ABOUT CHARACTERS AND THE DIRICHLET KERNEL ON VILENKIN GROUPS

MEDO PEPIĆ

Dedicated to Professor Fikret Vajzović on the occasion of his 80th birthday

Abstract. In this paper we study properties of characters and the Dirichlet kernel on Vilenkin groups which are of key importance for analyzing convergence, integrability and summability of Fourier-Vilenkin series.

1. Introduction and preliminaries

In this paper we work on a Vilenkin group \( G \) (i.e. an infinite, totally unconnected, compact Abelian group which satisfies the second axiom of countability). We can introduce the topology on \( G \) using the zero neighborhood chain

\[
G = G_0 \supset G_1 \supset \cdots \supset G_n \supset \cdots, \cap_{n=0}^\infty G_n = \{0\},
\]

which consists of open subgroups of group \( G \), such that quotient group \( G_n/G_{n+1} \) is a cyclic group of prime order \( p_{n+1}, \forall n \in \mathbb{N}_0 \). \( G \) is called bounded iff a sequence

\[(p_n)_{n \in \mathbb{N}} = (p_1, p_2, \ldots),\]

is bounded. For \( n \in \mathbb{N} \) we denote

\[m_n := p_1 p_2 \ldots p_n \ (m_0 := 1).\]

A classical example of a Vilenkin group is the product space

\[
\prod_{k=1}^\infty G_k,
\]

2000 Mathematics Subject Classification. 42C10, 43A40.

Key words and phrases. Vilenkin group, the character of the group, Dirichlet kernel on Vilenkin Group, Fourier-Vilenkin series.

This research was partially supported by the grant No. 11–14–21618.1/2007, Government of the Municipality of Sarajevo, Ministry of Science and Education.
where $G_k = \{0, 1\}$ is a cyclic group of the second order for all $k \in \mathbb{N}$, equipped with the discrete topology, with component addition (note that addition in each component is done by module 2). Its direct generalization is the group

$$G = \prod_{k=0}^{\infty} \mathbb{Z}(n_k),$$

where $\mathbb{Z}(n_k) := \{0, 1, \ldots, n_k - 1\}$, $n_k \geq 2$, is cyclic group of order $n_k$ ($k \in \mathbb{N}_0$) equipped with the discrete topology. An arbitrary $x \in G$ has a unique representation in the following form

$$x = \sum_{n=0}^{\infty} a_n x_n, \quad a_n \in \{0, 1, 2, \ldots, p_{n+1} - 1\}$$

where $x_n \in G_n \setminus G_{n+1}$ are previously arbitrary chosen and fixed, and for all $n \in \mathbb{N}_0$ and

$$G_n = \left\{ x \in G : \sum_{i=0}^{\infty} a_i x_i, \ a_i = 0, \text{ for } 0 \leq i < n \right\}.$$  

$G$ can be equipped with a Haar measure which is normalized (in the sense of $\mu(G) = 1$) and for all $n \in \mathbb{N}_0$ and for all $x \in G$:

$$\mu(x + G_n) = \frac{1}{m_n}.$$  

The class of all continuous functions $f : G \to C$ is denoted by $C(G)$. For $1 \leq p < \infty$, $L^p(G)$ is the set of all functions $f$ on $G$, measurable (according to $\mu$) such that

$$\int_G |f(x)|^p \, d\mu(x) < \infty.$$  

$L^\infty(G)$ is the set of all essentially bounded functions on $G$. All of these sets become Banach spaces in the case when we define the norm in the usual way. From compactness of the group $G$ follows

$$L^q(G) \subset L^p(G), \text{ for } 1 \leq p < q \leq \infty;$$

and using the fact that Haar measure is invariant with respect to translation it follows that norm $\|\cdot\|_p$ is invariant in respect to translation for all $1 \leq p \leq \infty$. If functions $f, \varphi$ on $G$ are such as that the function $f(x-h)\varphi(h)$ on $G$ is integrable for almost all $x \in G$, then $\int_G f(x-h)\varphi(h) \, d\mu(h)$ is called the convolution of functions $f$ and $\varphi$ and we write

$$f \ast \varphi(x) := \int_G f(x-h)\varphi(h) \, d\mu(h).$$
For all sequences of numbers \((b_n)_{n \in \mathbb{N}}\), with the property \(b_n \downarrow 0\), we can introduce metrics in \(G\), invariant with respect to translation, by
\[
d(x, y) = \begin{cases} 
  b_n, & x - y \in G_n \setminus G_{n+1}; \\
  0, & x = y.
\end{cases}
\]

It is a common to take \(b_n = \mu(G_n)\) or \(b_n = \mu(G_{n+1})\).

