d(Z) \neq \{0\}$.

4. Commutativity of near-rings

In [4], Bell studied commutativity in prime near-rings with a non-zero derivation d for which $d(xy) = d(yx)$ for all x, y in some non-zero one sided ideal. In this section, we continue this study and obtain some more general results for (σ,σ)-derivations in near-rings.

Theorem 4.1. *Let N be a 2-torsion free prime near-ring. Suppose d is a non-zero (σ,σ)-derivation on N such that $d([x, y]) = 0$, for all $x, y \in N$. Then N is a commutative ring.*

Proof. In view of our hypothesis, we have

$$d(xy) = d(yx), \text{ for all } x, y \in N. \tag{4.1}$$

Equation (4.1) can be written as,

$$\sigma(x)d(y) + d(x)\sigma(y) = \sigma(y)d(x) + d(y)\sigma(x), \text{ for all } x, y \in N. \tag{4.2}$$

Replacing x by yx in (4.2) and using (4.1), we obtain,

$$\sigma(yx)d(y) + d(yx)\sigma(y) = \sigma(y)d(xy) + d(y)\sigma(yx), \text{ for all } x, y \in N. \tag{4.3}$$

This implies that

$$\begin{aligned}\sigma(yx)d(y) &+ (\sigma(y)d(x) + d(y)\sigma(x))\sigma(y)\\ &= \sigma(yx)d(y) + \sigma(y)d(x)\sigma(y) + d(y)\sigma(yx),\end{aligned}$$

for all $x, y \in N$. Now, application of the Lemma 2.2 yields that

$$d(y)\sigma(x)\sigma(y) = d(y)\sigma(y)\sigma(x), \text{ for all } x, y \in N. \tag{4.4}$$

Again replace x by xz in (4.4) and use (4.4), to get

$$d(y)\sigma(x)\sigma(z)\sigma(y) = d(y)\sigma(xy)\sigma(z), \text{ for all } x, y, z \in N. \tag{4.5}$$

This implies that

$$d(y)\sigma(x)\sigma([y, z]) = 0, \text{ for all } x, y, z \in N. \tag{4.6}$$

Since σ is an automorphism of N, we get

$$\sigma^{-1}(d(y))N[y, z] = \{0\}, \text{ for all } y, z \in N. \tag{4.7}$$

This yields that for each fixed $y \in N$ either $d(y) = 0$ or $[y, z] = 0$, for all $z \in N$ i.e., for each fixed $y \in N$, we have either $d(y) = 0$ or $y \in Z$. But $y \in Z$ also implies that $d(y) \in Z$, for all $y \in N$. Therefore, in both the cases we find that $d(y) \in Z$, for all $y \in N$ and hence $d(N) \subseteq Z$. Thus by Lemma 2.3, N is a commutative ring. This completes the proof of our theorem. □

Theorem 4.2. *Let N be a 2-torsion free prime near-ring. Suppose d is a non-zero (σ, σ)-derivation on N such that $d(xoy) = 0$, for all $x, y \in N$, then N is a commutative ring.*

Proof. For all $x, y \in N$, we have,

$$d(xy) = -d(yx), \text{ for all } x, y \in N \tag{4.8}$$

that is,

$$\sigma(x)d(y) + d(x)\sigma(y) = -(\sigma(y)d(x) + d(y)\sigma(x)), \text{ for all } x, y \in N. \tag{4.9}$$

Replacing x by yx in (4.9), we get,

$$\sigma(yx)d(y)+d(yx)\sigma(y) = -(\sigma(y)d(yx)+d(y)\sigma(yx)), \text{ for all } x, y \in N. \tag{4.10}$$

By using (4.8) and Lemma 2.2, we find that

$$\begin{aligned}\sigma(yx)d(y) + (\sigma(y)d(x) + d(y)\sigma(x))\sigma(y) &= -(\sigma(y)(-d(xy)) + d(y)\sigma(yx))\\ &= -(\sigma(-y)d(xy) + d(y)\sigma(yx))\\ &= -(-\sigma(y)d(xy) + d(y)\sigma(yx))\\ &= \sigma(y)(\sigma(x)d(y) + d(x)\sigma(y)) - d(y)\sigma(yx)\\ &= \sigma(y)\sigma(x)d(y) + \sigma(y)d(x)\sigma(y) - d(y)\sigma(yx).\end{aligned}$$

The above expression yields that

$$d(y)\sigma(x)\sigma(y) = -d(y)\sigma(yx), \text{ for all } x, y \in N. \tag{4.11}$$

Again replace x by xz in (4.11) and use (4.11), we obtain

$$d(y)\sigma(x)\sigma(z)\sigma(y) = d(y)\sigma(x)\sigma(y)\sigma(z), \text{ for all } x, y, z \in N. \tag{4.12}$$

That is,

$$d(y)\sigma(x)\sigma([y, z]) = 0, \text{ for all } x, y, z \in N. \tag{4.13}$$

Now, using the same arguments as used after equation(4.6) in the last paragraph of the proof of Theorem 4.1, we get the required result. □

Corollary 4.1. *Let N be a 2-torsion free prime near-ring. Suppose that d is a non-zero derivation on N such that $d([x, y]) = 0$, for all $x, y \in N$ or $d(xoy) = 0$, for all $x, y \in N$. Then N is a commutative ring.*

The following example demonstrates that the conclusion of the above results need not be true even in the case of arbitrary rings.

Example 4.1. Let $R = R_1 \oplus R_2$, where R_1 is a non-commutative ring and R_2 is a commutative domain of characteristic 2 with identity admitting a non-zero derivation δ. Define a map $d : R \to R$ such that $d((a, b)) = (0, \delta(b))$. Then it is easy to see that d satisfies the properties: $d([x, y]) = 0$ and $d(xoy) = 0$ for all $x, y \in R$. However, R is not a commutative ring.

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References

[1] M. Ashraf and Ali, Shakir, *On (σ,τ)-derivations of prime near-rings,* Trends in Theory of Rings and Modules, (S. Tariq Rizvi & S. M. A. Zaidi, (Eds.)), Anamaya Publishers, New Delhi, (2005), 5–10.

[2] M. Ashraf, A. Ali and Ali, Shakir, (σ,τ)- *derivations on prime near-rings,* Arch. Math. (Brno), 40 (2004), 281–286.

[3] K. I. Beidar, Y. Fong and X.K., Wang, *Posner and Herstein theorems for derivations of 3-prime near-rings* , Comm. Algebra, 24 (5)(1996), 1581–1589.

[4] H. E. Bell, *On derivations in near-rings, II* , Kluwer Academic Publishers Netherlands (1997), 191-197.
[5] H. E. Bell and G. Mason, *On derivations in near-rings*, Near-Rings and Near-Fields (G. Betsch, ed.) North-Holland, Amsterdam (1987), 31–35.
[6] J. D. P. Meldrum, *Near-rings and their Link with Groups*, Pitman, 1985.
[7] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc., 8 (1957), 1093–1100.
[8] X. K. Wang, *Derivations in prime near-rings*, Proc. Amer. Math. Soc., 121 (1994), 361–366.

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