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ON A NONLINEAR VOLTERRA-FREDHOLM INTEGRAL EQUATION

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Abstract. In this paper we study the existence, uniqueness and other properties of solutions of a certain nonlinear Volterra-Fredholm integral equation. The well known Banach fixed point theorem and the new integral inequality with explicit estimate are used to establish the results.

1. Introduction

Consider the system of Volterra-Fredholm integral equations

$$x(t) = f(t) + \int_0^t g(t, s, x(s))\, ds + \int_0^\infty h(t, s, x(s))\, ds, \tag{VF}$$

for $0 \leq t < \infty$, where x, f, g, h are in R^n, the n-dimensional Euclidean space with appropriate norm denoted by $|.|$. We denote by $R_+ = [0, \infty)$, the given subset of R, the set of real numbers and throughout assume that $f \in C(R_+, R^n)$, and for $0 \leq s \leq t < \infty$; $g \in C(R_+^2, R^n)$ and $h \in C(R_+^2, R^n)$. Equations of the form (VF) arise naturally in the study of boundary value problems on the infinite half line, see [7]. In [11] Nohel has studied the special version of (VF) with $g(t, s, x(s)) = v(t, s)\, g_1(s, x(s))$ $h(t, s, x(s)) = k(t, s)\, h_1(s, x(s))$ by using the Schauder-Tychonoff fixed point theorem, when g_1, h_1 are in R^n and v, k are given n by n matrices (see also [1,4,9,12]). The advantage of the technique used in [11] (see also [9]) is that the solution to (VF) is obtained as the fixed point on the entire interval, avoiding the necessity of using continuation theorem. Frequently, it happens that the method which works very efficiently to establish existence does not yield other properties of the solutions in any ready fashion. The main objective of this paper is to study the existence, uniqueness and other properties of the solutions of equation (VF) by using Banach fixed point

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theorem (see [6, p.37]) coupled with Bielecki type norm [2,13] and the new integral inequality recently established by the present author in [15] (see also [16, p. 41]).

2. Existence and uniqueness

Let E be the space of those functions $\phi : R_+ \to R^n$ which are continuous and fulfill the condition

$$|\phi(t)| = O(\exp(\lambda t)), \tag{2.1}$$

where λ is a positive constant. In the space E we define the norm (see [2, 13])

$$|\phi|_E = \sup_{t\in R_+} [|\phi(t)| \exp(-\lambda t)]. \tag{2.2}$$

It is easily seen that E with norm defined in (2.2) is a Banach space. We note that the condition (2.1) implies that there exists a constant $N \geq 0$ such that $|\phi(t)| \leq N \exp(\lambda t)$. Using this fact in (2.2) we observe that

$$|\phi|_E \leq N. \tag{2.3}$$

We need the following slight variant of the new integral inequality established by Pachpatte in [15] (see also [16, Theorem 1.5.1 part (a_2)), to study various properties of solutions of equation (VF). For a detailed account on such inequalities, see [14,16].

Lemma. *Let $u(t), a(t), b(t), c(t), p(t), q(t) \in C(R_+, R_+)$ and suppose*

$$u(t) \leq a(t) + b(t) \int_0^t p(s) u(s)\, ds + c(t) \int_0^\infty q(s) u(s)\, ds, \tag{2.4}$$

for $t \in R_+$. If

$$d = \int_0^\infty q(s) M(s)\, ds < 1, \tag{2.5}$$

then

$$u(t) \leq L(t) + K M(t), \tag{2.6}$$

for $t \in R_+$, where

$$L(t) = a(t) + b(t) \int_0^t p(\tau) a(\tau) \exp\left(\int_\tau^t p(\sigma) b(\sigma)\, d\sigma\right) d\tau, \tag{2.7}$$

$$M(t) = c(t) + b(t) \int_0^t p(\tau) c(\tau) \exp\left(\int_\tau^t p(\sigma) b(\sigma)\, d\sigma\right) d\tau, \tag{2.8}$$

and

$$K = \frac{1}{1-d} \int_0^\infty q(s) L(s)\, ds. \tag{2.9}$$

Our main result in this section is given in the following theorem.