In \(G\) there is countable collection of \(\Gamma\) characters - continuous complex value functions \(\chi\), which satisfies the following conditions
\[
|\chi(x)| = 1 (\forall x \in G), \chi(x + y) = \chi(x)\chi(y) (\forall x, y \in G).
\]
The characters form an Abelian group with respect to the pointwise product of functions. We topologize \((\Gamma, \cdot)\) by defining a neighborhood basis around the unit
\[
\chi_0 \in \Gamma (\chi_0(x) = 1, \forall x \in G)
\]
using the collection of all sets
\[
U(A, \varepsilon) := \{\chi \in \Gamma : |\chi(a) - 1| < \varepsilon, \forall a \in A\},
\]
where \(A\) denotes the collection of all compact subsets in \(G\) and \(\varepsilon\) ranges over all positive numbers. It is known that [4, Th. 24.15 and Th. 24.26] \((\Gamma, \cdot)\) is discrete, countable Abelian group with torsion. Additionally, Vilenkin proved [8, Chapters 1.1, 1.2, 1.3 and 1.4] that in \(\Gamma\) there exists a chain
\[
\Gamma_0 = \{\chi_0\} \subset \Gamma_1 \subset \Gamma_2 \subset \cdots \Gamma_n \subset \cdots
\]
consists of subgroups \(\Gamma_n = G_n^\perp\) of the group \(\Gamma\), with following properties:
\[
(\forall n \in \mathbb{N}_0) G_n^\perp := \{\chi \in \Gamma : \chi(x) = 1, \forall x \in G_n\} = \Gamma(G/G_n);
\]
\[
\bigcup_{n=0}^{\infty} \Gamma_n = \Gamma; \ (\forall n \in \mathbb{N}_0) \Gamma_{n+1}/\Gamma_n \text{ is cyclic group with prime order } p_{n+1}.
\]
If for \(n \in \mathbb{N}_0\) we arbitrary choose \(\chi \in \Gamma_{n+1}/\Gamma_n\) and we assign to it the index \(m_n\), then for \(x \in G_n \setminus G_{n+1}\) is \(\chi_{m_n}^{p_{n+1}}(x) = 1\) and therefore
\[
\chi_{m_n}^{p_{n+1}} \in \Gamma_n \land \chi_{m_n}(x) \in \left\{e^{2\pi ik/p_{n+1}} : 1 \leq k < p_{n+1}\right\}.
\]
It is possible in (2) \((\forall n \in \mathbb{N}_0)\) to choose
\[
x_n \in G_n \setminus G_{n+1} \text{ such that } \chi_{m_n}(x_n) = e^{2\pi i / p_{n+1}}.
\]
In the main part of this paper e will assume that this holds. Hence, further we will \(\forall n \in \mathbb{N}_0\), by \(x_n\) denote element from \(G_n \setminus G_{n+1}\) for which we have
\( \chi_{m_n}(x_n) = e^{2\pi i / p_{n+1}}. \) Each \( n \in \mathbb{N} \) we can uniquely represent in the following form

\[
n = \sum_{i=0}^{N} a_i m_i, \quad a_i \in \{0, 1, 2, \ldots, p_{i+1} - 1\} \land a_N \neq 0 \land N = N(n).
\]  

(9)

Note that (9) is equivalent to

\[
m_N \leq n < m_{N+1}.
\]

Putting

\[
\chi_n = \prod_{i=0}^{N} \chi_{a_i}^{m_i}
\]

(10)

we obtain complete numeration of elements in group \((\Gamma, \cdot)\). According to this numeration we have

\[
(\forall n \in \mathbb{N}_0) \Gamma_n = \{\chi_0, \chi_1, \ldots, \chi_{m_n-1}\}.
\]

(11)

From the invariance of Haar measure under translation it follows that for all \( \chi \neq \chi_0 \), is \( \int \chi(x) d\mu(x) = 0. \)

Besides, the group \((\Gamma, \cdot)\) forms a complete and orthonormal system (in \( L^2(G) \)) according to Haar measure on \( G \) [1, p.77].

The order relation \(<\) on \( \Gamma \) we introduce by:

\[
(\forall m, n \in \mathbb{N}_0)(\chi_m < \chi_n \iff m < n).
\]

Let

\[
x = \sum_{n=0}^{\infty} a_n x_n, \quad a_n \in \{0, 1, 2, \ldots, p_{n+1} - 1\},
\]

\[
y = \sum_{n=0}^{\infty} b_n x_n, \quad b_n \in \{0, 1, 2, \ldots, p_{n+1} - 1\},
\]

are arbitrarily chosen elements from group \( G \). Order relation \(<\) in \( G \) we introduce by putting

\[
x < y \iff (\exists k \in \mathbb{N}_0)(a_i = b_i, \forall i \in \{0, 1, 2, \ldots, k\}) \land (a_{k+1} < b_{k+1}).
\]

