SARAJEVO JOURNAL OF MATHEMATICS
Vol.4 (16) (2008), 109–123

ABOUT CHARACTERS AND THE DIRICHLET KERNEL ON VILENKIN GROUPS

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Dedicated to Professor Fikret Vajzović on the occasion of his 80th birthday

Abstract. In this paper we study properties of characters and the Dirichlet kernel on Vilenkin groups which are of key importance for analyzing convergence, integrability and summability of Fourier-Vilenkin series.

1. Introduction and preliminaries

In this paper we work on a Vilenkin group G (i.e. an infinite, totally unconnected, compact Abelian group which satisfies the second axiom of countability). We can introduce the topology on G using the zero neighborhood chain

$$G = G_0 \supset G_1 \supset \cdots \supset G_n \supset \ldots, \cap_{n=0}^{\infty} G_n = \{0\}\,, \tag{1}$$

which consists of open subgroups of group G, such that quotient group $G_n \diagup G_{n+1}$ is a cyclic group of prime order $p_{n+1}, \forall n \in \mathbb{N}_0$. G is called **bounded** iff a sequence

$$(p_n)_{n\in\mathbb{N}} = (p_1, p_2, \ldots),$$

is bounded. For $n \in \mathbb{N}$ we denote

$$m_n := p_1 p_2 \ldots p_n \quad (m_0 := 1).$$

A classical example of a Vilenkin group is the product space

$$\prod_{k=1}^{\infty} G_k,$$

2000 *Mathematics Subject Classification.* 42C10, 43A40.

Key words and phrases. Vilenkin group, the character of the group, Dirichlet kernel on Vilenkin Group, Fourier-Vilenkin series.

This research was partially supported by the grant No. 11–14–21618.1/2007, Government of the Municipality of Sarajevo, Ministry of Science and Education.

where $G_k = \{0,1\}$ is a cyclic group of the second order for all $k \in \mathbb{N}$, equipped with the discrete topology, with component addition (note that addition in each component is done by module 2). Its direct generalization is the group

$$G = \prod_{k=0}^{\infty} \mathbb{Z}(n_k),$$

where $\mathbb{Z}(n_k) := \{0, 1, 2, \ldots, n_k - 1\}, n_k \geq 2$, is cyclic group of order n_k ($k \in \mathbb{N}_0$) equipped with the discrete topology. An arbitrary $x \in G$ has a unique representation in the following form

$$x = \sum_{n=0}^{\infty} a_n x_n,\ a_n \in \{0, 1, 2, \ldots, p_{n+1} - 1\} \tag{2}$$

where $x_n \in G_n \diagdown G_{n+1}$ are previously arbitrary chosen and fixed, and for all $n \in \mathbb{N}_0$ and

$$G_n = \left\{ x \in G : \sum_{i=0}^{\infty} a_i x_i,\ a_i = 0,\ \text{for } 0 \leq i < n \right\}. \tag{3}$$

G can be equipped with a Haar measure which is normalized (in the sense of $\mu(G) = 1$) and for all $n \in \mathbb{N}_0$ and for all $x \in G$:

$$\mu(x + G_n) = \frac{1}{m_n}. \tag{4}$$

The class of all continuous functions $f : G \to C$ is denoted by $C(G)$. For $1 \leq p < \infty$, $L^p(G)$ is the set of all functions f on G, measurable (according to μ) such that

$$\int_G |f(x)|^p \, d\mu(x) < \infty.$$

$L^\infty(G)$ is the set of all essentially bounded functions on G. All of these sets become Banah spaces in the case when we define the norm in the usual way. From compactness of the group G follows

$$L^q(G) \subset L^p(G),\ \text{for } 1 \leq p < q \leq \infty;$$

and using the fact that Haar measure is invariant with respect to translation it follows that norm $\|.\|_p$ is invariant in respect to translation for all $1 \leq p \leq \infty$. If functions f, φ on G are such as that the function $f(x-h)\ \varphi(h)$ on G is integrable for almost all $x \in G$, then $\int_G f(x-h)\ \varphi(h)\, d\mu(h)$ is called the **convolution** of functions f and φ and we write

$$f * \varphi(x) := \int_G f(x-h)\varphi(h)\, d\mu(h).$$

For all sequences of numbers $(b_n)_{n\in\mathbb{N}}$, with the property $b_n \downarrow 0$, we can introduce metrics in G, invariant with respect to translation, by

$$d(x,y) = \begin{cases} b_n, & x-y \in G_n \diagdown G_{n+1}; \\ 0, & x=y. \end{cases}$$

It is a common to take $b_n = \mu(G_n)$ or $b_n = \mu(G_{n+1})$.