Theorem 1. *Assume that*

(i) *g, h satisfy the conditions*

$$|g(t,s,x(s)) - g(t,s,y(s))| \leq v(t,s)\,|x(s) - y(s)|, \tag{2.10}$$
$$|h(t,s,x(s)) - h(t,s,y(s))| \leq k(t,s)\,|x(s) - y(s)|, \tag{2.11}$$

where $v, k \in C\left(R_+^2, R_+\right)$,

(ii) *for λ as in (2.1), there exist nonnegative constants α_1, α_2 such that $\alpha_1 + \alpha_2 < 1$ and*

$$\int_0^t v(t,s)\exp(\lambda s)\,ds \leq \alpha_1 \exp(\lambda t), \tag{2.12}$$
$$\int_0^\infty k(t,s)\exp(\lambda s)\,ds \leq \alpha_2 \exp(\lambda t), \tag{2.13}$$

(iii) *for λ as in (2.1), there exists a nonnegative constant β such that*

$$|f(t)| + \int_0^t |g(t,s,0)|\,ds + \int_0^\infty |h(t,s,0)|\,ds \leq \beta \exp(\lambda t). \tag{2.14}$$

Then the equation (VF) has a unique solution $x(t)$, $t \in R_+$ in E.

Proof. Let $x \in E$ and define the operator T by

$$(Tx)(t) = f(t) + \int_0^t g(t,s,x(s))\,ds + \int_0^\infty h(t,s,x(s))\,ds. \tag{2.15}$$

First we shall show that Tx maps E into itself. Evidently, Tx is continuous on R_+ and $Tx \in R^n$. We verify that (2.1) is fulfilled. From (2.15) and using the hypotheses we have

$$|(Tx)(t)| \leq |f(t)| + \int_0^t |g(t,s,x(s)) - g(t,s,0)|\,ds$$
$$+ \int_0^\infty |h(t,s,x(s)) - h(t,s,0)|\,ds + \int_0^t |g(t,s,0)|\,ds + \int_0^\infty |h(t,s,0)|\,ds$$
$$\leq \beta \exp(\lambda t) + \int_0^t |v(t,s)|\,|x(s)|\,ds + \int_0^\infty |k(t,s)|\,|x(s)|\,ds$$
$$\leq \beta \exp(\lambda t) + \int_0^t |v(t,s)| \exp(\lambda s)\,|x|_E\,ds + \int_0^\infty |k(t,s)| \exp(\lambda s)\,|x|_E\,ds$$
$$\leq [\beta + N(\alpha_1 + \alpha_2)] \exp(\lambda t). \tag{2.16}$$

From (2.16) it follows that $Tx \in E$. This proves that T maps E into itself.

Next, we verify that the operator T is a contraction map. Let $x, y \in E$. From (2.15) and using the hypotheses we have

$$\begin{aligned}
|(Tx)(t) - (Ty)(t)| &\le \int_0^t |g(t,s,x(s)) - g(t,s,y(s))|\, ds \\
&\quad + \int_0^\infty |h(t,s,x(s)) - h(t,s,y(s))|\, ds \\
&\le \int_0^t v(t,s)\, |x(s) - y(s)|\, ds + \int_0^\infty k(t,s)\, |x(s) - y(s)|\, ds \\
&\le \int_0^t v(t,s) \exp(\lambda s)\, |x-y|_E ds + \int_0^\infty k(t,s) \exp(\lambda s)\, |x-y|_E ds \\
&\le |x-y|_E (\alpha_1 + \alpha_2) \exp(\lambda t).
\end{aligned}$$

Consequently, we have

$$|Tx - Ty|_E \le (\alpha_1 + \alpha_2)\, |x-y|_E .$$

Since $\alpha_1 + \alpha_2 < 1$, it follows from Banach fixed point theorem (see [6, p. 37]) that T has a unique fixed point in E . The fixed point of T is however a solution of equation (VF). The proof is complete. □

Remark 1. We note that the norm $|.|_E$ defined by (2.2) was first used by Bielecki [2] (see [5] for developments related to this topic), and has the role of improving the length of the interval on which the existence is assured. In [11] Nohel has obtained the global existence of solutions of the special version of (VF) by giving up uniqueness. Here, Theorem 1 yields the existence and uniqueness of solutions of equation (VF) in E.

