NEW INEQUALITIES FOR MEANS

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Dedicated to Professor Fikret Vajzović on the occasion of his 80th birthday

Abstract. In this paper we present new inequalities for means. If

\[ \begin{align*}
A(a, b) &= \frac{a + b}{2}, \\
G(a, b) &= \sqrt{ab}, \\
H(a, b) &= \frac{2ab}{a + b}, \\
L(a, b) &= \frac{b - a}{\ln b - \ln a}, \\
I(a, b) &= \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{\frac{1}{b-a}},
\end{align*} \]

we will prove seven new theorems concerning these means.

1. Introduction

For \( x, y > 0 \) \((x \neq y)\) the following well known inequality holds clearly:

\[ H(x, y) < G(x, y) < L(x, y) < I(x, y) < A(x, y), \]

where \( A(x, y), G(x, y) \) and \( H(x, y) \) are the arithmetic, geometric and harmonic means of two positive numbers \( x, y \) respectively, and \( L(x, y), I(x, y) \) are special cases of the generalized logarithmic mean \( L_r(x, y) \) of two positive numbers \( x \) and \( y \), i.e.

\[ \begin{align*}
(1) \quad L_r(x, y) &= \left( \frac{y^{r+1} - x^{r+1}}{(r+1)(y-x)} \right)^{\frac{1}{r}}, \quad r \neq -1, 0; \\
(2) \quad L_{-1}(x, y) &= \frac{y-x}{\ln y - \ln x} = L(x, y); \\
(3) \quad L_0(x, y) &= \frac{1}{2} \left( \frac{y^y}{x^x} \right)^{\frac{1}{y-x}} = I(x, y).
\end{align*} \]

\( L(x, y) \) and \( I(x, y) \) are respectively called the logarithmic and exponential mean of two positive numbers \( x \) and \( y \). When \( x \neq y \), \( L_r(x, y) \) is a strictly increasing function of \( r \in (-\infty, +\infty) \). In particular, \( \lim_{r \to -\infty} L_r(x, y) = \min\{x, y\} \), \( \lim_{r \to +\infty} L_r(x, y) = \max\{x, y\} \)

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2. New results

**Theorem 1.** If \( 0 < a < b \), \( A(a, b) = \frac{a + b}{2} \), \( G(a, b) = \sqrt{ab} \), \( H(a, b) = \frac{2ab}{a + b} \), \( L(a, b) = \frac{b - a}{\ln b - \ln a} \) and \( I(a, b) = \frac{1}{e} \left( \frac{b}{a} \right)^{\frac{1}{b-a}} \), then the following inequalities hold.

\[
\begin{align*}
\frac{2L}{I+G} < A - L < \frac{L}{\sqrt{GI}} \\
\left( \frac{I}{G} \right)^{\frac{2L}{I+G}} < A < \left( \frac{I}{G} \right)^{\frac{L}{\sqrt{GI}}}.
\end{align*}
\]

(A)

and

(B)

Proof. We have

\[ \ln I = \frac{a \ln a - b \ln b}{a - b} - 1 = \frac{a}{L} + \ln b - 1, \]

therefore

\[ \ln \frac{I}{b} = \frac{a}{L} - 1 \quad \text{and} \quad \ln \frac{I}{a} = \frac{b}{L} - 1, \]

and from here

\[ \ln \frac{I}{G} = \frac{A}{L} - 1. \] (1)

1. If \( x > 1 \), then we have (See 3.6.18, page 273, using \( 1 + \frac{1}{x} = t \) in [7]):

\[ \frac{2(x - 1)}{x + 1} < \ln x < \frac{x - 1}{\sqrt{x}}. \] (2)

In (2) we take \( x = \frac{I}{G} \), then we have

\[ \frac{2 \left( \frac{I}{G} - 1 \right)}{\frac{I}{G} + 1} < \ln \frac{I}{G} < \frac{\frac{I}{G} - 1}{\sqrt{\frac{I}{G}}} \]

or from (1)

\[ \frac{2(I - G)}{I + G} < \frac{A}{L} - 1 < \frac{I - G}{\sqrt{IG}}, \]

and finally

\[ \frac{2L}{I+G} < \frac{A - L}{I - G} < \frac{L}{\sqrt{GI}}, \]

and this is the inequality (A).

