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THE BOUNDEDNESS OF THE B-RIESZ POTENTIAL IN THE B-MORREY SPACES

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Abstract. We consider the generalized shift operator (B shift operator), generated by the Laplace-Bessel differential operator $\Delta_B = \sum_{i=1}^{k} B_i + \sum_{j=k+1}^{n} \frac{\partial^2}{\partial x_j^2}$, $B=(B_1,\dots,B_k)$, $B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$, $\gamma_i > 0$, $i=1,\dots,k$, $|\gamma| = \gamma_1 + \dots + \gamma_k$. The B-maximal functions and the B-Riesz potentials, generated by the Laplace-Bessel differential operator Δ_B are investigated. We study the B-Riesz potentials in the B-Morrey spaces. The inequality of Sobolev-Morrey type is established for the B-Riesz potentials.

Introduction

The classical Riesz potential is an important technical tool in harmonic analysis, theory of functions and partial differential equations. The maximal function, singular integral, potential and related topics associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{k} B_i + \sum_{j=k+1}^{n} \frac{\partial^2}{\partial x_j^2}, \quad B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad \gamma_i > 0, \quad i = 1, \dots, k$$

have been investigated by many researchers, see B. Muckenhoupt and E. Stein [15], I. Kipriyanov [14], K. Trimeche [19], L. Lyakhov [13], K. Stempak [17],[18], A.D. Gadjiev and I.A. Aliev [3], I.A. Aliev and S. Bayrakci [1], I. Ekincioglu and A. Serbetci [11], V.S. Guliyev [4]-[7], V.S. Guliyev and J.J. Hasanov [9] and others.

In this paper we consider the generalized shift operator, generated by the Laplace-Bessel differential operator Δ_B in terms of which the B-maximal functions and B-Riesz potentials are investigated. We study the B-Riesz

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potential in the B-Morrey spaces. The inequality of Sobolev-Morrey type is established for the B-Riesz potentials.

The structure of the paper is as follows. In Section 1 we present some definitions, auxiliary results and study some embeddings into the function spaces (B-function spaces), associated with the Laplace-Bessel differential operator. In Section 2 the boundedness of the B-maximal operator on B-Morrey spaces $L_{p,\lambda,\gamma}$ is proved. The main result of the paper is the inequality of Sobolev-Morrey type for the B-Riesz potentials, established in Section 3. Note that all results of the paper in the case $k=1$ have been obtained in [9].

1. Definitions, notation and preliminaries

Suppose that $\mathbb{R}^n$ is n-dimensional Euclidean space, $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, $|x|^2=\sum_{i=1}^n x_i^2$, $1\le k\le n$, $n\ge 2$, $x'=(x_1,\dots,x_k)\in\mathbb{R}^k$, $x''=(x_{k+1},\dots,x_n)\in\mathbb{R}^{n-k}$, $x=(x',x'')\in\mathbb{R}^n$, $\mathbb{R}^n_{k,+}=\{x=(x',x'')\in\mathbb{R}^n; x_1>0,\dots,x_k>0\}$, $E_+(x,r)=\{y\in\mathbb{R}^n_{k,+}\ ;\ |x-y|<r\}$, $\gamma=(\gamma_1,\dots,\gamma_k)$, $\gamma_1>0,\dots,\gamma_k>0$, $|\gamma|=\gamma_1+\ldots+\gamma_k$, $(x')^\gamma=x_1^{\gamma_1}\cdots x_k^{\gamma_k}$.

