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PERIOD-TWO TRICHOTOMIES OF A DIFFERENCE EQUATION OF ORDER HIGHER THAN TWO

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Dedicated to Professor Fikret Vajzović on the occasion of his 80th birthday

Abstract. We investigate the period-two trichotomies of solutions of the equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}), \quad n = 0, 1, \dots$$

where the function f satisfies certain monotonicity conditions. We give fairly general conditions for period-two trichotomies to occur and illustrate the results with numerous examples.

1. Introduction and preliminaries

We investigate the period-two trichotomies of solutions of the equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}), \quad n = 0, 1, \dots \tag{1}$$

where the function f satisfies some monotonicity conditions and the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary non-negative real numbers.

The period-two trichotomy was discovered in [1], in the case of equation

$$x_{n+1} = p + \frac{x_{n-1}}{x_n} \quad n = 0, 1, \dots , \tag{2}$$

where $p > 0$ and $x_{-1}, x_0 > 0$, and can be stated as the following result:

Theorem 1. *The following period-two trichotomy result holds for Eq.* (2)

$p < 1 \Rightarrow$ *there exist unbounded solutions*
$p = 1 \Rightarrow$ *every solution converges to a period-two solution*
$p > 1 \Rightarrow$ *every solution converges to the equilibrium.*

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Recently, this result, which is not global, has been improved in the sense that the statement "there exist unbounded solutions" was replaced by the statement that "every solution off the global stable manifold of the unique equilibrium is unbounded", see [15]. Further period-two trichotomy results were obtained in a sequence of papers [7], [8], and [11] for the second order rational difference equations. More precisely all period-two trichotomies were described for the second order linear fractional difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}} \quad n = 0, 1, \dots ,$$

with non-negative parameters $\alpha, \beta, \gamma, A, B$, and C and non-negative initial conditions x_{-1}, x_0 such that $A + Bx_n + Cx_{n-1} > 0$ for all n.

Also Theorem 1 was generalized to the global bifurcation result for the general second order equation of the form

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots \tag{3}$$

where f satisfies some monotonicity and differentiability conditions. See [2].

The only bifurcation results obtained for Eq. (3) are period-doubling bifurcation of Selgrade and Roberds [18] and Naimark-Sacker bifurcation [6] and [12]. Both results are local as they guarantee the existence and stability of bifurcating periodic solution in a neighborhood of the critical value(s) of the parameter(s). Some global bifurcation results were obtained for monotone discrete dynamical systems that includes Eq. (3) when f is decreasing in first and increasing in second variable. Actually, we have shown that period-two trichotomy described above is exactly global period-doubling bifurcation. See [16]. Unfortunately, the tools of monotone discrete dynamical systems that we used in [2] and [16] are two-dimensional and do not extend to higher dimensions. The phenomenon of period-two trichotomy extends to third order equation (1) as is seen from the following result:

Theorem 2. (see [5]) *Consider the following equation*

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{A + Bx_n + x_{n-2}}, \quad n = 0, 1, \dots \tag{4}$$

where all parameters are non-negative and the initial conditions are positive.

The following period-two trichotomy result holds for Eq. (4)*:*

$\gamma > A \;\Rightarrow\;$ *there exist unbounded solutions*
$\gamma = A \;\Rightarrow\;$ *every solution converges to a period-two solution*
$\gamma < A \;\Rightarrow\;$ *every solution converges to the equilibrium.*

In this paper we extend the period-two trichotomy result for Eq. (4) to the case of general third order difference equation (1), where f satisfies certain monotonicity conditions. In particular we generalize first and third statement of Theorem 2 to the case of general third equation (1) and also to the difference equation of any order. We illustrate our results with numerous examples and conclude with two conjectures that give an indication of the connection between the above mentioned period-two trichotomy result and the global period-doubling bifurcation described in one of the conjectures.

Here we give some necessary definitions and results that we will use later.

Let I be an interval of real numbers and let $f \in C^1[I^3, I]$. Let $\bar{x} \in I$ be an equilibrium point of the difference equation (1) that is $\bar{x} = f(\bar{x}, \bar{x}, \bar{x})$.

Let

$$r = \frac{\partial f}{\partial u}(\bar{x}, \bar{x}, \bar{x}), \quad s = \frac{\partial f}{\partial v}(\bar{x}, \bar{x}, \bar{x}), \quad t = \frac{\partial f}{\partial w}(\bar{x}, \bar{x}, \bar{x})$$

denote the partial derivatives of $f(u, v, w)$ evaluated at an equilibrium $\bar{x}$ of Eq. (1). Then the equation

$$y_{n+1} = ry_n + sy_{n-1} + ty_{n-2}, \quad n = 0, 1, \dots \tag{5}$$

is called the **linearized equation** associated with Eq. (1) about the equilibrium point $\bar{x}$. The following result is well known, see [11]

Theorem 3. (**Linearized Stability**)

(a) *If all roots of the cubic equation*

$$\lambda^3 - r\lambda^2 - s\lambda - t = 0 \tag{6}$$

lie in the open unit disk $|\lambda| < 1$ of the complex plane C, then the equilibrium $\bar{x}$ of Eq.(1) is locally asymptotically stable.

