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NEW INEQUALITIES FOR MEANS

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Dedicated to Professor Fikret Vajzović on the occasion of his 80th birthday

Abstract. In this paper we present new inequalities for means. If

$$0 < a < b,\ A(a,b) = \frac{a+b}{2},\ G(a,b) = \sqrt{ab},\ H(a,b) = \frac{2ab}{a+b},$$

$$L(a,b) = \frac{b-a}{\ln b - lna},\ I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}},$$

we will prove seven new theorems concerning these means.

1. Introduction

For $x, y > 0$ $(x \neq y)$ the following well known inequality holds clearly:

$$H(x,y) < G(x,y) < L(x,y) < I(x,y) < A(x,y),$$

where $A(x,y), G(x,y)$ and $H(x,y)$ are the arithmetic, geometric and harmonic means of two positive numbers x, y respectively, and $L(x,y)$, $I(x,y)$ are special cases of the generalized logarithmic mean $L_r(x,y)$ of two positive numbers x and y, i.e.

(1) $L_r(x,y) = \left(\frac{y^{r+1}-x^{r+1}}{(r+1)(y-x)}\right)^{\frac{1}{r}},\ r \neq -1, 0;$
(2) $L_{-1}(x,y) = \frac{y-x}{\ln y - \ln x} = L(x,y);$
(3) $L_0(x,y) = \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}} = I(x,y).$

$L(x,y)$ and $I(x,y)$ are respectively called the logarithmic and exponential mean of two positive numbers x and y. When $x \neq y, L_r(x,y)$ is a strictly increasing function of $r \in (-\infty, +\infty)$. In particular, $\lim_{r\to-\infty} L_r(x,y) = \min\{x,y\}$, $\lim_{r\to+\infty} L_r(x,y) = \max\{x,y\}$

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2. New results

Theorem 1. *If* $0 < a < b$, $A(a,b) = \frac{a+b}{2}$, $G(a,b) = \sqrt{ab}$, $H(a,b) = \frac{2ab}{a+b}$, $L(a,b) = \frac{b-a}{\ln b - \ln a}$ *and* $I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$, *then the following inequalities hold.*

$$\frac{2L}{I+G} < \frac{A-L}{I-G} < \frac{L}{\sqrt{GI}} \tag{A}$$

and

$$\left(\frac{I}{G}\right)^{\frac{2L}{A+L}} < \frac{A}{L} < \left(\frac{I}{G}\right)^{\sqrt{\frac{L}{A}}}. \tag{B}$$

Proof. We have

$$\ln I = \frac{a\ln a - b\ln b}{a-b} - 1 = \frac{a}{L} + \ln b - 1,$$

therefore

$$\ln\frac{I}{b} = \frac{a}{L} - 1 \text{ and } \ln\frac{I}{a} = \frac{b}{L} - 1,$$

and from here

$$\ln\frac{I}{G} = \frac{A}{L} - 1. \tag{1}$$

1. If $x > 1$, then we have (See 3.6.18, page 273, using $1 + \frac{1}{x} = t$ in [7]):

$$\frac{2(x-1)}{x+1} < \ln x < \frac{x-1}{\sqrt{x}}. \tag{2}$$

In (2) we take $x = \frac{I}{G}$, then we have

$$\frac{2\left(\frac{I}{G}-1\right)}{\frac{I}{G}+1} < \ln\frac{I}{G} < \frac{\frac{I}{G}-1}{\sqrt{\frac{I}{G}}}$$

or from (1)

$$\frac{2(I-G)}{I+G} < \frac{A}{L} - 1 < \frac{I-G}{\sqrt{IG}},$$

and finally

$$\frac{2L}{I+G} < \frac{A-L}{I-G} < \frac{L}{\sqrt{GI}},$$

and this is the inequality (A).

2. In (2) we take $x = \frac{A}{L}$, therefore

$$\frac{2\left(\frac{A}{L}-1\right)}{\frac{A}{L}+1} < \ln\frac{A}{L} < \frac{\frac{A}{L}-1}{\sqrt{\frac{A}{L}}}.$$

Using (1), we obtain

$$\frac{2\ln\frac{I}{G}}{\frac{A}{L}+1}<\ln\frac{A}{L}<\frac{\ln\frac{I}{G}}{\sqrt{\frac{A}{L}}}$$

or

$$\left(\frac{I}{G}\right)^{\frac{2L}{A+L}}<\frac{A}{L}<\left(\frac{I}{G}\right)^{\sqrt{\frac{L}{A}}}$$

and this is the inequality (B).