Indeed, the following theorem is true concerning the uniqueness of solutions of equation (VF) in R^n.

Theorem 2. *Assume that the functions g, h in equation* (VF) *satisfy the conditions* (2.10), (2.11) *with $v(t,s) = b(t)\, p(s), k(t,s) = c(t)\, q(s)$ where $b, p, c, q \in C(R_+, R_+)$. Let $d, M(t)$ be as in* (2.5), 2(8). *Then the equation* (VF) *has at most one solution on R_+.*

Proof. Let $x_1(t)$ and $x_2(t)$ be two solutions of equation (VF). Then we have

$$\begin{aligned}
x_1(t) - x_2(t) &= \int_0^t \{g(t,s,x_1(s)) - g(t,s,x_2(s))\}\, ds \\
&\quad + \int_0^\infty \{h(t,s,x_1(s)) - h(t,s,x_2(s))\}\, ds. \qquad (2.17)
\end{aligned}$$

From (2.17) and using the hypotheses we have

$$|x_1(t) - x_2(t)| \le b(t) \int_0^t p(s) |x_1(s) - x_2(s)| \, ds \\ + c(t) \int_0^\infty q(s) |x_1(s) - x_2(s)| \, ds. \quad (2.18)$$

Here, it is easy to observe that $L(t)$ and K defined by (2.7) and (2.9) reduces to $L(t) = 0$ and $K = 0$. Now an application of Lemma to (2.18) yields $|x_1(t) - x_2(t)| \le 0$, and hence $x_1(t) = x_2(t)$. Thus there is at most one solution to the equation (VF). □

3. Estimates on solutions

In this section we obtain estimates on the solutions of the equation (VF) under some suitable assumptions on the functions involved in equation (VF).

Theorem 3. *Assume that the functions g, h satisfy the conditions*

$$|g(t, s, x(s))| \le b(t) p(s) |x(s)|, \quad (3.1)$$

$$|h(t, s, x(s))| \le c(t) q(s) |x(s)|, \quad (3.2)$$

where $b, p, c, q \in C(R_+, R_+)$. Let $d, M(t)$ be as in (2.5), (2.8) *and*

$$K_1 = \frac{1}{1-d} \int_0^\infty q(s) L_1(s) \, ds, \quad (3.3)$$

where $L_1(t)$ is defined by the right hand side of (2.7) *by replacing $a(t)$ by $|f(t)|$. If $x(t), t \in R_+$ is any solution of equation* (VF)*, then*

$$|x(t)| \le L_1(t) + K_1 M(t), \quad (3.4)$$

for $t \in R_+$.

Proof. By using the fact that $x(t)$ is a solution of equation (VF) and the hypotheses we have

$$|x(t)| \le |f(t)| + \int_0^t |g(t, s, x(s))| \, ds + \int_0^\infty |h(t, s, x(s))| \, ds \\ \le |f(t)| + b(t) \int_0^t p(s) |x(s)| \, ds + c(t) \int_0^\infty q(s) |x(s)| \, ds. \quad (3.5)$$

Now an application of Lemma to (3.5) yields (3.4). □

Next, we shall obtain the estimation on the solution of equation (VF) assuming that the functions g, h satisfy Lipschitz type conditions.

Theorem 4. *Suppose that the functions g, h be as in Theorem 2. Let $d, M(t)$ be as in (2.5), (2.8) and*

$$F(t) = \int_0^t g(t, s, f(s))\, ds + \int_0^\infty h(t, s, f(s))\, ds, \tag{3.6}$$

$$K_2 = \frac{1}{1-d} \int_0^\infty q(s)\, L_2(s)\, ds, \tag{3.7}$$

where $L_2(t)$ is defined by the right hand side of (2.7) by replacing $a(t)$ by $|F(t)|$. If $x(t), t \in R_+$ is any solution of equation (VF), then

$$|x(t) - f(t)| \leq L_2(t) + K_2 M(t), \tag{3.8}$$

for $t \in R_+$.