(b) *If at least one of the roots of Eq.(6) has modulus greater than one, then the equilibrium $\bar{x}$ of Eq.(1) is unstable.*

(c) *A necessary and sufficient condition for all roots of Eq.(6) to lie in the open unit disk $|\lambda| < 1$, is*

$$|r + t| < 1 - s,$$
$$|r - 3t| < 3 + s,$$
$$t^2 - s - rt < 1.$$

In this case the locally asymptotically stable equilibrium $\bar{x}$ is also called a **sink**.

We now give the definitions of positive and negative semicycle of a solution of Eq. (1) relative to an equilibrium point $\bar{x}$.

A *positive semicycle* of a solution $\{y_n\}$ of Eq. (1) consists of a string of terms $\{y_l, y_{l+1}, \dots, y_m\}$, all greater than or equal to the equilibrium $\overline{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either} \quad l = -1, \quad \text{or} \quad l > -1 \text{ and } y_{l-1} < \overline{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \text{ and } y_{m+1} < \overline{x}.$$

A *negative semicycle* of a solution $\{y_n\}$ of Eq. (1) consists of a string of terms $\{y_l, y_{l+1}, \dots, y_m\}$, all less than the equilibrium $\overline{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either} \quad l = -1, \quad \text{or} \quad l > -1 \text{ and } y_{l-1} \geq \overline{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \text{ and } y_{m+1} \geq \overline{x}.$$

In a similar way one can define the oscillation and semicycles around the interval $[L, U]$, $L < U$.

The following result holds.

Theorem 4. *Let I be an interval in $\mathbb{R}$. Let $f : I^3 \to I$ be a continuous function such that*

(i) *f is non-increasing in first and third variable and non-decreasing in second variable.*

(ii) *there exist numbers L, U, $0 < L < \bar{x} < U$ such that*

$$f(U, L, U) \geq L, \quad f(L, U, L) \leq U. \tag{7}$$

If Eq. (1) does not have prime period-two solution, then there exists exactly one equilibrium $\bar{x}$ of Eq. (1), and every solution of Eq. (1) converges to $\bar{x}$.

Proof. Notice that the condition (7) implies that $[L, U]$ is an invariant interval for a function f, that is $f : [L, U]^3 \to [L, U]$. Indeed, by using the monotonicity of f we have

$$L \leq f(U, L, U) \leq f(x, y, z) \leq f(L, U, L) \leq U, \text{ for every } x, y, z \in [L, U].$$

Consequently, Theorem 4 follows from Theorem A.0.6 in [11] or the general result in [12].

□

2. Existence of unbounded solutions

In this section we present some results on the existence of unbounded solutions of Eq. (1).

Theorem 5. *Let $\bar{x}$ be the unique equilibrium of Eq. (1).*

(i) *Assume that $f : I^3 \to I$, where $I \subset R$ is an interval is a continuous function which is non-increasing in first and third variable and non-decreasing in second variable.*

(ii) *Assume that there exist numbers* L, U, $0 < L < \bar{x} < U$ *and such that the following holds*

$$f(U,L,U) \leq L, \quad f(L,U,L) \geq U \tag{8}$$

and at least one inequality in (8) *is strict.*

If

$$x_{-1} \leq L \quad \text{and} \quad x_0, x_{-2} \geq U,$$

then the corresponding solution $\{x_n\}$ *satisfies*

$$x_{2n-1} \leq L \quad \text{and} \quad x_{2n} \geq U, \ n = 0, 1, \dots .$$

In other words, every solution of Eq. (1) *oscillates around interval* $[L, U]$, $0 \leq L < \bar{x} < U$ *with semicycles of length one.*

Proof. Assume that $x_{-1} \leq L$ and $x_0, x_{-2} \geq U$. Then by using the monotonicity of f and condition (8), we have

$$\begin{aligned} x_1 &= f(x_0, x_{-1}, x_{-2}) \leq f(U,L,U) \leq L \\ x_2 &= f(x_1, x_0, x_{-1}) \ \ \geq f(L,U,L) \geq U \\ x_3 &= f(x_2, x_1, x_0) \quad \leq f(U,L,U) \leq L \\ &\vdots \end{aligned}$$

By using induction we complete the proof of Theorem 5. □

Adding another assumption in Theorem 5 leads to a stronger conclusion.

