SARAJEVO JOURNAL OF MATHEMATICS
Vol.4 (16) (2008), 91–95

A MATRIX CHARACTERIZATION OF STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES

HARRY I. MILLER AND LEILA MILLER-VAN WIEREN

Dedicated to Professor Fikret Vajzović on the occasion of his 80th birthday

Abstract. Fridy and Miller have given a characterization of statistical convergence for bounded single sequences using a family of matrix summability methods. In this paper we prove the analogous result for double sequences.

1. Introduction

The concept of the statistical convergence of a sequence of reals $x = (x_n)$ was first introduced by H. Fast [3].

The sequence $x = (x_n)$ is said to converge statistically to L and we write

$$\lim_{n\to\infty} x_n = L \text{ statistically if for every } \epsilon > 0,$$

$$\lim_{n\to\infty} n^{-1}|\{k \leq n : |x_k - L| \geq \epsilon\}| = 0,$$

where $|A|$ denotes the cardinality of the set A.

Properties of statistically convergent sequences were studied in [1], [2], [4] and [7]. In [5], Fridy and Miller gave a characterization of statistical convergence for bounded sequences using a family of matrix summability methods.

For recent results on double sequences one should consult other references in [6] which can be found online.

Concretely, in [5], the following is proved.

Theorem 1.1. (Fridy, Miller) *Suppose $x = (x_n)$ is a bounded sequence of reals, then $\lim_{n\to\infty} x_n = L$ statistically if and only if the transformed*

2000 *Mathematics Subject Classification.* 40D25, 40G99, 28A12.

sequence Ax converges, in the ordinary sense, to L for every $A \in \tau$, where

$$\tau = \{A = (a_{nk}) : A \text{ is triangular, non-negative}, \sum_k a_{nk} = 1 \text{ for each } n$$

$$\text{and } \lim_{n\to\infty} \sum_{k\in T} a_{nk} = 0 \quad \text{for each } T \text{ having natural density } 0\}$$

and (Ax) denotes the sequence whose n-th term is $\sum_{k=1}^{\infty} a_{nk}x_k$. Here $A = (a_{nk})$. By a set $T, T \subseteq N$, having natural density 0, we man that $\lim_{n\to\infty} \frac{1}{n}|\{t : t \le n, t \in T\}| = 0$.

We remark here that Fridy, earlier, showed that statistical convergence is not equivalent to any regular summability method.

In this paper we consider double sequences $x = (x_{ij})_{i=1,j=1}^{\infty,\infty}$ of real numbers.

Definition 1.2. *We say that x is convergent to L (in the sense of Pringsheim) if for each $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that $|x_{ij} - L| < \epsilon$ whenever $i, j \ge N$.*

Definition 1.3. *We say x is statistically convergent to L if for each $\epsilon > 0$ the double sequence (y^{ϵ}_{mn}) where $y^{\epsilon}_{mn} = \frac{1}{mn}|\{(i,j) : i \le m, j \le n, |x_{ij} - L| \ge \epsilon\}|$ converges (in the sense of Pringsheim) to zero.*

Definition 1.4. *If $A = (a_{m,n,i,j})$ is a 4-dimensional matrix and $x = (x_{ij})$ is a double sequence then the double (transformed) sequence, $Ax := (y_{mn})$, is defined by $y_{mn} := \sum_{i=1,j=1}^{\infty,\infty} a_{m,n,i,j} \cdot x_{ij}$, where it is assumed that the summation exists as a Pringsheim limit (of the partial sums) for each $m, n \in N$.*

Definition 1.5. *τ denotes the collection of all 4-dimensional matrices $A = (a_{m,n,i,ij})$ satisfying:*

(i) $a_{m,n,i,j} \ge 0$, $\forall m, n, i, j \in N$,
(ii) $a_{m,n,i,j} = 0$ *if either $i > m$ or $j > n$,*
(iii) $\sum_{i,j} a_{m,n,i,j} = 1$ *for every $m, n \in N$,*
(iv) *if $T \in NXN$ and T has density 0, then the sequence $(z_{m,n}) := (\sum_{(i,j)\in T} a_{m,n,i,j})$ has Pringsheim limit zero.*

Here T having density zero means $\frac{1}{mn}|\{(i,j) : (i,j) \in T, i \le m, j \le n\}|$ has Pringsheim limit zero as $m, n \to \infty$.

2. Result

We now prove the double sequence analogoue of the Fridy and Miller theorem.

Theorem 2.1. *If $x = (x_{ij})$ is a bounded double sequence, then x is statistically convergent to L if and only if the transformed sequence Ax is convergent to L (in the sense of Pringsheim) for each 4-dimensional matrix $A \in \tau$.*

Proof. a) Suppose $x = (x_{ij})$ is bounded and x is statistically convergent to L. Suppose further that $A = (a_{m,n,i,j}) \in \tau$ and $\epsilon > 0$ is given. Then

$$(Ax)_{m,n} = \sum_{(i,j)\in T} a_{m,n,i,j}x_{ij} + \sum_{\substack{\{(i,j):(i,j)\notin T,\\ i\leq m, j\leq n\}}} a_{m,n,i,j}x_{ij} \tag{1}$$

where $T := \{(i,j) : |x_{ij} - L| \geq \epsilon\}$. This yields

$$\begin{aligned}|(Ax)_{m,n} - L| &= \Big|\sum_{(i,j)} a_{m,n,i,j}(x_{ij} - L)\Big| \\ &\leq \Big|\sum_{(i,j)\in T} a_{m,n,i,j}(x_{ij} - L)\Big| + \Big|\sum_{\substack{\{(i,j):(i,j)\notin T,\\ i\leq m, j\leq n\}}} a_{m,n,i,j}(x_{ij} - L)\Big| \\ &\leq (\sup_{(i,j)} |x_{ij} - L|) \sum_{(i,j)\in T} a_{m,n,i,j} + \epsilon \sum_{\substack{\{(i,j):(i,j)\notin T,\\ i\leq m, j\leq n\}}} a_{m,n,i,j}.\end{aligned}$$

Since T has density zero and ϵ is arbitrary, property (iv) implies that $\lim_{m,n\to\infty} |(Ax)_{m,n} - L| = 0$.
b) Now we prove the reverse implication.