2. In (2) we take \( x = \frac{A}{L} \), therefore

\[ \frac{2 \left( \frac{A}{L} - 1 \right)}{\frac{A}{L} + 1} < \ln \frac{A}{L} < \frac{\frac{A}{L} - 1}{\sqrt{\frac{A}{L}}}. \]
Using (1), we obtain
\[
\frac{2 \ln \frac{I}{G}}{\frac{A}{L} + 1} < \ln \frac{A}{L} < \frac{\ln \frac{I}{G}}{\sqrt[3]{\frac{A}{L}}}
\]
or
\[
\left( \frac{I}{G} \right)^{\frac{2L}{A+L}} < \frac{A}{L} < \left( \frac{I}{G} \right)^{\sqrt[3]{\frac{L}{A}}}
\]
and this is the inequality (B).

\[\square\]

**Theorem 2.** The following inequalities
\[
\frac{(A - L)G}{(I - G)L} < \frac{G + \sqrt[3]{IG^2}}{I + \sqrt[3]{IG^2}}
\]
and
\[
\left( \frac{A}{L} \right)^{A + \sqrt[3]{AL^2}} < \left( \frac{I}{G} \right)^{L + \sqrt[3]{AL^2}}
\]
hold.

**Proof.** If \(x > 1\), then we have (See 3.6.16, page 272 in [7]):
\[
\ln \frac{x}{x - 1} < 1 + \frac{\sqrt{x}}{x + \sqrt{x}}.
\]

1. In (3) we take \(x = \frac{I}{G}\), then
\[
\ln \frac{I}{G} \left( \frac{1}{G} - 1 \right) < 1 + \frac{\sqrt{I}}{G}.
\]

but using (1), we have
\[
\frac{A}{L} - 1 < \frac{1 + \sqrt{L}}{L + \sqrt{L}}
\]
or
\[
\frac{(A - L)G}{(I - G)L} < \frac{G + \sqrt[3]{IG^2}}{I + \sqrt[3]{IG^2}},
\]
i.e. the inequality (C) is proved.

2. In (3) we take \(x = \frac{A}{L}\), then
\[
\ln \frac{A}{L} \left( \frac{1}{L} - 1 \right) < 1 + \frac{\sqrt{A}}{L}.
\]
but from (1) we have
\[
\frac{\ln \frac{A}{L}}{\ln \frac{I}{G}} < \frac{1 + \sqrt[3]{\frac{A}{L}}}{\frac{A}{L} + \sqrt[3]{\frac{A}{L}}}
\]
or
\[
\left( \frac{A}{L} \right)^{\frac{A}{L} + \sqrt[3]{\frac{A}{L}^2}} < \left( \frac{I}{G} \right)^{L + \sqrt[3]{\frac{A}{L}^2}},
\]
i.e. inequality (D) is proved. \(\Box\)

**Theorem 3.** The following inequalities
\[
H^{L-G}L^{H-G} \leq G^{H+L-2G}, \tag{E}
\]
\[
G^{L-L}I^{G-L} \leq L^{G+I-2L}, \tag{F}
\]
\[
L^{A-I}A^{L-I} \leq I^{L+A-2I} \tag{G}
\]
hold.

**Proof.** If \(0 < a \leq x \leq b\), then
\[
a^{b-x}b^{a-x} \leq x^{a+b-2x}. \tag{4}
\]
If \(f(x) = (a + b - 2x) \ln x - (b - x) \ln a - (a - x) \ln b\), then
\[
f'(x) = -2 \ln x + \frac{a + b}{x} - 2 + \ln a + \ln b
\]
and
\[
f''(x) = -\frac{2}{x} + \frac{a + b}{x} < 0
\]
\(\Rightarrow f\) is concave
\(\Rightarrow f(x) \geq \min\{f(a), f(b)\} = 0.\)
In (4) we take \((a, x, b) \in \{(H, G, L), (G, L, I), (L, I, A)\}\) and because \(H \leq G \leq L \leq I \leq A\) we get the inequalities (E), (F) and (G). \(\Box\)

**Theorem 4.** If \(f, g : (0, +\infty) \mapsto (0, +\infty)\) are monotonic functions in the same sense, \(f\) is convex and \(g\) is concave, then the inequalities
\[
\frac{f(L) - f(G)}{f(I) - f(G)} \leq \frac{L - G}{I - G} \leq \frac{g(L) - g(G)}{g(I) - g(G)} \tag{H}
\]
and
\[
\frac{f(I) - f(L)}{f(A) - f(L)} \leq \frac{I - L}{A - L} \leq \frac{g(I) - g(L)}{g(A) - g(L)} \tag{I}
\]
hold.
Proof. Because $f$ is convex, we get that
\[
\begin{vmatrix}
  x_1 & f(x_1) & 1 \\
  x_2 & f(x_2) & 1 \\
  x_3 & f(x_3) & 1
\end{vmatrix} \geq 0
\]
for $x_1 \leq x_2 \leq x_3$. This is equivalent with
\[
\frac{f(x_2) - f(x_1)}{f(x_3) - f(x_1)} \leq \frac{x_2 - x_1}{x_3 - x_1}.
\]
(5)