Corollary 1. *If in addition to the hypotheses of Theorem* 5 *we assume that there exists constant* $K > 0$ *such that*

$$f(L, v, L) \geq K + v, \ \textit{for all } v \geq U, \tag{9}$$

then $\{x_{2n}\}$ *is unbounded sequence and*

$$\lim_{n\to\infty} x_{2n} = +\infty.$$

Proof. If condition (9) holds, then we have

$$\begin{aligned} x_2 &= f(x_1, x_0, x_{-1}) \geq f(L, x_0, L) \geq x_0 + K \\ x_4 &= f(x_3, x_2, x_1) \geq f(L, x_0 + K, L) \geq x_0 + K + K = x_0 + 2K \\ &\vdots \end{aligned}$$

Continuing this process and using mathematical induction, we obtain

$$x_{2n} \geq x_0 + nK \ \text{for} \ n = 1, 2 \dots$$

and so

$$\lim_{n\to\infty} x_{2n} = \infty.$$

□

Remark 1. Notice that conditions (7) and (8) are separated by the boundary condition

$$f(U, L, U) = L, \quad f(L, U, L) = U, \tag{10}$$

which is equivalent to the existence of period-two solutions, and therefore can be used to detect equations with the properties described in Theorems 4 and 5. An additional problem will be to show that when the monotonicity conditions on f in Theorem 4 and (10) are satisfied, every solution of Eq. (1) converges to a period two solution. It seems that such a result, for some classes of function f has been established in the coming monograph [3]. The conditions (7) of Theorem 4 and (8) of Theorem 5 can be checked effectively in many special cases of linear fractional equations of the form

$$x_{n+1} = \frac{A + A_0 x_n + A_1 x_{n-1} + \ldots + A_k x_{n-k}}{B + B_0 x_n + B_1 x_{n-1} + \cdots + B_k x_{n-k}}, \quad n = 0, 1, \ldots \tag{11}$$

where all parameters $A, B, A_i, B_i, i = 0, \ldots, k$ and the initial conditions $x_{-i}, i = 0, \ldots, k$ are non-negative and the denominator is non-zero.

3. Applications

Now we present three applications of Theorems 4, 5, and Corollary 1.

Example 1. Consider Eq. (4) where all parameters are non-negative, $\alpha > 0$ and the initial conditions are positive. Eq. (4) has been investigated in [5], [7] and [17].

We check conditions of Theorem 5:

$$f(U, L, U) = \frac{\alpha + \gamma L}{A + (B+1)U} \leq L \Rightarrow \alpha + \gamma L \leq AL + UL(B+1). \tag{12}$$

$$f(L, U, L) = \frac{\alpha + \gamma U}{A + (B+1)L} \geq U \Rightarrow \alpha + \gamma U \geq AU + UL(B+1). \tag{13}$$

From (12) and (13), we have

$$UL(B+1) \geq \alpha + \gamma L - AL,$$
$$UL(B+1) \leq \alpha + \gamma U - AU,$$

which implies

$$\alpha + \gamma L - AL \leq \alpha + \gamma U - AU,$$

that is,

$$\gamma(L - U) - A(L - U) \leq 0 \Leftrightarrow (L - U)(\gamma - A) \leq 0.$$

The previous inequality holds if and only if $\gamma > A$ for $L < U$.

Now choose $L = \frac{\gamma - A}{B+1}$, $U = \frac{\gamma - A}{B+1} + \frac{\alpha}{\gamma - A}$. We have

$$f(U, L, U) = L$$

$$f(L,U,L) = \frac{\alpha}{\gamma} + \frac{\gamma - A}{B+1} + \frac{\alpha}{\gamma - A} \geq U.$$

Next, we show that $L < \bar{x} < U$. The equilibrium $\bar{x}$ is a positive solution of quadratic equation

$$f(t) := (B+1)t^2 + (A-\gamma)t - \alpha = 0.$$

The graph of quadratic polynomial $f(t)$ is a parabola open upward with the properties $f(0) = -\alpha, f(\bar{x}) = 0$. Thus in order to check that $L < \bar{x} < U$ it is enough to show that $f(L) < 0$ and $f(U) > 0$. Indeed $f(L) = -\alpha < 0$ and $f(U) = \frac{\alpha^2(B+1)}{(\gamma-A)^2} > 0$.

If we choose K such that $0 < K < \frac{\alpha}{\gamma}$, then we have

$$f(L,v,L) = \frac{\alpha}{\gamma} + \frac{\gamma - A}{B+1} + \frac{\alpha}{\gamma - A} > K + v, \text{ for all } v \geq U.$$

We conclude: If $\gamma > A$, then the corresponding solution $\{x_n\}$ satisfies

$$x_{2n-1} \leq L, \ x_{2n} \geq U, \ \forall n.$$

By Corollary 1, the sequence $\{x_{2n}\}$ is unbounded and

$$x_{2n} \to \infty.$$

Finally, we check that the condition (7) is satisfied if $\gamma < A$. Choose $L = 0$ and $U = \frac{\alpha}{A-\gamma}$. Then (7) is clearly satisfied and $L < \bar{x}$. Furthermore $\bar{x} < U$ because $f(U) = \frac{\alpha^2(B+1)}{(\gamma-A)^2} > 0$ An immediate checking shows that Eq. (4) does not have a prime period-two solution if $\gamma < A$, which by Theorem 4 implies that every solution of Eq. (4) converges to the unique equilibrium. Finally, it can be seen that Eq. (4) has an infinite number of prime period-two solutions if and only if $\gamma = A$.