In G there is countable collection of Γ **characters** - continuous complex value functions χ, which satisfies the following conditions

$$|\chi(x)| = 1 (\forall x \in G), \chi(x+y) = \chi(x)\chi(y) (\forall x, y \in G).$$

The characters form an Abelian group with respect to the pointwise product of functions. We topologize $(\Gamma, \cdot)$ by defining a neighborhood basis around the unit

$$\chi_0 \in \Gamma (\chi_0(x) = 1, \forall x \in G)$$

using the collection of all sets

$$U(A,\varepsilon) := \{\chi \in \Gamma : |\chi(a) - 1| < \varepsilon, \forall a \in A\},$$

where A denotes the collection of all compact subsets in G and ε rangs over all positive numbers. It is known that [4, Th. 24.15 and Th. 24.26] $(\Gamma, \cdot)$ is discrete, countable Abelian group with torsion. Additionally, Vilenkin proved [8, Chapters 1.1, 1.2, 1.3 and 1.4] that in Γ there exists a chain

$$\Gamma_0 = \{\chi_0\} \subset \Gamma_1 \subset \Gamma_2 \subset \cdots \Gamma_n \subset \cdots$$

consists of subgroups $\Gamma_n = G_n^{\perp}$ of the group Γ, with following properties:

$$(\forall n \in \mathbb{N}_0) G_n^{\perp} := \{\chi \in \Gamma : \chi(x) = 1, \forall x \in G_n\} = \Gamma(G/G_n); \tag{5}$$

$$\bigcup_{n=0}^{\infty} \Gamma_n = \Gamma; \;\; (\forall n \in \mathbb{N}_0) \Gamma_{n+1} \diagup \Gamma_n \text{ is cyclic group with prime order } p_{n+1}\,. \tag{6}$$

If for $n \in \mathbb{N}_0$ we arbitrary choose $\chi \in \Gamma_{n+1} \diagdown \Gamma_n$ and we assign to it the index m_n, then for $x \in G_n \diagdown G_{n+1}$ is $\chi_{m_n}^{p_{n+1}}(x) = 1$ and therefore

$$\chi_{m_n}^{p_{n+1}} \in \Gamma_n \wedge \chi_{m_n}(x) \in \left\{ e^{2\pi i k \,/\, p_{n+1}} : 1 \le k < p_{n+1} \right\}. \tag{7}$$

It is possible in (2) $(\forall n \in \mathbb{N}_0)$ to choose

$$x_n \in G_n \diagdown G_{n+1} \text{ such that } \chi_{m_n}(x_n) = e^{2\pi i \,/\, p_{n+1}}. \tag{8}$$

In the main part of this paper e will assume that this holds. Hence, further we will $\forall n \in \mathbb{N}_0$, by x_n denote element from $G_n \diagdown G_{n+1}$ for which we have

$\chi_{m_n}(x_n) = e^{2\pi i \,/\, p_{n+1}}$. Each $n \in \mathbb{N}$ we can uniquely represent in the following form

$$n = \sum_{i=0}^{N} a_i m_i,\ a_i \in \{0, 1, 2, \ldots, p_{i+1} - 1\} \wedge a_N \neq 0 \wedge N = N(n). \tag{9}$$

Note that (9) is equivalent to

$$m_N \leq n < m_{N+1}.$$

Putting

$$\chi_n = \prod_{i=0}^{N} \chi_{m_i}^{a_i} \tag{10}$$

we obtain complete numeration of elements in group $(\Gamma, \cdot)$. According to this numeration we have

$$(\forall n \in \mathbb{N}_0)\Gamma_n = \{\chi_0, \chi_1, \ldots, \chi_{m_n - 1}\}. \tag{11}$$

From the invariance of Haar measure under translation it follows that for all $\chi \neq \chi_0$, is $\int_G \chi(x)\, d\mu(x) = 0$.

Besides, the group $(\Gamma, \cdot)$ forms a complete and orthonormal system (in $L^2(G)$) according to Haar measure on G [1, p.77].