Function $g$ is concave and we get that
\[
\begin{vmatrix}
  x_1 & g(x_1) & 1 \\
  x_2 & g(x_2) & 1 \\
  x_3 & g(x_3) & 1
\end{vmatrix} \leq 0
\]
for $x_1 \leq x_2 \leq x_3$. This is equivalent to
\[
\frac{g(x_2) - g(x_1)}{g(x_3) - g(x_1)} \geq \frac{x_2 - x_1}{x_3 - x_1}.
\]
(6)

It follows from (5) and (6):
\[
\frac{f(x_2) - f(x_1)}{f(x_3) - f(x_1)} \leq \frac{x_2 - x_1}{x_3 - x_1} \leq \frac{g(x_2) - g(x_1)}{g(x_3) - g(x_1)}.
\]
(7)

1. In (7) we take $x_1 = G$, $x_2 = L$ and $x_3 = I$ and we obtain the inequality (H).
2. In (7) we take $x_1 = L$, $x_2 = I$ and $x_3 = A$ and we obtain the inequality (I).

\[\Box\]

Theorem 5. The following inequalities
\[
\sum_{k=0}^{n-1} \frac{1}{(k+1)I + (n-k-1)G} < \frac{A-L}{L(I-G)} < \sum_{k=0}^{n-1} \frac{1}{kI + (n-k)G}
\]
(7)

and
\[
\left( \frac{I}{G} \right)^\alpha < \frac{A}{L} < \left( \frac{I}{G} \right)^\beta,
\]
(K)

where
\[
\alpha = L \sum_{k=0}^{n-1} \frac{1}{(k+1)A + (n-k-1)L}
\]
and
\[
\beta = L \sum_{k=0}^{n-1} \frac{1}{kA + (n-k)L}
\]
hold.

Proof. If \( x > 1 \), then we have

\[
(x - 1) \sum_{k=0}^{n-1} \frac{1}{n + (k + 1)(x - 1)} < \ln x < (x - 1) \sum_{k=0}^{n-1} \frac{1}{n + k(x - 1)}. \tag{8}
\]

(See 3.6.23, page 276 in [7]).

1. If in (8) we take \( x = \frac{A}{L} \) and using (1), then we obtain the inequality (J).

2. If in (8) we take \( x = \frac{4}{L} \) and using (1) \( (x - 1 = \frac{A}{L} - 1 = \ln \frac{L}{G} ) \), then we obtain the inequality (K). \( \square \)

Theorem 6. The following inequalities

\[
\frac{2cL}{I + G} < \frac{A - L}{I - G} < \frac{2dL}{I + G}, \tag{L}
\]

where

\[
c = 1 + \frac{(I - G)^2}{12IG} - \frac{(I^3 - G^3)(I - G)^3}{360I^3G^3},
\]

\[
d = 1 + \frac{(I - G)^2}{12IG} - \frac{(I^3 - G^3)(I - G)^3}{(360 + \varepsilon)I^3G^3},
\]

\[
\varepsilon = \frac{30 + (I^2 + 5GI + G^2)(I - G)^2}{I^2G^2},
\]

and

\[
\left( \frac{I}{G} \right)^{\frac{2\alpha}{4 + x}} < \frac{A}{L} < \left( \frac{I}{G} \right)^{\frac{2\beta}{4 + x}} \tag{M}
\]

where

\[
\alpha = 1 + \frac{(A - L)^2}{12AL} - \frac{(A^3 - L^3)(A - L)^3}{360A^3L^3},
\]

\[
\beta = 1 + \frac{(A - L)^2}{12AL} - \frac{(A^3 - L^3)(A - L)^3}{(360 + \gamma)A^3L^3},
\]

\[
\gamma = 30 \left( 1 + \frac{5L}{A} + \frac{L^2}{A^2} \right)
\]

hold.