Example 2. Consider the following equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_{n-2}}, \ n = 0, 1, \ldots, \tag{14}$$

where α and the initial conditions are positive. This equation has been investigated in [7].

We now check conditions of Theorem 5:

$$f(U,L,U) = \alpha + \frac{L}{U} \leq L \Rightarrow \alpha U + L \leq UL,$$

$$f(L,U,L) = \alpha + \frac{U}{L} \geq U \Rightarrow \alpha L + U \geq UL.$$

This implies

$$\alpha U + L \leq L\alpha + U \Rightarrow (U-L)(\alpha - 1) \leq 0.$$

The last inequality holds if and only if $\alpha \leq 1$.

Now choose L and U such that $L \in (\alpha, 1)$ and $U = \frac{L}{L-\alpha}$. An immediate checking shows that (8) is satisfied and $L < \bar{x} = \alpha + 1$. Finally, $\bar{x} < U$ is equivalent to $L < \bar{x}$ and so is satisfied. Next, by choosing $0 < K < \alpha$ we have

$$f(L, v, L) = \alpha = \frac{v}{L} > \alpha + v > K + v, \text{ for all } v \geq U.$$

By using Theorem 5, we conclude: If $\alpha < 1$, then the corresponding solution $\{x_n\}$ satisfies

$$x_{2n-1} \leq L, \ x_{2n} \geq U, \ \forall n.$$

By Corollary 1, the sequence $\{x_{2n}\}$ is unbounded and

$$x_{2n} \to \infty.$$

Next, we check that the condition (7) is satisfied if $\alpha > 1$. Choose $L = \alpha$ and $U = \frac{\alpha^2}{\alpha-1} = \alpha + 1 + \frac{1}{\alpha-1}$. Then (7) is satisfied and obviously $L < \bar{x} < U$. A straightforward checking shows that Eq.(14) does not have prime period-two solution if $\alpha > 1$, which by Theorem 4 implies that every solution of Eq. (14) converges to the unique equilibrium. Finally, an immediate checking shows that Eq. (14) has an infinite number of prime period-two solutions if and only if $\alpha = 1$.

Example 3. Consider equation

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{A + x_{n-2}}, \ n = 0, 1, \ldots . \tag{15}$$

where all parameters and initial conditions are nonnegative and $\alpha, A, \gamma > 0$. This equation has been investigated in [4], [7] and [17].

It was shown in [4] that Eq. (16) can be reduced to equation

$$y_{n+1} = \frac{a + y_{n-1}}{b + y_{n-2}}, \ n = 0, 1, \ldots . \tag{16}$$

Now, we check the conditions of Theorem 5:

$$f(L, U, L) = \frac{a + L}{b + U} \leq L \Rightarrow a + (1 - b)L \leq LU$$
$$f(L, U, L) = \frac{a + U}{b + L} \geq U \Rightarrow LU \leq a + (1 - b)U.$$

These inequalities hold if $b < 1$. Now choose: $L = 1 - b$, $U = \frac{a+(1-b)^2}{1-b} = 1 - b + \frac{a}{1-b}$. It can be checked that (8) is satisfied.

Next, we show that $L < \bar{x} < U$. The equilibrium $\bar{x}$ is a positive solution of quadratic equation

$$g(t) := t^2 + (b - 1)t - a = 0.$$

The graph of quadratic polynomial $g(t)$ is a parabola open upward with the properties $g(0) = -a,\ g(\bar{x}) = 0$. Thus in order to check that $L < \bar{x} < U$ it is enough to show that $g(L) < 0$ and $g(U) > 0$. Indeed $g(L) = -a < 0$ and $g(U) = \frac{a^2}{(1-b)^2} > 0$. By using Theorem 5, we conclude: If $b < 1$, then the corresponding solution $\{x_n\}$ satisfies

$$x_{2n-1} \le L,\ x_{2n} \ge U,\ \forall n.$$

Next, choose $0 < K < a$. Then

$$f(L, v, L) = a + v > K + v,\ \text{ for all } v \ge U.$$

By Corollary 1, the sequence $\{x_{2n}\}$ is unbounded and

$$x_{2n} \to \infty.$$

The condition (7) leads to the inequalities

$$a + (1-b)U \le LU \le a + (1-b)L$$

which are satisfied if and only if $b > 1$. Choose $L = \frac{a(b-1)}{a+(b-1)^2}$ and $U = \frac{a}{b-1}$. An immediate checking shows that (7) is satisfied. To show that $L < \bar{x} < U$ it is enough to show that $g(L) < 0$ and $g(U) > 0$. Indeed, $g(L) = -\frac{a^3}{(a+(b-1)^2)^2}$ and $g(U) = \frac{a^2}{(b-1)^2}$. An immediate checking shows that Eq. (16) does not have a prime period-two solution if $b > 1$, which by Theorem 4 implies that every solution of Eq. (16) converges to the unique equilibrium. Finally, it can be seen that Eq. (16) has an infinite number of prime period-two solutions if and only if $b = 1$.