For measurable $E\subset\mathbb{R}^n_{k,+}$ suppose $|E|_\gamma=\int_E (x')^\gamma dx$, then $|E_+(0,r)|_\gamma=\omega(n,k,\gamma)r^{n+|\gamma|}$, where

$$\omega(n,k,\gamma)=\int_{E_+(0,1)}(x')^\gamma dx=\frac{\pi^{\frac{n-k}{2}}}{2^k}\,\Gamma^{-1}\left(\frac{n+|\gamma|+2}{2}\right)\prod_{i=1}^k\Gamma\left(\frac{\gamma_i+1}{2}\right).$$

Denote by T^y the generalized shift operator (B–shift operator) acting according to the law

$$T^y f(x)=C_{\gamma,k}\int_0^\pi\cdots\int_0^\pi f\left((x',y')_\beta,x''-y''\right)\,d\nu(\beta),$$

where $(x_i,y_i)_{\beta_i}=(x_i^2-2x_iy_i\cos\beta_i+y_i^2)^{\frac12}$, $1\le i\le k$, $(x',y')_\beta=((x_1,y_1)_{\beta_1},\dots,(x_k,y_k)_{\beta_k})$, $d\nu\,(\beta)=\prod_{i=1}^k\sin^{\gamma_i-1}\beta_i\,d\beta_1\dots d\beta_k$, $1\le k\le n$ and

$$C_{\gamma,k}=\pi^{-\frac{k}{2}}\,\Gamma^{-1}\left(\frac{|\gamma|}{2}\right)\prod_{i=1}^k\Gamma\left(\frac{\gamma_i+1}{2}\right)=\frac{2^{k-1}|\gamma|}{\pi}\left(\frac{|\gamma|}{2}+1\right)\omega(2,k,\gamma).$$

We remark that the generalized shift operator T^y is closely connected with the Bessel differential operator B (for example, $n=k=1$ see [12], $n>1$, $k=1$ see [14] and $n,k>1$ see [13] for details).

Let $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ be the space of measurable functions on $\mathbb{R}^n_{k,+}$ with finite norm

$$\|f\|_{L_{p,\gamma}}=\|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})}=\left(\int_{\mathbb{R}^n_{k,+}}|f(x)|^p(x')^\gamma dx\right)^{1/p},\quad 1\le p<\infty.$$

For $p=\infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}}=\|f\|_{L_\infty}=\underset{x\in\mathbb{R}^n_{k,+}}{esssup}|f(x)|.$$

The translation operator T^y generates the corresponding B-convolution

$$(f\otimes g)(x)=\int_{\mathbb{R}^n_{k,+}} f(y)[T^y g(x)](y')^\gamma dy,$$

for which the Young inequality

$$\|f\otimes g\|_{L_{r,\gamma}}\le\|f\|_{L_{p,\gamma}}\|g\|_{L_{q,\gamma}},\quad 1\le p,q,r\le\infty,\quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$$

holds.

Lemma 1. *For all $x\in\mathbb{R}^n_{k,+}$ the following equality is valid*

$$\int_{E_+(0,t)} T^y g(x)(y')^\gamma dy=\int_{E((x,0),t)} g\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2},z''\right)d\mu(z,\overline{z'}),$$

where $E((x,0),t)=\{(z,\overline{z'})\in\mathbb{R}^n\times(0,\infty)^k\ :\ \left|(x-z,\overline{z'})\right|<t\}$, $d\mu(z,\overline{z'})=(\,\overline{z'}\,)^{\gamma-1}dzd\overline{z'}$, $d\overline{z'}=d\overline{z}_1\cdots d\overline{z}_k$, $(\overline{z'}\,)^{\gamma-1}=(\overline{z}_1)^{\gamma_1-1}\cdots(\overline{z}_k)^{\gamma_k-1}$.

The proof of Lemma 1 is straightforward via the following substitutions

$$z''=x'',\quad z_i=x_i\cos\alpha_i,\quad \overline{z_i}=x_i\sin\alpha_i,\quad 0\le\alpha_i<\pi,\ i=1,\dots,k,$$
$$x\in\mathbb{R}^n_{k,+},\ \overline{z'}=(\overline{z}_1,\dots,\overline{z}_k),\ (z,\overline{z'})\in\mathbb{R}^n\times(0,\infty)^k,\ 1\le k\le n.$$