In other words, order relation in \( G \) we introduce using lexicographical ordering of suitable sequences

\[(a_n) = (a_0, a_1, a_2, \ldots).\]

Using (3), we can also enumerate elements in factor group \( G/G_n \) according to the lexicographical ordering of their representatives

\[
\sum_{i=0}^{n-1} a_i x_i = z_j^{(n)} \quad (0 \leq j \leq m_n - 1).
\]

(12)
Each
\[ z_j^{(n)} = z_j^{(n)} + G_n \quad (1 \leq j < m_n), \]
is generating element of cyclic group \( G/G_n \) of order \( m_n \). In such a way each of groups \( G, \Gamma = \Gamma(G) \) and \( G/G_n (n \in \mathbb{N}_0) \) are equipped by an appropriate order relation.

When \( f \in L^1(G) \) and \( n \in \mathbb{N}_0 \), then:

1. \( c_n = c_n(f) = \hat{f}(\chi_n) = \hat{f}(n) := \int_G f(x)\chi_n(x) \, d\mu(x) \quad (n \in \mathbb{N}_0) \) are Fourier coefficients of function \( f \).
2. \( S(f) = S(f, x) := \sum_{n=0}^{\infty} \hat{f}(n)\chi_n(x) \) is Fourier series (precisely: Fourier-Vilenkin series) of function \( f \), and \( S_n(f) = S_n(f, x) := \sum_{i=0}^{n-1} \hat{f}(i)\chi_i(x) \) is partial sum index \( n \) of series \( S(f, x) \). Especially, \( S_0(f, x) := 0 \).
3. \( D_n(x) := \sum_{i=0}^{n-1} \chi_i(x) \) is the Dirichlet kernel of index \( n \). Especially, \( D_0(x) := 0 \).

For all \( n \in \mathbb{N}_0 \), all \( f \in L^1(G) \) and all \( x \in G \) holds
\[ D_n * f(x) = S_n(f, x), \text{ i.e. } D_n * f = S_n f, \]
since
\[
D_n * f(x) = \int_G D_n(x - t) f(t) \, d\mu(t) = \int_G \left( \sum_{i=0}^{n-1} \chi_i(x - t) \right) f(t) \, d\mu(t)
= \sum_{i=0}^{n-1} \chi_i(x) \int_G f(t) \chi_i(t) \, d\mu(t) = \sum_{i=0}^{n-1} \hat{f}(i)\chi_i(x) = S_n(f, x).
\]

If \( H \) is open and compact subgroup in \( G \) and if function \( f \in L^1(G) \) has property
\[ f(x) = \begin{cases} 1, & \forall x \in H; \\ 0, & \forall x \in G \setminus H, \end{cases} \]
then
\[ \hat{f}(\chi) = \mu(H) \cdot \zeta_{H^\perp}(\chi), \forall \chi \in \Gamma, \]
where \( \chi_A \) is the characteristic function of a set \( A \) [1, p.81]. Especially we have
\[
\int_{G_n} \chi(x) \, d\mu(x) = \mu(G_n) \cdot \zeta_{G_n^\perp}(\chi), \forall \chi \in \Gamma, \forall n \in \mathbb{N}_0.
\]
As consequences we have
\[
\int_{x_0 + G_n} \chi(x) \, d\mu(x) = \chi(x_0) \mu(G_n) \zeta_{G_n}^x(\chi), \forall \chi \in \Gamma, \forall n \in \mathbb{N}_0, \forall x_0 \in G \tag{14}
\]
and
\[
\int_{x_0 + G_n} \chi_k(x) \, d\mu(x) = 0, \forall n \in \mathbb{N}_0, \forall k \geq m_n. \tag{15}
\]

Properties of Dirichlet kernel on Vilenkin group are discussed in papers [8], [5] and monograph [1]. Results from these references we quote in a following theorem.

**Theorem 1.**

(D1) \( D_{m_n}(x) = \prod_{i=0}^{n-1} \frac{1 - \chi_{m_{i+1}}^{a_{i+1}}}{1 - \chi_{m_i}^{a_i}} \wedge D_{m_n}(x) = m_n \zeta_{G_n}(x) \) \[8, 2.2. p.4\].

(D2) If \( n \in \mathbb{N} \) is given by (9), then
\[
D_n = \chi_n \sum_{i=0}^{N} \frac{D_{m_i}^{a_i}}{1 - \chi_{m_i}} \quad [8, 2.3. p.5] \text{ and } [1, p.97 \text{ and } 98].
\]

(D3) \( x \in G \setminus G_n \Rightarrow (\forall k \in \mathbb{N}_0) \ |D_k(x)| \leq m_n \) \[8, 3.61. p.14\].

(D4) \( (\forall n \in \mathbb{N}_0) \int_D_n(x) \, d\mu(x) = 1 \) \[5, \text{Lemma 2, p.267}\].