□

Theorem 2. *The following inequalities*

$$\frac{(A-L)G}{(I-G)L}<\frac{G+\sqrt[3]{IG^2}}{I+\sqrt[3]{IG^2}} \tag{C}$$

and

$$\left(\frac{A}{L}\right)^{A+\sqrt[3]{AL^2}}<\left(\frac{I}{G}\right)^{L+\sqrt[3]{AL^2}} \tag{D}$$

hold.

Proof. If $x>1$, then we have (See 3.6.16, page 272 in [7]):

$$\frac{\ln x}{x-1}<\frac{1+\sqrt[3]{x}}{x+\sqrt[3]{x}}. \tag{3}$$

1. In (3) we take $x=\frac{I}{G}$, then

$$\frac{\ln\frac{I}{G}}{\frac{I}{G}-1}<\frac{1+\sqrt[3]{\frac{I}{G}}}{\frac{I}{G}+\sqrt[3]{\frac{I}{G}}},$$

but using (1), we have

$$\frac{\frac{A}{L}-1}{\frac{I}{G}-1}<\frac{1+\sqrt[3]{\frac{I}{G}}}{\frac{I}{G}+\sqrt[3]{\frac{I}{G}}}$$

or

$$\frac{(A-L)G}{(I-G)L}<\frac{G+\sqrt[3]{IG^2}}{I+\sqrt[3]{IG^2}},$$

i.e. the inequality (C) is proved.

2. In (3) we take $x=\frac{A}{L}$, then

$$\frac{\ln\frac{A}{L}}{\frac{A}{L}-1}<\frac{1+\sqrt[3]{\frac{A}{L}}}{\frac{A}{L}+\sqrt[3]{\frac{A}{L}}},$$

but from (1) we have

$$\frac{\ln \frac{A}{L}}{\ln \frac{I}{G}} < \frac{1+\sqrt[3]{\frac{A}{L}}}{\frac{A}{L}+\sqrt[3]{\frac{A}{L}}}$$

or

$$\left(\frac{A}{L}\right)^{A+\sqrt[3]{AL^2}} < \left(\frac{I}{G}\right)^{L+\sqrt[3]{AL^2}},$$

i.e. inequality (D) is proved. □

Theorem 3. *The following inequalities*

$$H^{L-G}L^{H-G} \le G^{H+L-2G}, \tag{E}$$

$$G^{I-L}I^{G-L} \le L^{G+I-2L}, \tag{F}$$

$$L^{A-I}A^{L-I} \le I^{L+A-2I} \tag{G}$$

hold.

Proof. If $0 < a \le x \le b$, then

$$a^{b-x}b^{a-x} \le x^{a+b-2x}. \tag{4}$$

If $f(x) = (a+b-2x)\ln x - (b-x)\ln a - (a-x)\ln b$, then

$$f'(x) = -2\ln x + \frac{a+b}{x} - 2 + \ln a + \ln b$$

and

$$f''(x) = -\frac{2}{x} - \frac{a+b}{x} < 0$$
$$\Rightarrow f \text{ is concave}$$
$$\Rightarrow f(x) \ge \min\{f(a), f(b)\} = 0.$$

In (4) we take $(a,x,b) \in \{(H,G,L),(G,L,I),(L,I,A)\}$ and because $H \le G \le L \le I \le A$ we get the inequalities (E), (F) and (G). □

Theorem 4. *If $f, g : (0,+\infty) \mapsto (0,+\infty)$ are monotonic functions in the same sense, f is convex and g is concave, then the inequalities*

$$\frac{f(L)-f(G)}{f(I)-f(G)} \le \frac{L-G}{I-G} \le \frac{g(L)-g(G)}{g(I)-g(G)} \tag{H}$$

and

$$\frac{f(I)-f(L)}{f(A)-f(L)} \le \frac{I-L}{A-L} \le \frac{g(I)-g(L)}{g(A)-g(L)} \tag{I}$$

hold.

