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PROPERTIES OF NEHARI DISKS

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Dedicated to Professor Fikret Vajzović on the occasion of his 80th birthday

Abstract. Let D be a simply connected plane domain and let B be the unit disk. The inner radius of D, $\sigma(D)$ is defined by $\sigma(D) = \sup\{a : a \geq 0, ||S_f||_D \leq a$ implies f is univalent in $D\}$. Here S_f is the Schwarzian derivative of f, ρ_D the hyperbolic density on D and $||S_f||_D = \sup_{z\in D} |S_f(z)|\rho_D^{-2}(z)$. Domains for which the value of $\sigma(D)$ is known include disks, angular sectors and regular polygons as well as certain classes of rectangles and equiangular hexagons.

When the inner radius for the above-mentioned domains, except non convex angular sectors, is computed it is seen that $\sigma(D) = 2 - ||S_h||_B$, where $h : B \longrightarrow D$ is the Riemann mapping and B the unit disk, a fact that yields a convenient method for computing $\sigma(D)$. We introduce the name Nehari disks for domains with the above property.

In this paper we generalize some results by Gehring, Pommerenke, Ahlfors and Minda that were proved in the unit disk, to analogous results for Nehari disks.

1. Introduction

We use the symbol C to denote the complex plane and $\overline{C}$ to denote the extended complex plane. Within $\overline{C}$, we use the symbol B to refer to the unit disk ($B = \{z : |z| < 1\}$). The symbol D will denote a domain in $\overline{C}$ with at least two points on its boundary.

For $z \in B$ the *hyperbolic density* of B at z is the quantity $\rho_B(z)$ given by $\rho_B(z) = 1/(1 - |z|^2)$. For a general simply connected domain D, the hyperbolic density ρ_D is then defined in terms of ρ_B and $h : B \longrightarrow D$ where h maps B conformally onto D (see [9]). It is not hard to verify that $\rho_D(z) \geq \rho_{D'}(z)$ for $z \in D$, when $D \subset D'$ are simply connected domains.

For f holomorphic in $D \subset C$, with $f'(z) \neq 0$ for $z \in D$, the *Schwarzian derivative* S_f, of f, is defined in D by $S_f(z) = (f''/f')'(z) - \frac{1}{2}(f''/f')^2(z)$. This definition can easily be extended to include locally univalent meromorphic functions. A detailed explanation of the extended definition can be found in

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[9]. To make our terminology more concise, locally univalent meromorphic functions will be referred to simply as locally univalent functions.

In order to discuss a univalence criterion for f we introduce a norm for S_f. Let D be a simply connected domain in $\overline{C}$. For f locally univalent in D, we define the *hyperbolic norm* of S_f with respect to D by $||S_f||_D = \sup_{z\in D} |S_f(z)|\rho_D^{-2}(z)$.

Now we introduce a domain constant known as the inner radius of a domain.

Definition 1.1. *Suppose D is a simply connected domain in $\overline{C}$. We define the inner radius of D, $\sigma(D)$, by*

$$\sigma(D) = \sup\{a : a \geq 0, ||S_f||_D \leq a \textit{ implies } f \textit{ is univalent in } D\}.$$

All images of D under Möbius transformations have the same inner radius as D.

Nehari [15] and Hille [6] proved that $\sigma(B) = 2$. Later, Lehtinen showed in [7] that $\sigma(D) \leq 2$ for all simply connected domains in $\overline{C}$ with equality occurring only when D is a disk in $\overline{C}$ (i.e. an image of B under a Möbius transformation). The inner radius of a domain has another important meaning that is not apparent from its definition. Ahlfors and Weill [2] proved that if f is locally univalent on D (a simply connected domain) with $||S_f||_D < \sigma(D)$, then f is univalent and can be extended to a quasiconformal mapping of $\overline{C}$. Ahlfors [1] and Gehring [4] proved that when D is a simply connected domain, $\sigma(D) > 0$ if and only if D is a quasidisk.

Next, we list some known values of $\sigma(D)$. Lehto and Lehtinen have calculated the inner radii of angular sectors in [8] and [7]. If $A_k = \{z : z \in C, 0 < \arg z < k\pi\}$, then $\sigma(A_k) = 2k^2$ for $0 < k < 1$ and $\sigma(A_k) = 4k - 2k^2$ for $1 < k < 2$. Another class of domains for which the inner radii have been calculated are regular polygons. Calvis [3] proved that $\sigma(P_n) = 2(n-2)^2/n^2$ where P_n is an open regular n-sided polygon.