(D5) \( m_k \leq n < m_{k+1} \wedge n = a_k m_k + r \wedge 0 < a_k < p_{k+1} \wedge 0 \leq r < m_k \Rightarrow D_n = \frac{1 - \chi_{m_k}^{a_k}}{1 - \chi_{m_k}} D_{m_k} + \chi_{m_k}^{a_k} D_r \) \[5, \text{Lemma 3, p.267}\].

2. **Results**

**Theorem 2.**

(D6) \( (\forall n \in \mathbb{N}_0)(\forall j \in \{0, 1, 2, \ldots, p_{n+1} - 1\}) \sum_{i=j m_n}^{(j+1)m_n - 1} \chi_i = D_{m_n} \chi_{m_n}^{j} \).

(D7) \( (\forall n \in \mathbb{N}_0)(\forall a_n \in \{0, 1, 2, \ldots, p_{n+1} - 1\}) D_{a_n, m_n} = D_{m_n} \frac{1 - \chi_{m_n}^{a_n}}{1 - \chi_{m_n}} \).

(D8) \( (\forall n \in \mathbb{N}_0) \sum_{i=m_n}^{m_{n+1} - 1} \chi_i = D_{m_n} \sum_{j=1}^{p_{n+1} - 1} \chi_{m_n}^{j} \).

(D9) \( (\forall n \in \mathbb{N}_0) D_{m_{n+1}} = D_{m_n} \frac{1 - \chi_{m_n}^{p_{n+1} - 1}}{1 - \chi_{m_n}} \) (Recurrent formulae).
\( (D_{10}) \) \( n = \sum_{i=0}^{N} a_i m_i, \ a_i \in \{0, 1, 2, \ldots, p_{i+1} - 1\} \land a_N \neq 0 \implies \)
\[
D_n = \sum_{i=0}^{N} D_{m_i} \frac{1 - \chi_{m_i}^{a_i}}{1 - \chi_{m_i}} \chi(a_{i+1}m_{i+1} + \cdots + a_N m_N),
\]

with

\[
\chi(a_{i+1}m_{i+1} + \cdots + a_N m_N) := \begin{cases} 
\chi_{m_{i+1}}^{a_{i+1}} \cdots \chi_{m_N}^{a_N}, & i \in \{0, 1, \ldots, N - 1\}; \\
\chi_0, & i = N.
\end{cases}
\]

\( (D_{11}) \) \( \forall n = \sum_{i=0}^{N} a_i m_i, \ a_i \in \{0, 1, 2, \ldots, p_{i+1} - 1\} \land a_N \neq 0 \)
\( (\forall k \in \{1, 2, \ldots, N\}) (n = a_k m_k + r) \implies \)
\[
D_r = \sum_{i \in \{0, 1, \ldots, N\} \setminus \{k\}} D_{m_i} \frac{1 - \chi_{m_i}^{a_i}}{1 - \chi_{m_i}} \chi(a_{i+1}m_{i+1} + \cdots + a_N m_N),
\]

and

\[
D_n = D_r + D_{m_k} \frac{1 - \chi_{m_k}^{a_k}}{1 - \chi_{m_k}} \chi(a_{k+1}m_{k+1} + \cdots + a_N m_N).
\]

**Proof.** \( (D_6) \)
\[
\sum_{i=jm_n}^{(j+1)m_n-1} \chi_i = \chi_{jm_n} + \chi_{jm_n+1} + \cdots + \chi_{jm_n+(m_n-1)}
= \chi_{m_n}^j (\chi_0 + \chi_1 + \cdots + \chi_{(m_n-1)}) = D_{mn} \chi_{m_n}^j.
\]

\( (D_7) \) Using \( (D_6) \) in third step we have
\[
D_{an,m_n} = \sum_{i=0}^{a_n m_n - 1} \chi_i = \sum_{j=0}^{a_n - 1} \left( \sum_{i=jm_n}^{(j+1)m_n-1} \chi_i \right)
= D_{m_n} \sum_{j=0}^{a_n-1} \chi_{m_n}^j = D_{m_n} \frac{1 - \chi_{m_n}^{a_n}}{1 - \chi_{m_n}}.
\]

\( (D_8) \)
\[
\sum_{i=m_n}^{m_n+1-1} \chi_i = \sum_{j=1}^{p_{n+1}-1} \left( \sum_{i=jm_n}^{(j+1)m_n-1} \chi_i \right) = D_{m_n} \sum_{j=1}^{p_{n+1}-1} \chi_{m_n}^j \quad \text{(in last step we used \( (D_6) \)).}
\]