4. Higher order extensions

Now, we consider the fourth order difference equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, x_{n-3}),\ n = 0, 1, \dots \tag{17}$$

where $x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary non-negative real numbers. Similarly, as for Eq. (1), the following results hold.

Theorem 6. *Consider Eq.* (17) *and assume that $\bar{x}$ is the unique equilibrium of Eq.* (17)*.*

(i) *Assume that $f : I^4 \to I$ is a continuous function which is non-increasing in first and third variable and non-decreasing in second and fourth variable.*

(ii) *Assume that there exist numbers L, U, $0 < L < \bar{x} < U$ and such that the following holds*

$$f(U, L, U, L) \le L,\ \ f(L, U, L, U) \ge U \tag{18}$$

and at least one inequality in (18) *is strict.*

If

$$x_{-1},\ x_{-3} \leq L \ \textit{and}\ x_0,\ x_{-2} \geq U,$$

then the corresponding solution $\{x_n\}$ *satisfies*

$$x_{2n-1} \leq L \ \textit{and}\ x_{2n} \geq U,\ \ n = 0, 1, \ldots\ .$$

In other words, every solution of Eq. (17) *oscillates around interval* $[L, U]$, $0 \leq L < \bar{x} < U$ *with semicycle of length one.*

Proof. Assume that $x_{-1},\ x_{-3} \leq L$ and $x_0,\ x_{-2} \geq U$. Then by using the monotonicity of f and conditions (18), we have

$$\begin{aligned} x_1 &= f(x_0, x_{-1}, x_{-2}, x_{-3}) \leq f(U, L, U, L) \leq L \\ x_2 &= f(x_1, x_0, x_{-1}, x_{-2}) \ \ \geq f(L, U, L, U) \geq U \\ x_3 &= f(x_2, x_1, x_0, x_{-1}) \quad \leq f(U, L, U, L) \leq L \\ x_4 &= f(x_3, x_2, x_1, x_0) \qquad \geq f(L, U, L, U) \geq U \\ &\vdots \end{aligned}$$

By using induction we complete the proof of Theorem 6. □

Corollary 2. *Assume that all conditions of Theorem* 6 *hold. If in addition we assume that there exists constant* $K > 0$ *such that*

$$f(L, v, L, v) \geq K + v,\ \ \textit{for all}\ v \geq U, \tag{19}$$

then $\{x_{2n}\}$ *is an unbounded sequence and*

$$\lim_{n\to\infty} x_{2n} = +\infty.$$

Proof. If the condition (19) holds, then we have

$$\begin{aligned} x_2 &= f(x_1, x_0, x_{-1}, x_{-2}) \geq f(L, x_0, L, x_{-2}) \geq x_0 + K + x_{-2} > x_0 + K \\ x_4 &= f(x_3, x_2, x_1, x_0) \geq f(L, x_0 + K, L, x_0) \geq x_0 + K + K + x_0 > x_0 + 2K \\ &\vdots \end{aligned}$$

Continuing this process and using mathematical induction, we obtain

$$x_{2n} > x_0 + nK \ \text{for}\ n = 1, 2 \ldots$$

and so

$$\lim_{n\to\infty} x_{2n} = \infty.$$

□

Theorem 6 is complemented by the following result, which follows from a general attractivity theorem in [13].

Theorem 7. *Let I be an interval in $\mathbb{R}$.*

(i) *Assume that $f : I^4 \to I$ is a continuous function which is non-increasing in first and third variable and non-decreasing in second variable and fourth variable.*

(ii) *Assume that there exist numbers L and U, $0 < L < \bar{x} < U$ such that the following holds*

$$f(U,L,U,L) \geq L, \;\; f(L,U,L,U) \leq U. \tag{20}$$

If Eq. (17) *does not have prime period-two solution, then there exists exactly one equilibrium $\bar{x}$ of Eq.* (17), *and every solution of Eq.* (17) *converges to $\bar{x}$.*

Example 4. Consider the following equation

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-3}}{A + Bx_n + x_{n-2}}, \;\; n = 0, 1, \ldots \tag{21}$$

where all parameters are non-negative and the initial conditions are positive. To find the critical value of parameters we will attempt to find period-two solutions $\ldots, \Phi, \Psi, \Phi, \Psi, \ldots$ of Eq. (21). We obtain

$$\Phi = \frac{\alpha + (\gamma + \delta)\Phi}{A + (B+1)\Psi}, \;\; \Psi = \frac{\alpha + (\gamma + \delta)\Psi}{A + (B+1)\Phi}$$

which implies

$$(B+1)\Phi\Psi - \alpha = (\gamma + \delta - A)\Phi = (\gamma + \delta - A)\Psi.$$

Thus the necessary condition for the existence of a period-two solution becomes

$$\gamma + \delta = A \tag{22}$$

Straightforward checking shows that this is also a sufficient condition. When (22) is satisfied Eq. (21) has an infinite number of period-two solutions which satisfy $(B+1)\Phi\Psi = \alpha$. By using a similar method as in Example 1 and Theorem 7 we can show that for $\gamma + \delta < A$ every solution tend to an equilibrium. Finally, by using Theorem 5 and Corollary 2 for $\gamma + \delta > A$ we can show that Eq. (21) has unbounded solutions.