Proof. Because f is convex, we get that

$$\begin{vmatrix} x_1 & f(x_1) & 1 \\ x_2 & f(x_2) & 1 \\ x_3 & f(x_3) & 1 \end{vmatrix} \geq 0$$

for $x_1 \leq x_2 \leq x_3$. This is equivalent with

$$\frac{f(x_2)-f(x_1)}{f(x_3)-f(x_1)} \leq \frac{x_2-x_1}{x_3-x_1}. \tag{5}$$

Function g is concave and we get that

$$\begin{vmatrix} x_1 & g(x_1) & 1 \\ x_2 & g(x_2) & 1 \\ x_3 & g(x_3) & 1 \end{vmatrix} \leq 0$$

for $x_1 \leq x_2 \leq x_3$. This is equivalent to

$$\frac{g(x_2)-g(x_1)}{g(x_3)-g(x_1)} \geq \frac{x_2-x_1}{x_3-x_1}. \tag{6}$$

It follows from (5) and (6):

$$\frac{f(x_2)-f(x_1)}{f(x_3)-f(x_1)} \leq \frac{x_2-x_1}{x_3-x_1} \leq \frac{g(x_2)-g(x_1)}{g(x_3)-g(x_1)}. \tag{7}$$

1. In (7) we take $x_1 = G, x_2 = L$ and $x_3 = I$ and we obtain the inequality (H).
2. In (7) we take $x_1 = L, x_2 = I$ and $x_3 = A$ and we obtain the inequality (I). □

Theorem 5. *The following inequalities*

$$\sum_{k=0}^{n-1} \frac{1}{(k+1)I+(n-k-1)G} < \frac{A-L}{L(I-G)} < \sum_{k=0}^{n-1} \frac{1}{kI+(n-k)G} \tag{J}$$

and

$$\left(\frac{I}{G}\right)^{\alpha} < \frac{A}{L} < \left(\frac{I}{G}\right)^{\beta}, \tag{K}$$

where

$$\alpha = L\sum_{k=0}^{n-1} \frac{1}{(k+1)A+(n-k-1)L}$$

and

$$\beta = L\sum_{k=0}^{n-1} \frac{1}{kA+(n-k)L}$$

hold.

Proof. If $x > 1$, then we have

$$(x-1)\sum_{k=0}^{n-1}\frac{1}{n+(k+1)(x-1)} < \ln x < (x-1)\sum_{k=0}^{n-1}\frac{1}{n+k(x-1)}\,. \tag{8}$$

(See 3.6.23, page 276 in [7]).

1. If in (8) we take $x = \frac{I}{G}$ and using (1), then we obtain the inequality (J).

2. If in (8) we take $x = \frac{A}{L}$ and using (1) ($x - 1 = \frac{A}{L} - 1 = \ln\frac{I}{G}$), then we obtain the inequality (K). □

Theorem 6. *The following inequalities*

$$\frac{2cL}{I+G} < \frac{A-L}{I-G} < \frac{2dL}{I+G} \tag{L}$$

where

$$c = 1 + \frac{(I-G)^2}{12IG} - \frac{(I^3-G^3)(I-G)^3}{360I^3G^3}\,,$$
$$d = 1 + \frac{(I-G)^2}{12IG} - \frac{(I^3-G^3)(I-G)^3}{(360+\varepsilon)I^3G^3}\,,$$
$$\varepsilon = \frac{30+(I^2+5GI+G^2)(I-G)^2}{I^2G^2}\,,$$

and

$$\left(\frac{I}{G}\right)^{\frac{2\alpha L}{A+L}} < \frac{A}{L} < \left(\frac{I}{G}\right)^{\frac{2\beta L}{A+L}} \tag{M}$$

where

$$\alpha = 1 + \frac{(A-L)^2}{12AL} - \frac{(A^3-L^3)(A-L)^3}{360A^3L^3}\,,$$
$$\beta = 1 + \frac{(A-L)^2}{12AL} - \frac{(A^3-L^3)(A-L)^3}{(360+\gamma)A^3L^3}\,,$$
$$\gamma = 30\left(1+\frac{5L}{A}+\frac{L^2}{A^2}\right)$$

hold.