In [12] we computed the inner radii for some classes of rectangles and equiangular hexagons. We proved that if R is a rectangle whose ratio of longer over shorter side is bounded from above by a specific constant $(1.52346\dots)$, then $\sigma(R) = 1/2$ and if H is an equiangular hexagon whose sides form the sequence *baabaa* with $b/a \leq 1.67117\dots$, then $\sigma(H) = 8/9$. In [10] we generalized the above mentioned results to classes of even-sided equiangular polygons. Namely, we proved that if two parallel sides of a n-sided regular polygon are stretched (shrunk) slightly, the inner radius remains constant (equals $2(n-2)^2/n^2$).

2. Nehari disks

In calculating the inner radii of domains mentioned earlier (angular sectors, regular polygons and some classes of even-sided equiangular polygons) the following simple but insightful lemma plays a key role.

Lemma 2.1. *If D is a simply connected domain and if h maps B conformally onto D, then $\sigma(D) \geq 2 - ||S_h||_B$.*

Proof. Suppose f is locally univalent on D with $||S_f||_D \leq 2 - ||S_h||_B$. Then, $f \circ h$ is locally univalent on B and $||S_{f \circ h}||_B = ||S_f - S_{h^{-1}}||_D \leq ||S_f||_D + ||S_h||_B \leq 2$ (see [11]). This implies that $f \circ h$ is univalent in B and hence f is univalent in D. Thus $\sigma(D) \geq 2 - ||S_h||_B$. □

It turns out that the lower bound $2 - ||S_h||_B$ is equal to $\sigma(D)$ in the case of many domains for which $\sigma(D)$ is known—disks, parallel strips, convex angular sectors and regular polygons, as well as the above-mentioned classes of even-sided equiangular polygons (see [11], [12] and [10]). Moreover, this yields a good method for computing $\sigma(D)$ for many domains. This prompted us to introduce a special name for these domains (we first introduced it in [12]). A simply connected domain D in $\overline{C}$ is called a *Nehari disk* if

$$\sigma(D) = 2 - ||S_h||_B,$$

where h maps B conformally onto D. Hence disks, parallel strips, convex angular sectors and the above-mentioned even-sided equiangular polygons are all Nehari disks. Of course, there exist many simply connected domains which are not (non convex angular sectors). By restricting our attention to regulated domains with convex corners (a class wide enough to include the domains mentioned earlier), we have proved the following characterization of Nehari disks (see [13]).

Theorem 2.2. *Suppose that D is a regulated domain with convex corners and that h maps B conformally onto D. Then, D is a Nehari disk if and only if* $\limsup_{|z| \to 1} |S_h(z)|(1 - |z|^2)^2 = ||S_h||_B$.

Thus, computing $\sigma(D)$ for some domains can be based on merely understanding the behavior of the Riemann mapping h.

There are a number of known results, specifically proved about univalence and extensions of mappings on B. Some of them clearly cannot be generalized for arbitrary, simply connected domains. For others, the question remains unanswered. Nehari disks allow generalizations of some of these results. In this paper we state and prove some of them.

3. Theorems of Gehring and Pommerenke

In view of the Ahlfors-Weill theorem mentioned in the introduction, we know that the image of B under a locally univalent function f with $||S_f||_B < 2$ is a quasidisk. In [5] Gehring and Pommerenke were able to show what happens to $f(B)$ when $||S_f||_B = 2$. Let S denote the parallel strip $S = \{z : 0 < \text{Im } z < \pi\}$.

Theorem 3.1. (Gehring-Pommerenke) *If f is a locally univalent function on B and if $||S_f||_B \leq 2$, then $f(B)$ is a Jordan domain or the image of the parallel strip S under a Möbius transformation.*

It is natural to ask whether the conclusion of the above theorem still holds if B is replaced by an arbitrary domain D and 2 by $\sigma(D)$. In general, the answer is no. For example, if $1 < k < 2$ and if $f : A_k \longrightarrow B_k$ is conformal, where $B_k = \left\{z : |\arg z| < \dfrac{k\pi}{2}\right\} \bigcap \left\{z : |\arg(1-z)| < \dfrac{k\pi}{2}\right\}$, then $||S_f||_{A_k} = \sigma(A_k)$, and B_k is clearly not a Jordan domain (see [9]). However, if we include the assumption that D is a Nehari disk, a generalization of Theorem 3.1 is easily obtainable.