The following result extends and unifies Theorems 5 and 6.

Theorem 8. *Let $f : I^{k+1} \to I$ be a continuous function, where k is a positive integer, and where I is an interval of real numbers. Consider the difference equation of $k+1$ order*

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, \ldots x_{n-k}), \;\; n = 0, 1, \ldots \tag{23}$$

where $x_{-k}, x_{-k+1}, \ldots, x_{-1}$ are arbitrary non-negative real numbers. Assume that there exists unique equilibrium $\bar{x}$ of Eq. (23).

Suppose that f satisfies the following conditions :

(i) *For each integer* $0 \leq i \leq k+1$, *the function* $f(y_1, y_2, \ldots, y_{k+1})$ *is non-increasing in first variable* y_1, *non-decreasing in second variable* y_2, *and then alternately non-increasing and non-decreasing in remaining variables.*

(ii) *Assume that there exist a numbers* L *and* U, $0 < L < \bar{x} < U$, *such that the following holds*

$$f(U, L, U, \ldots, L) \leq L \ \text{ and } \ f(L, U, L, \ldots, U) \geq U, \ \text{ if } k \text{ odd}, \tag{24}$$

or

$$f(U, L, U, \ldots, U) \leq L \ \text{ and } \ f(L, U, L, \ldots, L) \geq U, \ \text{ if } k \text{ even}. \tag{25}$$

If

$$x_{-1},\ x_{-3}, \ldots, x_{-(2k-1)} \leq L \ \text{ and } \ x_0,\ x_{-2}, \ldots, x_{-(2k-2)} \geq U,\ k = 1, 2, \ldots$$

then the corresponding solution $\{x_n\}$ *satisfies*

$$x_{2n-1} \leq L \ \text{ and } \ x_{2n} \geq U,\ \ n = 0, 1, \ldots .$$

In other words, every solution of Eq. (23) *oscillates around interval* $[L, U]$, $0 \leq L < \bar{x} < U$ *with semicycle of length one.*

Proof. The proof of this Theorem is similar to the proofs of Theorems 5 and 6, and will be omitted. □

The following Corollary extends and unifies Corollaries 1 and 2.

Corollary 3. *Assume that all conditions of Theorem* 8 *hold. If in addition we assume that there exists constant* $K > 0$ *such that*

$$f(L, v, L, \ldots, L) \geq K + v, \ \text{ for all } v \geq U, \ \text{ if } k \text{ even} \tag{26}$$

or

$$f(L, v, L, \ldots, v) \geq K + v, \ \text{ for all } v \geq U, \ \text{ if } k \text{ odd} \tag{27}$$

then $\{x_{2n}\}$ *is unbounded sequence and*

$$\lim_{n \to \infty} x_{2n} = +\infty.$$

Proof. The proof is similar to the proofs of the Corollaries 1 and 2, and will be omitted. □

Theorem 8 has the following counterpart for a global attractivity, which is an extension of Theorems 4 and 7. See [13].

Theorem 9. *Let* $f : I^{k+1} \to I$ *be a continuous function, where* k *is a positive integer, and where* I *is an interval of real numbers. Consider Eq.* (23) *where* $x_{-k}, x_{-k+1}, \ldots, x_{-1}$ *are arbitrary non-negative real numbers. Assume that there exists unique equilibrium* $\bar{x}$ *of Eq.* (23).

Suppose that f *satisfies the following conditions :*

(i) *For each integer* $0 \leq i \leq k+1$*, the function* $f(y_1, y_2, \dots, y_{k+1})$ *is non-increasing in first variable* y_1*, non-decreasing in second variable* y_2*, and then alternately non-increasing and non-decreasing in other variables.*

(ii) *Assume that there exist a numbers* L *and* U*,* $0 < L < \bar{x} < U$*, such that the following holds*

$$f(U, L, U, \dots, L) \geq L \text{ and } f(L, U, L, \dots, U) \leq U, \text{ if } k \text{ odd}, \tag{28}$$

or

$$f(U, L, U, \dots, U) \geq L \text{ and } f(L, U, L, \dots, L) \leq U, \text{ if } k \text{ even}. \tag{29}$$

If Eq. (23) *does not have prime period-two solution, then there exists exactly one equilibrium* $\bar{x}$ *of Eq.* (23)*, and every solution of Eq.* (23) *converges to* $\bar{x}$*.*