Proof. Using the fact that $x(t)$ is a solution of (VF) we observe that

$$\begin{aligned} x(t) - f(t) = &\int_0^t \{g(t, s, x(s)) - g(t, s, f(s))\}\, ds \\ &+ \int_0^\infty \{h(t, s, x(s)) - h(t, s, f(s))\}\, ds \\ &+ \int_0^t g(t, s, f(s))\, ds + \int_0^\infty h(t, s, f(s))\, ds. \end{aligned} \tag{3.9}$$

From (3.9) and using the hypotheses we have

$$\begin{aligned} |x(t) - f(t)| \leq |F(t)| &+ b(t) \int_0^t p(s)\, |x(s) - f(s)|\, ds \\ &+ c(t) \int_0^t q(s)\, |x(s) - f(s)|\, ds. \end{aligned} \tag{3.10}$$

Now an application of Lemma to (3.10) yields (3.8). □

Next, consider the system (VF) and the system of Volterra integral equations

$$y(t) = f(t) + \int_0^t g(t, s, y(s))\, ds, \tag{3.11}$$

for $t \in R_+$, where f, g are as in equation (VF).

The following theorem deals with the estimate on the difference between the solutions of equations (VF) and (3.11).

Theorem 5. *Suppose that the functions g, h be as in Theorem 2 and further assume that $h(t, s, 0) = 0$. Let $y(t), t \in R_+$ be a solution of equation (3.11) such that $|y(t)| \leq Q$, where $Q \geq 0$ is a constant. Let*

$$\bar{a}(t) = Qc(t) \int_0^\infty q(s)\, ds,$$

and $d, M(t)$ *be as in* (2.5), (2.8) *and*

$$K_3 = \frac{1}{1-d} \int_0^\infty q(s) L_3(s)\, ds, \tag{3.12}$$

where $L_3(t)$ *is defined by the right hand side of* (2.7) *by replacing* $a(t)$ *by* $\bar{a}(t)$. *If* $x(t), t \in R_+$ *is a solution of equation* (VF), *then*

$$|x(t) - y(t)| \le L_3(t) + K_3 M(t), \tag{3.13}$$

for $t \in R_+$.

Proof. Using the facts that $x(t)$ and $y(t)$ for $t \in R_+$ are the solutions of equations (VF) and (3.11) we observe that

$$\begin{aligned} x(t) - y(t) = &\int_0^t \{g(t,s,x(s)) - g(t,s,y(s))\}\, ds \\ &+ \int_0^\infty \{h(t,s,x(s)) - h(t,s,y(s))\}\, ds \\ &+ \int_0^\infty \{h(t,s,y(s)) - h(t,s,0)\}\, ds. \end{aligned} \tag{3.14}$$

From (3.14) and using the hypothes we have

$$\begin{aligned} |x(t) - y(t)| &\le b(t) \int_0^t p(s)\, |x(s) - y(s)| ds + c(t) \int_0^\infty q(s)\, |x(s) - y(s)| ds \\ &\quad + c(t) \int_0^\infty q(s)\, |y(s)| ds \\ &\le \bar{a}(t) + b(t) \int_0^t p(s)\, |x(s) - y(s)| ds \\ &\quad + c(t) \int_0^\infty q(s)\, |x(s) - y(s)| ds. \end{aligned} \tag{3.15}$$

Now as an application of Lemma to (3.15) yields (3.13). □

4. Continuous dependence

In this section we study the continuous dependence of solutions of equation (VF) on the functions involved on the right hand side of equation (VF) and also the continuous dependence of solutions of equations of the forms (VF) on parameters.