Theorem 3.2. *Suppose D is a Nehari disk. If f is a locally univalent function on D and if $||S_f||_D \leq \sigma(D)$, then $f(D)$ is a Jordan domain or the image of the parallel strip S under a Möbius transformation.*

Proof. Let $h : B \longrightarrow D$ denote the Riemann mapping and let $g = f \circ h$. Then g is locally univalent on B and we have

$$||S_f||_D = ||S_{g\circ h^{-1}}||_D = ||S_g - S_h||_B \geq ||S_g||_B - ||S_h||_B.$$

Since D is a Nehari disk, $\sigma(D) = 2 - ||S_h||_B$ and

$$||S_g||_B \leq ||S_f||_D + ||S_h||_B \leq \sigma(D) + ||S_h||_B = 2.$$

By Theorem 3.1, we conclude that $g(B)$ is a Jordan domain or the image of the parallel strip S under a Möbius transformation. Since $f(D) = g \circ h^{-1}(D) = g(B)$, the proof is complete. □

We recall another result from [5].

Theorem 3.3. (Gehring-Pommerenke) *Suppose f is a locally univalent function on B and* $\limsup_{|z|\to 1} |S_f(z)|(1 - |z|^2)^2 < 2$. *If $f(B)$ is a Jordan domain, then $f(B)$ is a quasidisk.*

As before, a generalization for Nehari disks is possible.

Theorem 3.4. *Suppose D is a Nehari disk, f is a locally univalent function on D, and* $\limsup_{z\to\delta D} |S_f(z)|\rho_D^{-2}(z) < \sigma(D)$. *If $f(D)$ is a Jordan domain, then $f(D)$ is a quasidisk.*

Proof. Let $h : B \longrightarrow D$ denote the Riemann mapping and let $g = f \circ h$. Then g is locally univalent on B and we have

$$\begin{aligned}\limsup_{|z|\to 1} |S_g(z)|(1-|z|^2)^2 &= \limsup_{w\to \delta D} |S_f(w) - S_h^{-1}(w)|\rho_D^{-2}(w) \\ &\leq \limsup_{w\to \delta D} |S_f(w)|\rho_D^{-2}(w) + \limsup_{w\to \delta D} |S_h^{-1}(w)|\rho_D^{-2}(w) \\ &< \sigma(D) + ||S_h^{-1}||_D = \sigma(D) + ||S_h||_B = 2.\end{aligned}$$

By Theorem 3.3 , we conclude that if $g(B)$ is a Jordan domain, then it is a quasidisk. Since $f(D) = g \circ h^{-1}(D) = g(B)$, the proof is complete. □

4. A theorem of Ahlfors

We consider next some well-known results on the role of the Schwarzian derivative in showing univalence and existence of quasiconformal extensions of locally univalent functions. We must introduce the notion of complex dilatation of a quasiconformal mapping.

Let $f : D \longrightarrow D'$ be a K-quasiconformal mapping. Then f is differentiable with $\partial f \neq 0$ almost everywhere in D and the complex dilatation of f, μ_f, defined almost everywhere in D by $\mu_f(z) = \frac{\overline{\partial} f(z)}{\partial f(z)}$, satisfies

$$||\mu_f||_\infty \leq \frac{K-1}{K+1} < 1$$

where $\partial f(z) = \frac{1}{2}(f_x - f_y)$ and $\overline{\partial} f(z) = \frac{1}{2}(f_x + f_y)$. Here $||\mu_f||_\infty$ is the essential supremum of μ_f on D. The quantity $||\mu_f||_\infty$ is an alternative to K and measures the deviation of f from a conformal mapping. For details on complex dilatation, see [9].

Now we are ready to state a theorem of Ahlfors [1].

Theorem 4.1. (Ahlfors) *Let D be a L-quasidisk. There is a constant $\epsilon(L) > 0$, depending only on L, such that if f is locally univalent in D with $||S_f||_D < \epsilon(L)$, then f is univalent in D and can be extended to a K-quasiconformal mapping of $\overline{C}$ where $K \leq \frac{\epsilon(L)+||S_f||_D}{\epsilon(L)-||S_f||_D}$.*

The next theorem addresses the special case when D is a disk or a half-plane. In fact, this theorem was proved earlier than Theorem 4.1, by Ahlfors and Weill [2].