Example 5. Consider the following equation

$$x_{n+1} = \frac{\alpha + \sum_{k=0}^{r} a_k x_{n-(2k+1)}}{\beta + \sum_{k=0}^{r} b_k x_{n-(2k)}}, \quad n = 0, 1, \dots \tag{30}$$

where all parameters are non-negative and the initial conditions are positive, such that the expression in denominator is positive. To find the critical value of parameters we will attempt to find period-two solutions $\dots, \Phi, \Psi, \Phi, \Psi, \dots$ of Eq. (30). We obtain

$$\Phi = \frac{\alpha + \Phi \sum_{k=0}^{r} a_k}{\beta + \Psi \sum_{k=0}^{r} b_k}, \quad \Psi = \frac{\alpha + \Psi \sum_{k=0}^{r} a_k}{\beta + \Phi \sum_{k=0}^{r} b_k}$$

which imply

$$\Phi\Psi \sum_{k=0}^{r} b_k - \alpha = \Phi \left(\sum_{k=0}^{r} a_k - \beta \right) = \Psi \left(\sum_{k=0}^{r} a_k - \beta \right).$$

Thus a necessary condition for the existence of a period-two solution becomes

$$\sum_{k=0}^{r} a_k = \beta \tag{31}$$

Straightforward checking shows that this is also a sufficient condition. When (31) is satisfied Eq. (30) has an infinite number of period-two solutions which satisfy $\Phi\Psi \sum_{k=0}^{r} b_k = \alpha$. By using a similar method as in Example 1 and Theorem 9 we can show that for $\sum_{k=0}^{r} a_k < \beta$ every solution of Eq. (30) tend to an equilibrium. Finally, by using Theorem 8 and Corollary 3 for $\sum_{k=0}^{r} a_k > \beta$ we can show that Eq. (30) has unbounded solutions.

5. Existence of period-two solution for linear fractional difference equation

In view of Examples 1-3 it seems that one can identify period-two trichotomies in linear fractional difference equation (11) where all parameters $A, B, A_i, B_i, i = 0, \ldots, k$ and the initial conditions $x_{-i}, i = 0, \ldots, k$ are non-negative and the denominator is non-zero by finding a condition for existence of period-two solutions of Eq. (11). To accomplish this task we consider two cases when k is odd and even. We have the following result

Proposition 1. *Let*

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots, \quad \Phi < \Psi \tag{32}$$

be a period-two solution of Eq. (11)*.*

(a) *If k is odd then the necessary and sufficient condition for the existence of a period-two solution* (32) *is*

$$A_1 + A_3 + \cdots + A_k - A_0 - B \geq 0, \tag{33}$$

$$(B_1 + B_3 + \cdots + B_k)(\Phi + \Psi) = A_1 + A_3 + \cdots + A_k - A_0 - B. \tag{34}$$

(b) *If k is even then the necessary and sufficient condition for the existence of a period-two solution* (32) *is*

$$A_1 + A_3 + \cdots + A_{k-1} \geq A_0 + A_2 + A_4 + \cdots + A_k + B, \tag{35}$$

$$(B_1+B_3+\cdots+B_{k-1})(\Phi+\Psi) = (A_1+A_3+\cdots+A_{k-1})-(A_0+A_2+\cdots+A_k+B). \tag{36}$$

Proof. Case (a). Let k be odd. Period-two solution (32) satisfies

$$\Psi = \frac{A + A_0\Phi + A_1\Psi + A_2\Phi + \cdots + A_k\Psi}{B + B_0\Phi + B_1\Psi + B_2\Phi + \cdots + B_k\Psi}$$

$$\Phi = \frac{A + A_0\Psi + A_1\Phi + A_2\Psi + \cdots + A_k\Phi}{B + B_0\Psi + B_1\Phi + B_2\Psi + \cdots + B_k\Phi}.$$

Simplifying these two fractions and subtracting them we obtain the following identity

$$(B_1 + B_3 + \cdots + B_k)(\Phi + \Psi) = A_1 + A_3 + \cdots + A_k - A_0 - B,$$

which implies (33) and (34). The converse statement can be obtained by reversing the steps.

Case (b). Let k be even. Period-two solution (32) satisfies

$$\Psi = \frac{A + A_0\Phi + A_1\Psi + A_2\Phi + \cdots + A_k\Phi}{B + B_0\Phi + B_1\Psi + B_2\Phi + \cdots + B_k\Phi}$$

$$\Phi = \frac{A + A_0\Psi + A_1\Phi + A_2\Psi + \cdots + A_k\Psi}{B + B_0\Psi + B_1\Phi + B_2\Psi + \cdots + B_k\Psi}.$$

Simplifying these two fractions and subtracting them we obtain the following identity

$$\begin{aligned}(\Phi + \Psi)&(B_1 + B_3 + \cdots + B - k - 1)\\ &= (A_1 + A_3 + \cdots + A_{k-1}) - (A_0 + A_2 + \cdots + A_k + B).\end{aligned}$$

Since $(B_1 + B_3 + \cdots + B - k - 1) \geq 0$, we must have $(A_1 + A_3 + \cdots + A_{k-1}) - (A_0 + A_2 + \cdots + A_k + B) \geq 0$ which implies (35) and (36). The converse statement can be obtained by reversing the steps. □

Proposition 1 gives a simple test for detecting period-two trichotomies in Eq. (11). We present two applications to the well known special cases of Eq. (11).