\(^1\)From \( (D_1) \), \( (10) \) and \( (11) \) follows that \( (D_{10}) \) is just another form of \( (D_2) \). Here we present our proof of this property, which is considerably shorter than the proof given in [1, p.97 and 98].
Using \((D_8)\) we have
\[
D_{m_{n+1}} - D_m = \sum_{i=m_n}^{m_{n+1}-1} \chi_i = D_m \cdot \sum_{j=1}^{p_{n+1}-1} \chi_{m_n}
\]
\[
\Rightarrow D_{m_{n+1}} = D_m \left( \chi_0 + \sum_{j=1}^{p_{n+1}-1} \chi_j \right) = D_m \frac{1-\chi_{m_{n+1}}}{1-\chi_{m_n}}.
\]

\((D_{10})\) For the proof we will use mathematical induction. If \(n = 1 = 1.m_0\), then \(N = 0 \land a_0 = 1\), so \((D_{10})\) obviously is true. Let us assume \((D_{10})\) holds for arbitrarily chosen \(n \in \mathbb{N}\) given by \((9)\), and let us prove that \((D_{10})\) also holds for \(n + 1\). Really, if \(n + 1 = (p_1 - 1)m_0 + \cdots + (p_s - 1)m_s + a_s m_s + \cdots + a_N m_N + 1\)

\[
= (a_s + 1)m_s + a_{s+1}m_{s+1} + \ldots + a_N m_N + 1 = \sum_{i=0}^{N} b_i m_i,
\]

where
\[
b_i := \begin{cases} 
0, & i < s; \\
a_s + 1, & i = s; \\
a_i, & s < i \leq N.
\end{cases}
\]

Thereby and by \(0 \leq b_i \leq p_{i+1} - 1\) \(\forall i \in \{0, 1, \ldots, N\}\), using inductive step we obtain
\[
D_{n+1} = \sum_{i=0}^{N} D_{m_i} \frac{1-\chi_{b_i}}{1-\chi_{m_i}} \chi(b_{i+1}m_{i+1} + \cdots + b_N m_N).
\]

If \(n\) is such that does not exist number \(s\), then \(a_i = p_{i+1} - 1\) \(\forall i \in \{0, 1, \ldots, N\}\), then
\[
n + 1 = (p_1 - 1)m_0 + \cdots + (p_{N+1} - 1)m_N + 1 = m_{n+1} = \sum_{i=0}^{N+1} c_i m_i,
\]

with
\[
c_i := \begin{cases} 
0, & 0 \leq i \leq N; \\
1, & i = N + 1.
\end{cases}
\]

Now
\[
\frac{1-\chi_{c_i}}{1-\chi_{m_i}} = \begin{cases} 
0, & 0 \leq i \leq N; \\
1, & i = N + 1.
\end{cases}
\]

then \((D_{10})\) becomes clearly true equality \(D_{m_{N+1}} = D_{m_{N+1}} \cdot \chi_0\).

\((D_{11})\) Let \(n\) be given by \((9)\). Then
\[
n = \sum_{i=0}^{N} a_i m_i = a_k m_k + r \Leftrightarrow r = \sum_{i \in \{0,1,2,\ldots,N\} \setminus \{k\}} a_i m_i.
\]
Then we have

\[ D_n = \sum_{i=0}^{n-1} \chi_i \]

\[ = \sum_{i=0}^{a_0 m_0 + \cdots + a_k m_k - 1} \chi_i + \sum_{i=a_0 m_0 + \cdots + a_k m_k - 1}^{a_0 m_0 + \cdots + a_N m_N - 1} \chi_i \]

\[ = S_1 + S_2 + S_3, \]

otherwise

\[ D_n = D_r + S_2 \land D_r = S_1 + S_3. \]  (16)

\[ S_1 = \sum_{i=0}^{a_0 m_0 + \cdots + a_k m_k - 1} \chi_i \]

\[ = \sum_{i=a_0 m_0 + \cdots + a_N m_N}^{a_0 m_0 + \cdots + a_N m_N - 1} \chi_i + \sum_{i=a_0 m_0 + \cdots + a_k m_k - 1}^{a_0 m_0 + \cdots + a_N m_N - 1} \chi_i \]

\[ = \sum_{j=1}^{L_{j-1}-1} \sum_{i=L_j}^{L_j + a_j - 1} \chi_i, \]

where we put

\[ L_j := \begin{cases} a_j m_j + \cdots + a_N m_N, & 0 \leq j \leq N; \\ 0, & j = N + 1. \end{cases} \]  (17)