Definition 1. *Let $1\le p<\infty$. By $WL_{p,\gamma}(\mathbb{R}^n_{k,+})$ we denote the weak $L_{p,\gamma}$ spaces defined as the set of locally integrable functions $f(x)$, $x\in\mathbb{R}^n_{k,+}$ with the finite norm*

$$\|f\|_{WL_{p,\gamma}}=\sup_{r>0} r\left|\left\{x\in\mathbb{R}^n_{k,+}\ :\ |f(x)|>r\right\}\right|_\gamma^{1/p}.$$

Definition 2. [5] *Let $1\le p<\infty$, $0\le\lambda\le n+|\gamma|$. We denote by $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ Morrey spaces ($\equiv$ B-Morrey spaces) as the set of locally integrable functions $f(x)$, $x\in\mathbb{R}^n_{k,+}$, with the finite norm*

$$\|f\|_{L_{p,\lambda,\gamma}}=\sup_{t>0,\,x\in\mathbb{R}^n_{k,+}}\left(t^{-\lambda}\int_{E_+(0,t)} T^y|f(x)|^p(y')^\gamma dy\right)^{1/p}.$$

Note that

$$L_{p,0,\gamma}(\mathbb{R}^n_{k,+})=L_{p,\gamma}(\mathbb{R}^n_{k,+}),$$
$$L_{p,n+|\gamma|,\gamma}\left(\mathbb{R}^n_{k,+}\right)=L_\infty\left(\mathbb{R}^n_{k,+}\right).$$

Definition 3. [5] *Let $1 \le p < \infty$, $0 \le \lambda \le n+|\gamma|$. We denote by $WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ the weak B-Morrey spaces as the set of locally integrable functions $f(x)$,$x \in \mathbb{R}^n_{k,+}$ with finite norm*

$$\|f\|_{WL_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \left(t^{-\lambda} \int_{\{y\in E_+(0,t):\, T^y|f(x)|>r\}} (y')^\gamma dy \right)^{1/p}.$$

Note that

$$WL_{p,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,0,\gamma}(\mathbb{R}^n_{k,+}),$$

$$L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \ \text{ and } \ \|f\|_{WL_{p,\lambda,\gamma}} \le \|f\|_{L_{p,\lambda,\gamma}}.$$

2. $L_{p,\lambda,\gamma}$-boundedness of the B-maximal operator

In this section we study the $L_{p,\lambda,\gamma}$-boundedness of the B-maximal operator (see [4])

$$M_\gamma f(x) = \sup_{r>0} |E_+(0,r)|_\gamma^{-1} \int_{E_+(0,r)} T^y|f(x)|(y')^\gamma dy.$$

Theorem 1. 1. *If $f \in L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $0 \le \lambda < n+|\gamma|$, then $M_\gamma f \in WL_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{WL_{1,\lambda,\gamma}} \le C_{1,\lambda,\gamma}\|f\|_{L_{1,\lambda,\gamma}}, \tag{1}$$

where $C_{1,\lambda,\gamma}$ depends only on λ,γ,k and n.

2. *If $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \infty, 0 \le \lambda < n+|\gamma|$, then $M_\gamma f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{L_{p,\lambda,\gamma}} \le C_{p,\lambda,\gamma}\|f\|_{L_{p,\lambda,\gamma}}, \tag{2}$$

where $C_{p,\lambda,\gamma}$ depends only on p,λ,γ,k and n.

Proof. We need to introduce the maximal operator defined on a space of homogeneous type (Y,d,ν). By this we mean a topological space $Y = \mathbb{R}^n \times (0,\infty)^k$ equipped with a continuous pseudometric d and a positive measure ν satisfying

$$\nu(E((x,\overline{x'}),2r)) \le C_1\nu(E((x,\overline{x'}),r)) \tag{3}$$

with a constant C_1 independent of $(x,\overline{x'})$ and $r>0$. Here $E((x,\overline{x'}),r) = \{(y,\overline{y'}) \in Y \ : \ d(((x,\overline{x'}),(y,\overline{y'})) < r\}$, $d\nu(y,\overline{y'}) = (\overline{y'})^{\gamma-1}dy\, d\overline{y'}$, $(\overline{y'})^{\gamma-1} = (\overline{y_1})^{\gamma_1-1}\cdots(\overline{y_k})^{\gamma_k-1}$, $d((x,\overline{x'}),(y,\overline{y'})) = |(x,\overline{x'})-(y,\overline{y'})| \equiv (|x-y|^2+(\overline{x'}-\overline{y'})^2)^{\frac12}$.