Theorem 4.2. (Ahlfors-Weill) *Suppose D is a disk or a half-plane and f is locally univalent on D. If $||S_f||_D < 2$, then f is univalent in D and can be extended to a K-quasiconformal mapping of $\overline{C}$ where $K \leq \frac{2+||S_f||_D}{2-||S_f||_D}$.*

Remark 4.3. In the proof of Theorem 4.2 it is shown that if $D = U$, where U is the upper half-plane, then the complex dilatation of this extension satisfies $\mu_f(\overline{z}) = -2y^2 S_f(z)$ for $z \in U$.

From the above we see that we may choose $\epsilon(L) = \sigma(D)$ whenever D is a disk or a half-plane. Lehto asked (in [9]) whether we may take $\epsilon(L) = \sigma(D)$ for other quasidisks D. We answer this question positively for Nehari disks. First, a preliminary lemma.

Lemma 4.4. *Suppose D is a quasidisk and h maps U conformally onto D. If f is locally univalent on D and if $||S_f||_D < 2 - |S_h||_U$, then f can be extended to a quasiconformal mapping of $\overline{C}$ whose complex dilatation satisfies $||\mu_f||_\infty \leq \frac{2||S_f||_D}{4-||S_{f\circ h}||_U \cdot ||S_h||_U}$.*

Proof. From our assumption clearly $||S_h||_U < 2$. Hence by Remark 4.3, h can be extended to quasiconformal mapping of $\overline{C}$, so that

$$\mu_h(\overline{z}) = -2y^2 S_h(z) \text{ for } z \in U \tag{1}$$

and consequently,

$$||\mu_h||_\infty \leq \frac{||S_h||_U}{2}. \tag{2}$$

Next let $g = f \circ h$. Then g is univalent in U and $||S_g||_U \leq ||S_f||_D + ||S_h||_U < 2$. Thus, by Remark 4.3, g can be extended to a quasiconformal mapping of $\overline{C}$, so that

$$\mu_g(\overline{z}) = -2y^2 S_g(z) \text{ for } z \in U \tag{3}$$

and consequently,

$$||\mu_g||_\infty \leq \frac{||S_g||_U}{2}. \tag{4}$$

Then $f = g \circ h^{-1}$ defines a quasiconformal extension of f on $\overline{C}$ with

$$\mu_f(w) = \mu_{g\circ h^{-1}}(w) = \frac{\mu_g(z) - \mu_h(z)}{1 - \mu_g(z)\overline{\mu_h(z)}} \cdot \left(\frac{\partial h(z)}{|\partial h(z)|}\right)^2$$

where $w = h(z)$ for almost all $z \in \overline{C}$ and hence for almost all $w \in \overline{C}$ (see [9]). Thus

$$|\mu_f(w)| \leq \frac{|\mu_g(z) - \mu_h(z)|}{1 - |\mu_g(z)||\overline{\mu_h(z)}|}$$

for almost all $w \in \overline{C}$, where $z = h^{-1}(w)$. Also from (2) and (4),

$$\frac{|\mu_g(z) - \mu_h(z)|}{1 - |\mu_g(z)||\overline{\mu_h(z)}|} \leq \frac{|\mu_g(z) - \mu_h(z)|}{1 - ||\mu_g||_\infty ||\mu_h||_\infty}$$

$$\leq \frac{|\mu_g(z) - \mu_h(z)|}{1 - \frac{||S_g||_U ||S_h||_U}{4}}$$
$$\leq \frac{4|\mu_g(z) - \mu_h(z)|}{4 - ||S_g||_U ||S_h||_U}$$

for almost all $z \in \overline{C}$, so we conclude that

$$|\mu_f(w)| \leq \frac{4|\mu_g(z) - \mu_h(z)|}{4 - ||S_g||_U ||S_h||_U} \tag{5}$$

for almost all $w \in \overline{C}$, where $z = h^{-1}(w)$. Finally, from (1) and (3),

$$\mu_g(z) - \mu_h(z) = -2y^2(S_g(\overline{z}) - S_h(\overline{z})) \tag{6}$$

for $z \in \overline{C} \setminus \overline{U}$. Then since h is a quasiconformal self-mapping of $\overline{C}$ and $h(U) = D$, $h(\overline{C} \setminus \overline{U}) = \overline{C} \setminus \overline{D}$, (5) and (6) imply that