Because

\[ \sum_{i=L_j}^{L_j + a_j - 1} \chi_i = \sum_{i=L_j}^{a_0 m_0 + \cdots + a_j m_j - 1} \chi_i = \sum_{i=0}^{a_0 m_0 + \cdots + a_j m_j - 1} \chi_i \]

\[ = \chi L_j \sum_{i=0}^{a_j - 1} \chi_i = \chi L_j \sum_{s=0}^{a_j - 1} \sum_{i=s m_j - 1}^{a_j - 1 (s+1) m_j - 1} \chi_i \]

\[ = \chi L_j \cdot D_{m_j - 1} \sum_{s=0}^{a_j - 1} \chi_{s m_j - 1} = \chi L_j \cdot D_{m_j - 1} \cdot \frac{1 - \chi^a_{m_j - 1}}{1 - \chi_{m_j - 1}}, \]

we have

\[ S_1 = \sum_{j=1}^k \chi L_j \cdot D_{m_j - 1} \cdot \frac{1 - \chi^a_{m_j - 1}}{1 - \chi_{m_j - 1}} = \sum_{j=0}^{k-1} D_{m_j} \cdot \frac{1 - \chi a_j}{1 - \chi_{m_j}} \chi_{(a_{j+1} + \cdots + a_N \cdot m_N)}, \]  (18)
\[ S_3 = \sum_{i=a_0 m_k+\cdots+a_{k-1} m_k}^{a_{k+1} m_k+\cdots+a_N m_N - 1} \chi_i = \sum_{i=a_0 m_k+\cdots+a_{k-1} m_k}^{a_{k+1} m_k+\cdots+a_N m_N - 1} \chi_i + \sum_{i=a_{k+1} m_k+\cdots+a_N m_N}^{a_{k+2} m_k+\cdots+a_N m_N - 1} \chi_i + \cdots + \sum_{i=a_N m_N}^{a_N m_N - 1} \chi_i + \sum_{i=0}^{i=a_N m_N} \chi_i \]

\[ = \sum_{j=k+2}^{N+1} \sum_{i=L_j}^{L_{j-1} - 1} \chi_i = \sum_{j=k+2}^{N+1} D_{m_{j-1}} \frac{1 - \chi_{m_{j-1}}}{1 - \chi_{m_{j-1}}} \chi L_j \]

\[ = \sum_{j=k+1}^{N} D_{m_j} \frac{1 - \chi_{m_j}}{1 - \chi_{m_j}} \chi L_{j+1}. \]

So,

\[ S_3 = \sum_{j=k+1}^{N} D_{m_j} \frac{1 - \chi_{m_j}}{1 - \chi_{m_j}} \chi (a_{j+1} m_{j+1} + \cdots + a_N m_N), \quad \text{(19)} \]

\[ S_2 = \sum_{i=a_0 m_k+\cdots+a_{k-1} m_k}^{a_k m_k+\cdots+a_N m_N - 1} \chi_i = \sum_{i=a_0 m_k+\cdots+a_{k-1} m_k}^{a_{k+1} m_k+\cdots+a_N m_N} \chi_i \]

\[ = \sum_{i=L_{k+1}}^{L_{k+1} + a_k m_k - 1} \chi_i = \sum_{i=L_{k+1}}^{L_{k+1} + a_k m_k - 1} \chi_i = \chi L_{k+1} \sum_{i=0}^{i=a_k m_k-1} \chi_i. \]

So,

\[ S_2 = D_{m_k} \frac{1 - \chi_{m_k}}{1 - \chi_{m_k}} \chi (a_{k+1} m_{k+1} + \cdots + a_N m_N), \quad \text{(20)} \]

\((D_{11})\) follows form \((16), (18), (19)\) and \((20)\).

\[ \square \]

**Remark 1.**

a) \((D_{11})\) can be written in the following form

\[ D_{n-a_k m_k} = \sum_{i \in \{0,1,2,\ldots,N\} \setminus \{k\}} D_{m_i} \frac{1 - \chi_{m_i}}{1 - \chi_{m_i}} \chi (a_{i+1} m_{i+1} + \cdots + a_N m_N), \]

b) Putting \(k = N\), in \((D_{11})\), we obtain \((D_5)\), i.e.

\[ D_n = \frac{1 - \chi_{m_N}}{1 - \chi_{m_N}} D_{m_N} + \chi_{m_N}^a D_r. \]

**Corollary 1.**

(i) \(x \in G_n \setminus G_{n+1} \Rightarrow D_{m_k}(x) = \begin{cases} m_k, & 0 \leq k \leq n; \\ 0, & k > n. \end{cases} \)
(ii) $D_{m_n}(x) = m_n \zeta_{G_n}(x)$ (see (D1)).

(iii) $x \in G \setminus G_n \Rightarrow \sum_{i=m_n}^{\infty} \chi_i(x) = 0$.

(iv) $(\forall x \in G \setminus \{0\})(\exists_{1 n = n(x) \in \mathbb{N}_0}) D_{m_k}(x) = \begin{cases} m_k, & 0 \leq k \leq n; \\ 0, & k > n. \end{cases}$

(v) $(\forall x \in G \setminus \{0\}) \sum_{n=0}^{\infty} \chi_n(x) = 0$.