Let (Y,d,ν) be a space of homogeneous type. Define

$$M_\nu \overline{f}(x,\overline{x'}) = \sup_{r>0} \nu(E((x,\overline{x'}),r))^{-1} \int_{E((x,\overline{x'}),r)} \left|\overline{f}(y,\overline{y'})\right| d\nu(y),$$

where $\overline{f}(x,\overline{x'}) = f\left(\sqrt{x_1^2+\overline{x}_1^2},\dots,\sqrt{x_k^2+\overline{x}_k^2},x''\right)$.

It is well known that the maximal operator M_ν is of weak type (1, 1) and is bounded on $L_p(Y,d\nu)$ for $1<p<\infty$ (see [2]). Here we are concerned with the maximal operator defined by $d\nu(y,\overline{y'}) = (\overline{y'})^{\gamma-1}dy\ d\overline{y'}$. It is clear that this measure satisfies the doubling condition (3).

It can be proved that

$$M_\gamma f\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2},z''\right) = M_\nu\overline{f}\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right),$$

or

$$M_\gamma f(x) = M_\nu \overline{f}(x,0). \tag{4}$$

Indeed, Lemma 1

$$\int_{E_+(0,r)} T^y\left|f\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2},z''\right)\right|(y')^\gamma\ dy = \int_{E\left(\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right),r\right)} \left|\overline{f}(y,\overline{y'})\right|\ d\nu(y,\overline{y'})$$

and

$$|E_+(0,r)|_\gamma = \nu E\left(\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right),r\right)$$

imply (2). Furthermore, taking $\overline{z}_k=0$ in (2) we get (4).

Using Lemma 1 and equality (2) we have

$$\int_{E_+(0,r)} T^y\left(M_\gamma f(x)\right)^p(y')^\gamma\ dy$$
$$= \int_{E((x,0),r)}\left(M_\gamma f\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2},z''\right)\right)^p\ d\nu(z,\overline{z'})$$
$$= \int_{E((x,0),r)}\left(M_\nu \overline{f}\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right)\right)^p\ d\nu(z,\overline{z'}).$$

In [10] there was proved that the analogue of the Fefferman-Stein theorem for the maximal operator defined on a space of homogeneous type is valid, if condition (3) is satisfied. Therefore

$$\int_{E((x,\overline{x'}),r)}\left(M_\nu\varphi(y,\overline{y'})\right)^p\psi(y,\overline{y'})d\nu(y,\overline{y'}) \le C_2\int_{E((x,\overline{x'}),r)}|\varphi(y,\overline{y'})|^p M_\nu\psi(y,\overline{y'})\ d\nu(y,\overline{y'}). \tag{5}$$

Then taking $\varphi(y,\overline{y'}) = \overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\dots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)$ and $\psi(y,\overline{y'}) \equiv 1$ we obtain from inequality (5) and Lemma 1 that

$$\int_{E_+(0,r)} T^y \left(M_\gamma f(x)\right)^p (y')^\gamma dy$$
$$= \int_{E((x,0),r)} \left(M_\nu \overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\dots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right)^p d\nu(y,\overline{y'})$$
$$\le C_2 \int_{E((x,0),r)} \left|\overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\dots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right|^p d\nu(y,\overline{y'})$$
$$= C_2 \int_{E((x,0),r)} \left|f\left(\sqrt{y_1^2+\overline{y}_1^2},\dots,\sqrt{y_k^2+\overline{y}_k^2},y''\right)\right|^p d\nu(y,\overline{y'})$$
$$= C_2 \int_{E_+(0,r)} T^y|f(x)|^p (y')^\gamma dy \le C_2\, r^\lambda\, \|f\|^p_{L_{p,\lambda,\gamma}}.$$

□

Corollary 1. *Let $f \in L^{loc}_{1,\gamma}(\mathbb{R}^n_{k,+})$, then*

$$\lim_{t\to 0} |E_+(0,t)|^{-1}_\gamma \int_{E_+(0,t)} T^y f(x)\ (y')^\gamma dy = f(x)$$

for almost all $x \in \mathbb{R}^n_{k,+}$.