$$|\mu_f(w)| \leq \frac{8y^2|S_g(\overline{z}) - S_h(\overline{z})|}{4 - ||S_g||_U ||S_h||_U}$$
$$\leq \frac{2||S_g - S_h||_U}{4 - ||S_g||_U ||S_h||_U}$$
$$\leq \frac{2||S_f||_D}{4 - ||S_{f\circ h}||_U ||S_h||_U}$$

for almost all $w \in \overline{C} \setminus \overline{D}$, where $z = h^{-1}(w)$.

As f is univalent in D and $\mu_f = 0$ in D, from the above we conclude that $|\mu_f(w)| \leq \frac{2||S_f||_D}{4 - ||S_{f\circ h}||_U ||S_h||_U}$ for almost all $w \in \overline{C}$. This shows that

$$||\mu_f||_\infty \leq \frac{2||S_f||_D}{4 - ||S_{f\circ h}||_U ||S_h||_U}, \tag{7}$$

so the lemma is proved. □

Now we can obtain the announced result.

Theorem 4.5. *Suppose D is a Nehari quasidisk and f is locally univalent on D. If $||S_f||_D < \sigma(D)$, then f is univalent in D and can be extended to a K-quasiconformal mapping of $\overline{C}$ where*

$$K \leq \frac{\sigma(D) + ||S_f||_D}{\sigma(D) - ||S_f||_D}.$$

Proof. Suppose that h maps U onto D conformally. Then $||S_f||_D < 2 - ||S_h||_U$. From Lemma 4.4 we conclude that f can be extended to a quasiconformal mapping of $\overline{C}$ whose complex dilatation satisfies

$$||\mu_f||_\infty \leq \frac{2||S_f||_D}{4 - ||S_{f\circ h}||_U ||S_h||_U}. \tag{8}$$

It also follows that $||S_{f\circ h}||_U \leq ||S_f||_D + ||S_h||_U < 2$ and hence, as $\sigma(D) = 2 - ||S_h||_U$, it can be seen that

$$\begin{aligned}\frac{2||S_f||_D}{4-||S_{f\circ h}||_U||S_h||_U} &= \frac{2||S_f||_D}{2(\sigma(D)+||S_h||_U)-||S_{f\circ h}||_U||S_h||_U} \\ &= \frac{2||S_f||_D}{2\sigma(D)+||S_h||_U(2-||S_{f\circ h}||_U)} \\ &\leq \frac{||S_f||_D}{\sigma(D)}.\end{aligned}$$

The above together with (8) yields $||\mu_f||_\infty \leq \frac{||S_f||_D}{\sigma(D)}$ and consequently for $K = \frac{1+||\mu_f||_\infty}{1-||\mu_f||_\infty}$, f is K-quasiconformal on $\overline{C}$ and

$$K \leq \frac{\sigma(D)+||S_f||_D}{\sigma(D)-||S_f||_D}.$$

completing the proof of the theorem. □

5. A theorem of Minda

Minda [14] generalized the classical result of Nehari that $\sigma(B) = 2$ by proving the following theorem.

Theorem 5.1. (Minda) *Suppose f is a locally univalent function on B. If $||S_f||_B \leq 2(1+\delta^2)$ for some $\delta \geq 0$, then f is univalent in the disk $\left\{z : |z| < \tanh\left(\frac{\pi}{2\delta}\right)\right\}$ and the radius of the disk is sharp for all $\delta \geq 0$.*

The example Minda used to show sharpness was the function

$$F_\delta(z) = \left(\frac{1+z}{1-z}\right)^{i\delta} = \exp\left(i\delta \log\left(\frac{1+z}{1-z}\right)\right)$$

defined on B for any fixed $\delta > 0$ (this function was first mentioned by Hille [6]). For $\delta > 0$, F_δ is univalent in $\left\{z : |z| < \tanh\left(\frac{\pi}{2\delta}\right)\right\}$ but is not injective on any larger disk centered at the origin.