Consider the system (VF) and the system of Volterra-Fredholm integral equations

$$y(t) = r(t) + \int_0^t G(t,s,y(s))\,ds + \int_0^\infty H(t,s,y(s))\,ds, \tag{4.1}$$

for $t \in R_+$, where r, G, H are in R^n, $r \in C(R_+, R^n)$ and for $0 \le s \le t < \infty$, $G, H \in C\left(R_+^2 \times R^n, R^n\right)$.

The following theorem shows the continuous dependence of solutions of equation (VF) on the right hand side of equation (VF).

Theorem 6. *Suppose that the functions g, h be as in Theorem 2. Assume that*

$$|f(t) - r(t)| + \int_0^t |g(t,s,y(s)) - G(t,s,y(s))|\,ds + \int_0^\infty |h(t,s,y(s)) - H(t,s,y(s))|\,ds \le \varepsilon, \tag{4.2}$$

where f, g, h and r, G, H are the functions involved in equations (VF) and (4.1), $y(t)$ is a solution of equation (4.1) and $\varepsilon > 0$ is an arbitrary small constant. Let $d, M(t)$ be as in (2.5), (2.8) and

$$K_4 = \frac{1}{1-d}\int_0^\infty q(s)\,L_4(s)\,ds, \tag{4.3}$$

where $L_4(t)$ is defined by the right hand side of (2.7) by replacing $a(t)$ by ε. Then the solution $x(t), t \in R_+$ of equation (VF) depends continuously on the functions involved on the right hand side of equation (VF).

Proof. Let $x(t)$ and $y(t)$ for $t \in R_+$ be the solutions of equations (VF) and (4.1) respectively. Then

$$\begin{aligned} x(t) - y(t) &= f(t) - r(t) \\ &+ \int_0^t \{g(t,s,x(s)) - g(t,s,y(s)) + g(t,s,y(s)) - G(t,s,y(s))\}\,ds \\ &+ \int_0^\infty \{h(t,s,x(s)) - h(t,s,y(s)) + h(t,s,y(s)) - H(t,s,y(s))\}\,ds. \end{aligned} \tag{4.4}$$

From (4.4) and using the hypotheses we have

$$|x(t)-y(t)| \leq |f(t)-r(t)| + \int_0^t |g(t,s,x(s)) - g(t,s,y(s))|\, ds$$
$$+\int_0^t |g(t,s,y(s)) - G(t,s,y(s))|\, ds + \int_0^\infty |h(t,s,x(s)) - h(t,s,y(s))|\, ds$$
$$+\int_0^\infty |h(t,s,y(s)) - H(t,s,y(s))|\, ds$$
$$\leq \varepsilon + b(t)\int_0^t p(s)|x(s)-y(s)|\, ds + c(t)\int_0^\infty q(s)|x(s)-y(s)|\, ds. \quad (4.5)$$

Now an application of Lemma to (4.5) yields

$$|x(t)-y(t)| \leq L_4(t) + K_4 M(t), \quad (4.6)$$

for $t \in R_+$. From (4.6) it follows that the solutions of equation (VF) depends continuously on the functions involved on the right hand side of equation (VF). □

Remark 2. From (4.6) it is easy to observe that, if $L_4(t)$ and $M(t)$ are bounded for $t \in R_+$ and $\varepsilon \to 0$, then $|x(t)-y(t)| \to 0$ on R_+.

We, next consider the following systems of Volterra-Fredholm integral equations

$$z(t) = f(t) + \int_0^t A(t,s,z(s),\mu)ds + \int_0^\infty B(t,s,z(s),\mu)ds, \quad (4.7)$$

and

$$z(t) = f(t) + \int_0^t A(t,s,z(s),\mu_0)ds + \int_0^\infty B(t,s,z(s),\mu_0)ds, \quad (4.8)$$

for $t \in R_+$, where f, A, B are in R^n; μ, μ_0 are real parameters, $f \in C(R_+, R^n)$ and for $0 \leq s \leq t < \infty$, $A, B \in C\left(R_+^2 \times R^n \times R, R^n\right)$.

The next theorem shows the dependency of solutions of equations (4.7) and (4.8) on parameters.