Corollary 2. [7]

1. *If $f \in L_{1,\gamma}(\mathbb{R}^n_{k,+})$, then $M_\gamma f \in WL_{1,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{WL_{1,\gamma}} \le C_{1,\gamma}\|f\|_{L_{1,\gamma}},$$

where $C_{1,\gamma}$ depends only on γ, k and n.

2. *If $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p \le \infty$, then $M_\gamma f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|M_\gamma f\|_{L_{p,\gamma}} \le C_{p,\gamma}\|f\|_{L_{p,\gamma}},$$

where $C_{p,\gamma}$ depends only on p,γ,k and n.

In the Theorem 1 if we take $\lambda = 0$, we obtain Corollary 2.

3. Hardy-Littlewood-Sobolev-Morrey type inequality for B-Riesz potential

Consider the B-Riesz potentials

$$I^\alpha_\gamma f(x) = \int_{\mathbb{R}^n_{k,+}} T^y|x|^{\alpha-n-|\gamma|} f(y)(y')^\gamma dy, \quad 0 < \alpha < n+|\gamma|.$$

For the B-Riesz potentials the following generalized Hardy–Littlewood–Sobolev theorem is valid.

Theorem 2. *Let* $0 < \alpha < n + |\gamma|$, $1 \le p < \frac{n+|\gamma|}{\alpha}$ *and* $0 \le \lambda < n + |\gamma| - \alpha p$.

If $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, *where* $1 < p < \frac{n+|\gamma|}{\alpha}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+|\gamma|-\lambda}$, *then* $I^\alpha_\gamma f \in L_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\left\| I^\alpha_\gamma f \right\|_{L_{q,\lambda,\gamma}} \le C_{p,\lambda} \left\| f \right\|_{L_{p,\lambda,\gamma}},$$

where $C_{p,\lambda}$ *is independent of* f.

If $f \in L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $1 - \frac{1}{q} = \frac{\alpha}{n+|\gamma|-\lambda}$, *then* $I^\alpha_\gamma f \in WL_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\left\| I^\alpha_\gamma f \right\|_{WL_{q,\lambda,\gamma}} \le C_\lambda \left\| f \right\|_{L_{1,\lambda,\gamma}},$$

where C_λ *is independent of* f.

Proof. Let $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. Then

$$I^\alpha_\gamma f(x) = \left(\int_{E_+(0,t)} + \int_{\mathbb{R}^n_{k,+} \setminus E_+(0,t)} \right) T^y f(x) |y|^{\alpha-n-|\gamma|} (y')^\gamma \, dy$$
$$\equiv A(x,t) + C(x,t). \tag{6}$$

For $A(x,t)$ we have

$$|A(x,t)| \le \int_{E_+(0,t)} T^y \left| f(x) \right| |y|^{\alpha-n-|\gamma|} (y')^\gamma \, dy$$
$$\le \sum_{k=-\infty}^{-1} \left(2^k t \right)^{\alpha-n-|\gamma|} \int_{E_+(0,2^{k+1}t) \setminus E_+(0,2^k t)} T^y \left| f(x) \right| y_n^\gamma dy.$$

Hence

$$|A(x,t)| \le C_3 t^\alpha M_\gamma f(x) \qquad \text{with} \qquad C_3 = \frac{\omega(n,\gamma) 2^{n+|\gamma|}}{2^\alpha - 1}. \tag{7}$$