There is another, more general formulation of this theorem in terms of hyperbolic distance (for details see [9]). The advantage of using hyperbolic distance is that it is invariant with respect to conformal mappings. We introduce the following notation (see [14]). If f is a holomorphic function on a domain D, then for $z \in D$ we define $r(z,f)$ to be the hyperbolic radius of the largest hyperbolic disk in D centered at z in which f is univalent. We define $r(f)$ as $\inf\{r(z,f) : z \in D\}$. By using the above notation and an appropriate Möbius transformation, Minda proved a more general formulation of Theorem 5.1. This can be stated as follows.

Theorem 5.2. (Minda) *Suppose f is a locally univalent function on D, where D is a disk or a half-plane. If $||S_f||_D \leq \sigma(D) + 2\delta^2$, then $r(f) \geq \frac{\pi}{2\delta}$ and this lower bound is sharp for all $\delta \geq 0$.*

The example used to show sharpness for disks in Theorem 5.2 is the same as in Theorem 5.1 up to composition with a Möbius transformation.

Remark 5.3. When $D = U$, the function $G_\delta(z) = (-iz)^{i\delta}$ on U demonstrates sharpness for $\delta \geq 0$.

Now, we prove a generalization of Theorem 5.2 for Nehari disks.

Theorem 5.4. *Suppose f is a locally univalent function on D, where D is a Nehari disk. If $||S_f||_D \leq \sigma(D) + 2\delta^2$, then $r(f) \geq \frac{\pi}{2\delta}$, for all $\delta \geq 0$.*

Proof. Let $g = f \circ h$ on B, where h maps B onto D conformally. Then g is locally univalent on B and

$$\begin{aligned} ||S_g||_B &= ||S_{f\circ h}||_B \\ &\leq ||S_f||_D + ||S_h||_B \\ &= ||S_f||_D + 2 - \sigma(D) \\ &\leq (\sigma(D) + 2\delta^2) + 2 - \sigma(D) \\ &= 2 + 2\delta^2 \end{aligned}$$

since D is a Nehari disk. Thus by Theorem 5.2, $r(f \circ h) = r(g) \geq \frac{\pi}{2\delta}$ and since hyperbolic distance is invariant under conformal mapping, we get

$$r(f) \geq \frac{\pi}{2\delta}.$$

□

In some specific cases, we can verify the sharpness for this result. This is shown in the following corollaries.

Corollary 5.5. *Suppose that $0 < k < 1$ and that f is locally univalent on the angular sector A_k. If $||S_f||_{A_k} \leq 2k^2 + 2\delta^2$, then $r(f) \geq \frac{\pi}{2\delta}$ and this lower bound is sharp for all $\delta \geq 0$.*

Proof. Let $h(z) = z^k$ for $z \in U$. Then h maps U conformally onto A_k and we know that $\sigma(A_k) = 2k^2 = 2 - ||S_h||_U$ so the first part of the statement follows from Theorem 5.4. To show sharpness, we look at the function $G_\delta \circ h^{-1}$ on A_k for $\delta \geq 0$; $G_\delta \circ h^{-1}$ is locally univalent on A_k and for $\delta \geq 0$,

$$||S_{G_\delta \circ h^{-1}}||_{A_k} = ||S_{G_\delta} - S_h||_U = \sup_{z\in U} 4y^2 \left|\frac{1+\delta^2}{2z^2} - \frac{1-k^2}{2z^2}\right| = 2k^2 + 2\delta^2.$$

From Remark 5.3, $r(G_\delta \circ h^{-1}) = r(G_\delta) = \frac{\pi}{2\delta}$ for $\delta \geq 0$ so sharpness is proved. □

Corollary 5.6. *Suppose f is locally univalent on the parallel strip S. If $||S_f||_S \leq 2\delta^2$, then $r(f) \geq \frac{\pi}{2\delta}$ and this lower bound is sharp for all $\delta \geq 0$.*

Proof. The proof is completely analogous to that of Corollary 5.5, with $h(z) = \log z$. □

Finally, Theorem 5.4 can be restated as follows, thus eliminating the use of the constant δ.

Theorem 5.7. *Suppose f is a locally univalent function on D, where D is a Nehari disk. If $||S_f||_D > \sigma(D)$, then*

$$r(f) \geq \frac{\pi}{\sqrt{2(||S_f||_D - \sigma(D))}}.$$

Proof. Let

$$\delta = \sqrt{\frac{||S_f||_D - \sigma(D)}{2}}.$$

The result immediately follows from Theorem 5.4. □

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