Suppose $x = (x_{ij})$ is a bounded double sequence that does not converge statistically to L. Then there exists an $\epsilon > 0$ such that T_ϵ does not have density zero, where $T_\epsilon := \{(i,j) : |x_{ij} - L| \geq \epsilon\}$. Then either:
Case 1. $T_\epsilon^+ := \{(i,j) : x_{ij} \geq L + \epsilon\}$ does not have density zero,
Case 2. $T_\epsilon^- := \{(i,j) : x_{ij} \leq L - \epsilon\}$ does not have density zero.

Suppose Case 1 holds. Then, there exists $\delta > 0$, and two strictly increasing sequences of natural numbers (m_k) and (n_k) such that:

$$\frac{1}{m_k n_k}|\{(i,j), i \leq m_k, j \leq n_k \text{ such that } x_{ij} \geq L + \epsilon\}| > \delta \tag{2}$$

for all $k \in N$. We now define a 4-dimensional matrix A as follows:

$$a_{m_k,n_k,i,j} = \left\{ \begin{array}{c} \frac{1}{\mu_k} \text{ if } i \leq m_k, j \leq n_k \text{ and } x_{ij} \geq L + \epsilon \\ \text{where } \mu_k = |\{(i,j) : i \leq m_k, j \leq n_k \text{ and } x_{ij} \geq L + \epsilon\}|, \\ 0 \text{ otherwise} \end{array} \right\},$$

$$a_{m,n,i,j} = \left\{ \begin{array}{cc} \frac{1}{mn} & \text{if } i \leq m, j \leq n \text{ and } (m,n) \neq (m_k,n_k) \forall k \in N, \\ 0 & \text{otherwise} \end{array} \right\}.$$

We now show that $A = (a_{m,n,i,j})$ given by (2) is in τ but Ax does not converge to L.

Clearly $\sum_{i,j} a_{m_k,n_k,i,j} x_{ij} \geq L + \epsilon$ for all $k \in N$, so Ax does not converge to L. It remains to show that A (given in (2)) is in τ. It is easy to see that A satisfies i), ii), and iii) from Definition 4. We now show that A satisfies condition iv). Suppose $T \in NXN$ has density zero. Then, for $(m,n) \neq (m_k, n_k)$ from (2) $z_{m,n} = \frac{1}{mn}|\{(i,j) \in T, i \leq m, j \leq n\}|$ tends to zero as $m, n \to \infty$.

For $k \in N$, from (2) we get

$$\sum_{(i,j)\in T} a_{m_k,n_k,i,j} = \frac{1}{\mu_k}|\{(i,j) : (i,j) \in T, i \leq m_k, j \leq n_k, x_{ij} \geq L + \epsilon\}|$$
$$\leq \frac{1}{\delta}\frac{1}{m_k n_k}|\{(i,j) : (i,j) \in T, i \leq m_k, j \leq n_k, x_{ij} \geq L + \epsilon\}|.$$

So, $\lim_{k\to\infty} \sum_{(i,j)\in T} a_{m_k,n_k,i,j} = 0$ since T has density zero. Hence $A = (a_{m,n,i,j})$ is in τ. The proof for Case 2 is completely analogous. □

Definition 2.2. *$\tau^\star \subset \tau$ is defined to consist of all 4-dimensional matrices $A = (a_{m,n,i,j})$ whose entries are all rational numbers.*

Corollary 2.3. *If $x = (x_{ij})$ is a bounded double sequence, then x is statistically convergent to L if and only if Ax is convergent to L for each $A \in \tau^\star$.*

Proof. a) If $x = (x_{ij})$ is bounded and is statistically convergent to L, then, by our Theorem, Ax is convergent to L for every $A \in \tau^\star$.
b) Suppose x is not statistically convergent to L. In the proof of b) in our Theorem, we see that there exists $A \in \tau^\star$, such that Ax is not convergent to L. □

We note that τ has cardinality of the continium (i.e. of the real line) while $\tau^\star$ is countable.

We conclude this note with an example. Namely, we show that our theorem cannot be extended to unbounded double sequences. This has already been noted for single sequences [5].

Example. Let $A = (a_{m,n,i,j})$ be the $(C, 1, 1)$ 4-dimensional Cesaro matrix, i.e.

$$a_{m,n,i,j} = \left\{ \begin{array}{c} \frac{1}{mn} \text{ if } i \leq m \text{ and } j \leq n \\ 0 \ otherwise \end{array} \right\}.$$

Define $x = (x_{ij})$ as follows:

$$x_{ij} = \left\{ \begin{array}{c} n^2 \text{ if } i = j = n \\ otherwise \end{array} \right\}.$$

Then x converges statistically to zero, but Ax does not converge to zero, and $A \in \tau^{\star}$.

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(Received: December 10, 2007)

Harry I. Miller
Department of Mathematics
University of Sarajevo
Faculty of Natural Sciences
71000 Sarajevo
Bosnia and Herzegovina
E–mail: himiller@hotmail.com

Leila Miller-Van Wieren
Department of Mathematics
Sarajevo School of
Science and Technology
71000 Sarajevo
Bosnia and Herzegovina
E–mail: lejla.miller@ssst.edu.ba