Proof. (i) $x \in G_n \Rightarrow x \in G_k (\forall k \leq n)$ (from (1)). Therefore and from (11) using fact that is $G_{1_k} \subseteq G_{1_n} (\forall k \leq n)$ we obtain $D_{m_k}(x) = m_k$.

By $x \notin G_{n+1} \Rightarrow x \notin G_k (\forall k \geq n+1)$. Therefore, using (5), follows $D_{m_k}(x) = 0 (\forall k \geq n)$. Therefore we have $\sum_{i=m_n}^{\infty} \chi_i(x) = \sum_{k=n}^{\infty} (D_{m_{k+1}}(x) - D_{m_k}(x)) = 0$.

(iv) By $G \setminus \{0\} = \bigcup_{n=0}^{\infty} (G_n \setminus G_{n+1})$ and fact that is $\{G_{n} \setminus G_{n+1}\}_{n \in \mathbb{N}_0}$ tassellation of set $G \setminus \{0\}$, we conclude that an arbitrarily chosen $x \in G \setminus \{0\}$ there exists one (and only one) $n = n(x) \in \mathbb{N}_0$ such that $x \in G_{n} \setminus G_{n+1}$. Hence and by (i) follows (iv).

(v) Let $n = n(x)$ from (iv). Then we have $\sum_{n=0}^{\infty} \chi_n(x) = \sum_{i=0}^{m_n-1} \chi_i(x) + \sum_{i=m_n}^{\infty} \chi_i(x) = D_{m_n}(x) + \sum_{k=n}^{\infty} (D_{m_{k+1}}(x) - D_{m_k}(x)) = m_n + (0 - m_n) = 0$. 

$\square$
Theorem 3. For all \( n = \sum_{i=0}^{N} a_i m_i \), \( a_N \neq 0 \) and for all \( k \in \mathbb{N}_0 \) and all \( x \in G_k \setminus G_{k+1} \)
\[ D_n(x) = \begin{cases} n, & k > N; \\ D_{n,k} \cdot \chi_{(a_{k+1} m_{k+1} + \cdots + a_N m_N)}(x), & k \leq N, \end{cases} \]
where
\[ D_{n,k} := \varepsilon^{a_k} \left( \sum_{i=0}^{k-1} a_i m_i \right) + m_k \frac{1 - \varepsilon^{a_k}}{1 - \varepsilon} (\forall k \leq N), \]
\( \varepsilon = \chi_{m_k}(x) \) is some primitive \( p_{k+1} \)-th root of unit.

Proof. a) \( k > N \Rightarrow G_k^\perp \supseteq G_i^\perp (\forall i \leq N) \). Hence and from (D10) we have
\[ D_n(x) = \sum_{i=0}^{N} D_{m_i}(x) \frac{1 - \chi_{m_i}(x)}{1 - \chi_{m_i}(x)} \chi_{m_{i+1}}(x) \cdots \chi_{m_N}(x) = \sum_{i=0}^{N} a_i m_i = n. \]
b) \( k \leq N \Rightarrow (G_k^\perp \supseteq G_i^\perp (\forall i \leq k)) \land \chi_{m_i} \notin G_k^\perp (\forall i \geq k) \land D_{m_i}(x) = 0 (\forall i \geq k) \land \chi_{m_k}(x) = \varepsilon \) (some primitive \( p_{k+1} \)-th root of unit). Hence and from (D10) we have
\[ D_n(x) = \left( \sum_{i=0}^{k-1} a_i m_i \right) \cdot \chi_{m_k}(x) \cdots \chi_{m_N}(x) \]
\[ + m_k \frac{1 - \chi_{m_k}(x)}{1 - \chi_{m_k}(x)} \chi_{m_{k+1}}(x) \cdots \chi_{m_N}(x) \]
\[ = D_{n,k} \cdot \chi_{m_{k+1}}(x) \cdots \chi_{m_N}(x). \]
\[ \square \]

Corollary 2. For all \( n = \sum_{i=0}^{N} a_i m_i \), \( a_N \neq 0 \) and for all \( k \in \mathbb{N}_0 \) and all \( x \in G_k \setminus G_{k+1} \)
\[ |D_n(x)| = \begin{cases} n, & k > N; \\ |D_{n,k}| \leq \sum_{i=0}^{k} a_i m_i, & k \leq N. \end{cases} \]