Theorem 7. *Assume that the functions A, B satisfy the conditions*

$$|A(t,s,z,\mu) - A(t,s,\bar{z},\mu)| \leq b(t)p(s)|z-\bar{z}|, \quad (4.9)$$
$$|A(t,s,z,\mu) - A(t,s,z,\mu_0)| \leq r_1(t,s)|\mu-\mu_0|, \quad (4.10)$$
$$|B(t,s,z,\mu) - B(t,s,\bar{z},\mu)| \leq c(t)q(s)|z-\bar{z}|, \quad (4.11)$$
$$|B(t,s,z,\mu) - B(t,s,z,\mu_0)| \leq r_2(t,s)|\mu-\mu_0|, \quad (4.12)$$

where $b, p, c, q \in C(R_+, R_+)$, $r_1, r_2 \in C(R_+^2, R_+)$. *Let*

$$a_0(t) = |\mu - \mu_0| \left[\int_0^t r_1(t,s)\,ds + \int_0^\infty r_2(t,s)\,ds \right],$$

and $d, M(t)$ *be as in* (2.5), (2.8) *and*

$$K_5 = \frac{1}{1-d} \int_0^\infty q(s)\,L_5(s)\,ds, \tag{2.13}$$

where $L_5(t)$ *is defined by the right hand side of* (2.7) *by replacing* $a(t)$ *by* $a_0(t)$ *as given above. Let* $z_1(t)$ *and* $z_2(t)$ *be the solutions of equations* (4.7) *and* (4.8) *respectively. Then*

$$|z_1(t) - z_2(t)| \le L_5(t) + K_5 M(t), \tag{4.14}$$

for $t \in R_+$.

Proof. Let $z(t) = z_1(t) - z_2(t)$. Since $z_1(t)$ and $z_2(t)$ are the solutions of equations (4.7) and (4.8) we have

$$\begin{aligned} z(t) = z_1(t) - z_2(t) = \int_0^t &\big\{ A(t,s,z_1(s),\mu) - A(t,s,z_2(s),\mu) \\ &+ A(t,s,z_2(s),\mu) - A(t,s,z_2(s),\mu_0) \big\}\,ds \\ &+ \int_0^\infty \big\{ B(t,s,z_1(s),\mu) - B(t,s,z_2(s),\mu) \\ &+ B(t,s,z_2(s),\mu) - B(t,s,z_2(s),\mu_0) \big\}\,ds. \end{aligned} \tag{4.15}$$

From (4.15) and using the hypotheses we have

$$\begin{aligned} |z(t)| &\le \int_0^t |A(t,s,z_1(s),\mu) - A(t,s,z_2(s),\mu)|\,ds \\ &\quad + \int_0^t |A(t,s,z_2(s),\mu) - A(t,s,z_2(s),\mu_0)|\,ds \\ &\quad + \int_0^\infty |B(t,s,z_1(s),\mu) - B(t,s,z_2(s),\mu)|\,ds \\ &\quad + \int_0^\infty |B(t,s,z_2(s),\mu) - B(t,s,z_2(s),\mu_0)|\,ds \\ &\le b(t) \int_0^t p(s)\,|z_1(s) - z_2(s)|\,ds + \int_0^t r_1(t,s)\,|\mu - \mu_0|\,ds \\ &\quad + c(t) \int_0^\infty q(s)\,|z_1(s) - z_2(s)|\,ds + \int_0^\infty r_2(t,s)\,|\mu - \mu_0|\,ds \\ &= a_0(t) + b(t) \int_0^t p(s)\,z(s)\,ds + +c(t) \int_0^\infty q(s)\,z(s)\,ds. \end{aligned} \tag{4.16}$$

Now an application of Lemma to (4.16) yields (4.14), which shows the dependence of solutions of equations (4.7) and (4.8) on parameters. □

Remark 3. We note that the equation considered in (VF) is of more general type and in the special case, it contains both Volterra and Fredholm type integral equations. The literature concerning such equations is particularly rich, and we refer the readers to the books [3,6,10].

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