Proof. If \( x > 1 \), then we have

\[
\frac{2(x - 1)}{x + 1} \left( 1 + \frac{x - 1}{12} - \frac{x - 1}{12x} - \frac{(x - 1)^3}{360} + \frac{(x - 1)^3}{360x^3} \right) < \ln x
\]

\[
< \frac{2(x - 1)}{x + 1} \left( 1 + \frac{x - 1}{12} - \frac{x - 1}{12x} - \frac{(x - 1)^3}{360 + u} + \frac{(x - 1)^3}{(360 + u)x^3} \right), \tag{9}
\]
where \( u = 30 \left( 1 + \frac{1}{x} + \frac{1}{x^2} \right) \). (See 3.6.19, page 274, using \( 1 + \frac{1}{x} = t \) in [7]).

1. In (9) we take \( x = \frac{L}{G} \) and using (1) we obtain the desired inequalities (L).

2. In (9) we take \( x = \frac{A}{L} \) and using (1) we obtain the desired inequalities (M), because
\[
\frac{2(x - 1)}{x + 1} = \frac{2(\frac{A}{L} - 1)}{\frac{A}{L} + 1} = \frac{2 \ln \frac{L}{G}}{\frac{A}{L} + 1}, \text{ etc.}
\]

\[\Box\]

**Theorem 7.** The following inequality
\[
L(a, b) < L\left(\frac{a + b}{2}, \sqrt{ab}\right) < \left( A(\sqrt{a}, \sqrt{b}) \right)^2 < A(a, b) \tag{N}
\]
holds.

**Proof.** The inequality \( \left( A(\sqrt{a}, \sqrt{b}) \right)^2 < A(a, b) \) is simply the Power-Mean Inequality. Applying the Hadamard’s Inequality to the convex function \( f(x) = \frac{1}{x} \), we get
\[
\frac{1}{L\left(\frac{1}{2}(a + b), \sqrt{ab}\right)} = \frac{1}{\frac{1}{2}(a + b) - \sqrt{ab}} \int_{\sqrt{ab}}^{\frac{1}{2}(a + b)} f(x) dx
\]
\[
> f\left(\frac{\frac{1}{2}(a + b) + \sqrt{ab}}{2}\right) = \frac{1}{\left( A(\sqrt{a}, \sqrt{b}) \right)^2}.
\]

This gives the inequality \( L\left(\frac{a + b}{2}, \sqrt{ab}\right) < \left( A(\sqrt{a}, \sqrt{b}) \right)^2 \).

The inequality \( L(a, b) < L\left(\frac{a + b}{2}, \sqrt{ab}\right) \) transforms successively into
\[
(b - a) \ln \left(\frac{1}{2}(a + b)\right) < (\sqrt{ab} - a) \ln a + (b - \sqrt{ab}) \ln b,
\]
\[
\left(\frac{1}{2}(a + b)\right)^{\sqrt{a} + \sqrt{b}} < a^{\sqrt{a}} b^{\sqrt{b}},
\]
\[
\frac{\left(\frac{1}{2}(a + b)\right)^{\sqrt{a} + \sqrt{b}}}{a^{\sqrt{a}} b^{\sqrt{b}}} < 1,
\]
\[
\left(\frac{\frac{1}{2}(a + b)}{a}\right)^{\sqrt{a}} \left(\frac{\frac{1}{2}(a + b)}{b}\right)^{\sqrt{b}} < 1,
\]
\[
\left(\frac{1}{2} \left(1 + \frac{b}{a}\right)\right)^{\sqrt{a}} \left(\frac{1}{2} \left(1 + \frac{a}{b}\right)\right)^{\sqrt{b}} < 1,
\]
\[
\frac{1}{2} \left( 1 + \frac{b}{a} \right) \left( \frac{1}{2} \left( 1 + \frac{a}{b} \right) \right)^{\frac{\sqrt{b}}{a-n}} < 1,
\]
\[
\frac{1}{2} \left( 1 + x^2 \right) \left( \frac{x^2 + 1}{2x^2} \right)^{x} < 1,
\]
with \( x = \sqrt{\frac{b}{a}} > 1 \) (because \( b > a \)).

Now, for \( x > 1 \), let
\[
f(x) = \ln \left( \frac{1}{2} (1 + x^2) \right) + x \ln \left( \frac{x^2 + 1}{2x^2} \right).
\]
Then
\[
f'(x) = \frac{2(x - 1)}{x^2 + 1} + \ln \left( \frac{x^2 + 1}{2x^2} \right)
\]
and
\[
f''(x) = -\frac{2(x - 1)^2(x + 1)}{x(x^2 + 1)^2} < 0.
\]
Hence \( f \) is strictly concave. Since \( f(1) = 0 \) and \( f'(1) = 0 \), we conclude that \( f(x) < 0 \). Therefore, the given inequality (N) is true. \( \square \)

References

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