Example 6. Consider equation

$$x_{n+1} = \frac{p + qx_{n-1}}{1 + x_n}, \ \ n = 0, 1, \ldots \tag{37}$$

where all parameters are positive and the initial conditions are nonnegative. Eq. (37) has been investigated in [8].
In this case, we have that

$$A = p, A_0 = 0, A_1 = q$$

and

$$B = 1, B_0 = 1, B_1 = 0.$$

Condition (33) in Proposition 1, becomes

$$q - 0 - 1 \geq 0 \Rightarrow q \geq 1$$

and the condition (34) becomes

$$(\Phi + \Psi) \cdot 0 = q - 1 \Leftrightarrow q = 1.$$

Intersection of these two conditions gives $q = 1$.
So, $q = 1$ is the necessary and sufficient condition for existence of period-two solutions of Eq.(37).

Example 7. Consider equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + x_n + x_{n-2}}, \ \ n = 0, 1, \ldots \tag{38}$$

where all parameters are positive and the initial conditions are nonnegative. Eq. (37) has been investigated in [7], p.167.
In this case , we have that

$$A = \alpha, A_0 = \beta, A_1 = \gamma, A_2 = \delta$$

and

$$B = A, B_0 = 1, B_1 = 0, B_2 = 1.$$

Conditions (35) and (36) in Proposition 1, become

$$\gamma \geq A + \beta + \delta$$

and

$$(a + b)0 = \gamma - (A + \beta + \delta)$$

which yields

$$\gamma = A + \beta + \delta.$$

So, $\gamma = A + \beta + \delta$ is the necessary and sufficient condition for existence of period-two solutions of Eq. (37).

6. Conjectures

We proved that the period-two trichotomy in the case of second-order equation is actually the period-doubling bifurcation, see [2]. We used the techniques of competitive systems in the plane which does not have higher dimensional analogue at this time, but we believe that the similar result holds for k-th order difference equation, $k \geq 3$. Thus we are offering the following conjectures:

Conjecture 1. *Consider Eq.* (11) *subject to the conditions of Corollary* 3. *Then every solution of Eq.* (11) *which starts in the complement of the union of the stable manifolds of the zero and the positive equilibrium is unbounded.*

This conjecture was proved for $k = 1$ in [2].

Conjecture 2. *Let* $\mathcal{I} = [a, \infty)$*, and let* $\mathcal{A}$ *be a connected subset of* $\mathbb{R}$*. Given a family of difference equations of* $k + 1$ *order*

$$x_{n+1} = f_\alpha(x_n, x_{n-1}, x_{n-2}, \ldots x_{n-k}), \;\; n = 0, 1, \ldots \tag{39}$$

where $x_{-k}, x_{-k+1}, \ldots, x_{-1}$ *are arbitrary non-negative real numbers with* f_α $(u_0, u_1, \ldots, u_{k+1})$ *continuous on* $\mathcal{I}^{k+1}$*, suppose that for each* $\alpha \in \mathcal{A}$*,*

- a_1. $f_\alpha(u_0, u_1, \ldots, u_{k+1})$ *is strictly decreasing in* u_{2m} *and strictly increasing in* u_{2m+1} *for* $m = 0, 1, \ldots$ *in the interior of* $\mathcal{I}^{k+1}$.
- a_2. $f_\alpha(u_0, u_1, \ldots, u_{k+1})$ *is smooth in* α *and* $(u_0, u_1, \ldots, u_{k+1})$
- a_3. *There is a unique interior equilibrium* $\overline{x}_\alpha$ *which varies continuously in* α.
- a_4. *There exists a continuous function* $\Gamma : \mathcal{A} \to \mathbb{R}$ *such that for* α *in the parametric region* $\{\alpha : \Gamma(\alpha) < 0\} \cup \{\alpha : \Gamma(\alpha) > 0\}$ *there are no prime period-two solutions. There exists a prime period-two solution for* α *in the parametric region* $\{\alpha : \Gamma(\alpha) = 0\}$.

a_5. *For* α *in the parametric region* $\{\alpha : \Gamma(\alpha) < 0\}$ *all solutions of Eq.* (39) *are bounded.*

Then the equilibrium $\overline{x}_\alpha$ *is globally asymptotically stable for* α *in the parametric region* $\{\alpha : \Gamma(\alpha) < 0\}$. *For* α *in the parametric region* $\{\alpha : \Gamma(\alpha) = 0\}$, *every solution of Eq.* (39) *converges to period-two solution (not necessarily prime). For* α *in the parametric region* $\{\alpha : \Gamma(\alpha) > 0\}$, *every solution of Eq.* (39) *is unbounded except for the solutions that belong to the the closure of the global stable manifold of the equilibrium.*

Conjecture 2 was proved for $k = 1$ in [2].

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