Theorem 4. a) \( \int_{G_k} D_n(x) \, d\mu(x) = \begin{cases} n \frac{1}{m_k}, & n < m_k; \\ 1, & n \geq m_k. \end{cases} \)
b) \( \int_{G_k \setminus G_{k+1}} D_n(x) \, d\mu(x) = \begin{cases} n \frac{1}{m_{k+1}}, & n \leq m_k; \\ 1 - \frac{m_k}{m_{k+1}}, & m_k < n < m_{k+1}; \\ 0, & n \geq m_{k+1}. \end{cases} \)
Proof. a) By (11) we have
\[ \int_{G_k} D_n(x) \, d\mu(x) = \int_{G_k} \sum_{i=0}^{n-1} \chi_i(x) \, d\mu(x) = n \int_{G_k} d\mu(x) = \frac{n}{m_k} \, (\text{if } n < m_k). \]
If \( n \geq m_k \), then
\[ \int_{G_k} D_n(x) \, d\mu(x) = \int_{G_k} \sum_{i=0}^{n-1} \chi_i(x) \, d\mu(x) \]
\[ = \int_{G_k} \left( \sum_{i=0}^{m_k-1} \chi_i(x) + \sum_{i=m_k}^{n-1} \chi_i(x) \right) d\mu(x) \]
\[ = m_k \int_{G_k} d\mu(x) = 1 \, (\text{by (15)}). \]

b) \[
\int_{G_k \setminus G_{k+1}} D_n(x) \, d\mu(x) = \int_{G_k \setminus G_{k+1}} \sum_{i=0}^{n-1} \chi_i(x) \, d\mu(x) \]
\[ = \int_{G_k \setminus G_{k+1}} \left( \sum_{i=0}^{m_{k+1}-1} \chi_i(x) \right) d\mu(x) \]
\[ = \sum_{j=1}^{m_k-1} \sum_{i=0}^{n-1} \chi_i(jx_k + x) \, d\mu(jx_k + x) \]
\[ = \sum_{j=1}^{m_k-1} \sum_{i=0}^{n-1} \chi_i(jx_k + x) \, d\mu(jx_k + x) \]
\[ = \sum_{j=1}^{m_k-1} \sum_{i=0}^{n-1} \chi_i^j(x_k) \, d\mu(G_{k+1}) \, \zeta_{G_{k+1}}^\perp(\chi_i). \]
So, \[
\int_{G_k \setminus G_{k+1}} D_n(x) \, d\mu(x) = \mu(G_{k+1}) \sum_{j=1}^{m_k-1} \sum_{i=0}^{n-1} \chi_i^j(x_k) \, \zeta_{G_{k+1}}^\perp(\chi_i). \quad (21) \]

Now we discuss separately each of the three (only) possible cases:
\[ b_1) n \leq m_k; \quad b_2) m_k < n < m_{k+1}; \quad b_3) n \geq m_{k+1}. \]
In the case \( b_1 \) we have \( \chi_i(x_k) = 1 \ (\forall i < m_k) \), then
\[
\int_{G_k \setminus G_{k+1}} D_n(x) \, d\mu(x) = \sum_{j=1}^{p_{k+1}-1} \sum_{i=0}^{m_k-1} \frac{1}{m_{k+1}} 1 = \frac{n(p_{k+1} - 1)}{m_{k+1}}.
\]

In the case \( b_2 \) we have
\[
\int_{G_k \setminus G_{k+1}} D_n(x) \, d\mu(x) = \sum_{j=1}^{p_{k+1}-1} \left( \sum_{i=0}^{m_k-1} 1^j + \sum_{i=m_k}^{n-1} \chi_i^j(x_k) \right) \cdot \frac{1}{m_{k+1}} 1
\]
\[= \frac{1}{m_{k+1}} \sum_{j=1}^{p_{k+1}-1} \left( m_k + \sum_{i=m_k}^{n-1} \varepsilon_i^j \right) = 1 - \frac{n}{m_{k+1}},
\]
because \( \sum_{j=1}^{p_{k+1}-1} \varepsilon_i^j = -1 \), for each fixed \( i \in \{ m_k, m_k + 1, \ldots, n - 1 \} \).

In the case \( b_3 \) we have
\[
\int_{G_k \setminus G_{k+1}} D_n(x) \, d\mu(x) = \frac{1}{m_{k+1}} \sum_{j=1}^{p_{k+1}-1} \left( m_k - \sum_{i=m_k}^{m_k + 1} \chi_i^j(x_k) \right)
\]
\[= \frac{m_k(p_{k+1} - 1)}{m_{k+1}} + \frac{1}{m_{k+1}} \sum_{i=m_k}^{m_k + 1} \sum_{j=1}^{p_{k+1}-1} \varepsilon_i^j = 0,
\]
because \( \varepsilon_i \) is some primitive \( p_{k+1} \)-th root of unit (we take in account that \( \zeta_{G_{k+1}}^i(\chi_i) = 0 \), for \( i \geq m_{k+1} \)).

\[\square\]

References


(Received: January 10, 2008) Department of Mathematics
(Revised: April 22, 2008) University of Sarajevo
71000 Sarajevo, Zmaja od Bosne 33-35
Bosnia and Herzegovina
E-mail: mpecic@lol.ba