From (6), for $C(x,t)$ by the Hölder's inequality we have

$$|C(x,t)| \le \left(\int_{\mathbb{R}^n_{k,+} \setminus E_+(0,t)} |y|^{-\beta} T^y \left| f\,(x) \right|^p (y')^\gamma dy \right)^{\frac{1}{p}}$$
$$\times \left(\int_{\mathbb{R}^n_{k,+} \setminus E_+(0,t)} |y|^{\left(\frac{\beta}{p} + \alpha - n - |\gamma| \right) p'} (y')^\gamma dy \right)^{\frac{1}{p'}} \le C_4 t^{\frac{\lambda-n-|\gamma|}{p} + \alpha} \left\| f \right\|_{L_{p,\lambda,\gamma}}. \tag{8}$$

Thus, from (7) and (8) we have

$$\left| I^\alpha_\gamma f(x) \right| \le C_5 \left(t^\alpha M_\gamma f(x) + t^{\frac{\lambda-n-|\gamma|}{q}} \left\| f \right\|_{L_{p,\lambda,\gamma}} \right).$$

Minimizing with respect to t, at $t = \left[(M_\gamma f(x))^{-1} \left\| f \right\|_{L_{p,\lambda,\gamma}} \right]^{p/(n+|\gamma|-\lambda)}$ we arrive at

$$\left| I^\alpha_\gamma f(x) \right| \le C_6 \left(M_\gamma f(x) \right)^{p/q} \left\| f \right\|^{1-p/q}_{L_{p,\lambda,\gamma}}.$$

Hence, by Theorem 1, we have

$$\int_{E_+(0,t)} T^y \left|I_\gamma^\alpha f(x)\right|^q (y')^\gamma dy \le C_6 \|f\|_{L_{p,\lambda,\gamma}}^{q-p} \int_{E_+(0,t)} T^y \left(M_\gamma f(x)\right)^p (y')^\gamma dy$$
$$\le C_7 t^\lambda \|f\|_{L_{p,\lambda,\gamma}}^{q-p} \|f\|_{L_{p,\lambda,\gamma}}^p \le C_7 t^\lambda \|f\|_{L_{p,\lambda,\gamma}}^q .$$

Let $f \in L_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. It suffices to prove the inequality (2) with 2β instead of β on the left-hand side of the inequality. So

$$\left|\left\{y \in E_+(0,t) \ : \ T^y \left|I_\gamma^\alpha f(x)\right| > 2\beta\right\}\right|_\gamma$$
$$\le |\{y \in E_+(0,t) \ : \ T^y |A(x,t)| > \beta\}|_\gamma$$
$$+ |\{y \in E_+(0,t) \ : \ T^y |C(x,t)| > \beta\}|_\gamma .$$

Taking into account inequality (7) and Theorem 1 we have

$$|\{y \in E_+(0,t) \ : \ T^y |A(x,t)| > \beta\}|_\gamma$$
$$\le \left|\left\{y \in E_+(0,t) \ : \ T^y (M_\gamma f(x)) > \frac{\beta}{C_5 t^\alpha}\right\}\right|_\gamma \le \frac{C_8 t^\alpha}{\beta} \cdot t^\lambda \|f\|_{L_{1,\lambda,\gamma}}$$

and thus if $C_4 t^{\frac{\lambda-n-|\gamma|}{q}} \|f\|_{L_{1,\lambda,\gamma}} = \beta$, then $|C(x,t)| \le \beta$ and consequently, $|\{y \in E_+(0,t) \ : \ T^y |C(x,t)| > \beta\}|_\gamma = 0$.

Finally

$$\left|\left\{y \in E_+(0,t) \ : \ T^y |I_\gamma^\alpha f(x)| > 2\beta\right\}\right|_\gamma$$
$$\le \frac{C_8}{\beta} t^\lambda t^\alpha \|f\|_{L_{1,\lambda,\gamma}} = C_9 t^\lambda \left(\frac{\|f\|_{L_{1,\lambda,\gamma}}}{\beta}\right)^q .$$

The theorem is proved. □

Corollary 3. [8] *Let* $0 < \alpha < n + |\gamma|$.