Proof. If $x > 1$, then we have

$$\frac{2(x-1)}{x+1}\left(1+\frac{x-1}{12}-\frac{x-1}{12x}-\frac{(x-1)^3}{360}+\frac{(x-1)^3}{360x^3}\right) < \ln x$$
$$< \frac{2(x-1)}{x+1}\left(1+\frac{x-1}{12}-\frac{x-1}{12x}-\frac{(x-1)^3}{360+u}+\frac{(x-1)^3}{(360+u)x^3}\right)\,, \tag{9}$$

where $u = 30\left(1+\frac{5}{x}+\frac{1}{x^2}\right)$. (See 3.6.19, page 274, using $1+\frac{1}{x}=t$ in [7]).

1. In (9) we take $x = \frac{I}{G}$ and using (1) we obtain the desired inequalities (L).

2. In (9) we take $x = \frac{A}{L}$ and using (1) we obtain the desired inequalities (M), because

$$\frac{2(x-1)}{x+1} = \frac{2(\frac{A}{L}-1)}{\frac{A}{L}+1} = \frac{2\ln\frac{I}{G}}{\frac{A}{L}+1}, \text{ etc.}$$

□

Theorem 7. *The following inequality*

$$L(a,b) < L(\frac{a+b}{2},\sqrt{ab}) < \left(A(\sqrt{a},\sqrt{b})\right)^2 < A(a,b) \tag{N}$$

holds.

Proof. The inequality $\left(A(\sqrt{a},\sqrt{b})\right)^2 < A(a,b)$ is simply the Power-Mean Inequality. Applying the Hadamard's Inequality to the convex function $f(x) = \frac{1}{x}$, we get

$$\frac{1}{L\left(\frac{1}{2}(a+b),\sqrt{ab}\right)} = \frac{1}{\frac{1}{2}(a+b)-\sqrt{ab}}\int_{\sqrt{ab}}^{\frac{1}{2}(a+b)} f(x)dx$$

$$> f\left(\frac{\frac{1}{2}(a+b)+\sqrt{ab}}{2}\right) = \frac{1}{\left(A(\sqrt{a},\sqrt{b})\right)^2}.$$

This gives the inequality $L(\frac{a+b}{2},\sqrt{ab}) < \left(A(\sqrt{a},\sqrt{b})\right)^2$.

The inequality $L(a,b) < L(\frac{a+b}{2},\sqrt{ab})$ transforms successively into

$$(b-a)\ln\left(\frac{1}{2}(a+b)\right) < (\sqrt{ab}-a)\ln a + (b-\sqrt{ab})\ln b,$$

$$\left(\frac{1}{2}(a+b)\right)^{\sqrt{a}+\sqrt{b}} < a^{\sqrt{a}}b^{\sqrt{b}},$$

$$\frac{\left(\frac{1}{2}(a+b)\right)^{\sqrt{a}+\sqrt{b}}}{a^{\sqrt{a}}b^{\sqrt{b}}} < 1,$$

$$\left(\frac{\frac{1}{2}(a+b)}{a}\right)^{\sqrt{a}}\left(\frac{\frac{1}{2}(a+b)}{b}\right)^{\sqrt{b}} < 1,$$

$$\left(\frac{1}{2}\left(1+\frac{b}{a}\right)\right)^{\sqrt{a}}\left(\frac{1}{2}\left(1+\frac{a}{b}\right)\right)^{\sqrt{b}} < 1,$$

$$\frac{1}{2}\left(1+\frac{b}{a}\right)\left(\frac{1}{2}\left(1+\frac{a}{b}\right)\right)^{\sqrt{\frac{b}{a}}} < 1,$$
$$\frac{1}{2}(1+x^2)\left(\frac{x^2+1}{2x^2}\right)^{x} < 1,$$

with $x=\sqrt{\frac{b}{a}}>1$ (because $b>a$).

Now, for $x>1$, let

$$f(x)=\ln\left(\frac{1}{2}(1+x^2)\right)+x\ln\left(\frac{x^2+1}{2x^2}\right).$$

Then

$$f'(x)=\frac{2(x-1)}{x^2+1}+\ln\left(\frac{x^2+1}{2x^2}\right)$$

and

$$f''(x)=\frac{-2(x-1)^2(x+1)}{x(x^2+1)^2}<0.$$

Hence f is strictly concave. Since $f(1)=0$ and $f'(1)=0$, we conclude that $f(x)<0$. Therefore, the given inequality (N) is true. □

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