If $1 < p < \frac{n+|\gamma|}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+|\gamma|}$, $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, *then* $I_\gamma^\alpha f \in L_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\left\|I_\gamma^\alpha f\right\|_{L_{q,\gamma}} \le C_{p,\gamma} \|f\|_{L_{p,\gamma}}, \tag{9}$$

where C_p *is independent of* f.

If $f \in L_{1,\gamma}(\mathbb{R}^n_{k,+})$, $\frac{1}{q} = 1 - \frac{\alpha}{n+|\gamma|}$, *then* $I_\gamma^\alpha f \in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\left\|I_\gamma^\alpha f\right\|_{WL_{q,\gamma}} \le C_\lambda \|f\|_{L_{1,\gamma}}, \tag{10}$$

where C_λ *is independent of* f.

Theorem 3. *Let* $0 < \alpha < n + |\gamma|$.

If $1 < p < \frac{n+|\gamma|}{\alpha}$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+|\gamma|}$ *is necessary for inequality* (9) *to be valid.*

If $p = 1$, *then the condition* $1 - \frac{1}{q} = \frac{\alpha}{n+|\gamma|}$ *is necessary for inequality* (10) *to hold.*

Proof. Let $1 < p < \frac{n+|\gamma|}{\alpha}$, $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and inequality (9) hold.

Define $f_t(x) =: f(tx)$. Then

$$\|f_t\|_{L_{p,\gamma}} = t^{-\frac{n+|\gamma|}{p}} \|f\|_{L_{p,\gamma}}$$

and

$$\left\|I^\alpha_\gamma f_t\right\|_{L_{q,\gamma}} = t^{-\alpha-\frac{n+|\gamma|}{q}} \left\|I^\alpha_\gamma f\right\|_{L^\gamma_q(\mathbb{R}^n_{k,+})}.$$

By the inequality (9)

$$\left\|I^\alpha_\gamma f\right\|_{L_{q,\gamma}} \le C_{p,q} t^{\alpha+\frac{n+|\gamma|}{q}-\frac{n+|\gamma|}{p}} \|f\|_{L_{p,\gamma}}.$$

If $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{n+|\gamma|}$, then in the case $t \to 0$ we have $\left\|I^\alpha_\gamma f\right\|_{L_{q,\gamma}} = 0$ for all $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$.

As well as if $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{n+|\gamma|}$, then at $t \to \infty$ we obtain $\left\|I^\alpha_\gamma f\right\|_{L_{q,\gamma}} = 0$ for all $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$.

Therefore $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n+|\gamma|}$.

Now, let $f \in L_{1,\gamma}(\mathbb{R}^n_{k,+})$ and inequality (10) hold. We have

$$\left\|I^\alpha_\gamma f_t\right\|_{WL_{q,\gamma}} = t^{-\alpha-\frac{n+|\gamma|}{q}} \left\|I^\alpha_\gamma f\right\|_{WL_{q,\gamma}}.$$

By inequality (10)

$$\left\|I^\alpha_\gamma f\right\|_{WL_{q,\gamma}} \le C_q t^{\alpha+\frac{n+|\gamma|}{q}-n-|\gamma|} \|f\|_{L_{1,\gamma}}.$$

If $1 > \frac{1}{q} + \frac{\alpha}{n+|\gamma|}$, then in the case $t \to 0$ we have $\left\|I^\alpha_\gamma f\right\|_{WL_{q,\gamma}} = 0$ for all $f \in L_{1,\gamma}(\mathbb{R}^n_{k,+})$.

Similarly, if $1 < \frac{1}{q} + \frac{\alpha}{n+|\gamma|}$, then for $t \to \infty$ we obtain $\left\|I^\alpha_\gamma f\right\|_{WL_{q,\gamma}} = 0$ for all $f \in L_{1,\gamma}(\mathbb{R}^n_{k,+})$.

Therefore $1 = \frac{1}{q} + \frac{\alpha}{n+|\gamma|}$. □

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