The order relation $<$ on Γ we introduce by:

$$(\forall m, n \in \mathbb{N}_0)(\chi_m < \chi_n \Leftrightarrow m < n).$$

Let

$$x = \sum_{n=0}^{\infty} a_n x_n, \quad a_n \in \{0, 1, 2, \ldots, p_{n+1} - 1\},$$

$$y = \sum_{n=0}^{\infty} b_n x_n, \quad b_n \in \{0, 1, 2, \ldots, p_{n+1} - 1\},$$

are arbitrarily chosen elements from group G. Order relation $<$ in G we introduce by putting

$$x < y \Leftrightarrow (\exists k \in \mathbb{N}_0)(a_i = b_i, \forall i \in \{0, 1, 2, \ldots, k\}) \wedge (a_{k+1} < b_{k+1}).$$

In other words, order relation in G we introduce using lexicographical ordering of suitable sequences

$$(a_n) = (a_0, a_1, a_2, \ldots).$$

Using (3), we can also enumerate elements in factor group G/G_n according to the lexicographical ordering of their representatives

$$\sum_{i=0}^{n-1} a_i x_i = z_j^{(n)} \quad (0 \leq j \leq m_n - 1). \tag{12}$$

Each

$$\left[z_j^{(n)}\right] = z_j^{(n)} + G_n \quad (1 \leq j < m_n),$$

is generating element of cyclic group G/G_n of order m_n. In a such way each of groups

$$G, \Gamma = \Gamma(G) \text{ and } G/G_n(n \in \mathbb{N}_0)$$

are equipped by an appropriate order relation.

When $f \in L^1(G)$ and $n \in \mathbb{N}_0$, then:

a) $c_n = c_n(f) = \hat{f}(\chi_n) = \hat{f}(n) := \int_G f(x)\overline{\chi_n(x)}\, d\mu(x) \quad (n \in \mathbb{N}_0)$ are **Fourier coefficients** of function f.

b) $S(f) = S(f,x) := \sum_{n=0}^{\infty} \hat{f}(n)\chi_n(x)$ is **Fourier series** (precisely: **Fourier-Vilenkin series**) of function f, and $S_n(f) = S_n(f,x) := \sum_{i=0}^{n-1} \hat{f}(i)\chi_i(x)$ is **partial sum** index n of series $S(f,x)$. Especially, $S_0(f,x) := 0$.

c) $D_n(x) := \sum_{i=0}^{n-1} \chi_i(x)$ is the **Dirichlet kernel** of index n. Especially, $D_0(x) := 0$.

For all $n \in \mathbb{N}_0$, all $f \in L^1(G)$ and all $x \in G$ holds

$$D_n * f(x) = S_n(f,x), \text{ i.e. } D_n * f = S_n f,$$

since

$$D_n * f(x) = \int_G D_n(x-t) f(t)\, d\mu(t) = \int_G \left(\sum_{i=0}^{n-1} \chi_i(x-t)\right) f(t)\, d\mu(t)$$
$$= \sum_{i=0}^{n-1} \chi_i(x) \int_G f(t)\overline{\chi_i(t)}\, d\mu(t) = \sum_{i=0}^{n-1} \hat{f}(i)\chi_i(x) = S_n(f,x).$$

If H is open and compact subgroup in G and if function $f \in L^1(G)$ has property

$$f(x) = \begin{cases} 1, & \forall x \in H; \\ 0, & \forall x \in G \setminus H, \end{cases}$$

then

$$\hat{f}(\chi) = \mu(H).\zeta_{H^\perp}(\chi), \forall \chi \in \Gamma,$$

where χ_A is the characteristic function of a set A [1, p.81]. Especially we have

$$\int_{G_n} \chi(x)\, d\mu(x) = \mu(G_n).\zeta_{G_n^\perp}(\chi), \forall \chi \in \Gamma, \forall n \in \mathbb{N}_0. \tag{13}$$

As consequences we have

$$\int_{x_0+G_n} \chi(x)\, d\mu(x) = \chi(x_0)\mu(G_n).\zeta_{G_n^\perp}(\chi), \forall \chi \in \Gamma, \forall n \in \mathbb{N}_0, \forall x_0 \in G \tag{14}$$

and

$$\int_{x_0+G_n} \chi_k(x)\, d\mu(x) = 0, \forall n \in \mathbb{N}_0, \forall k \geq m_n. \tag{15}$$

Properties of Dirichlet kernel on Vilenkin group are discussed in papers [8], [5] and monograph [1]. Results from these references we quote in a following theorem.

Theorem 1.

(D_1) $D_{m_n}(x) = \prod\limits_{i=0}^{n-1} \frac{1-\chi_{m_i}^{p_{i+1}}}{1-\chi_{m_i}} \wedge D_{m_n}(x) = m_n.\zeta_{G_n}(x)$ [8, 2.2. p.4].

(D_2) *If* $n \in \mathbb{N}$ *is given by* (9) , *then*

$$D_n = \chi_n. \sum_{i=0}^{N} \frac{D_{m_i}}{\chi_{m_i}^{a_i}}.\frac{1-\chi_{m_i}^{a_i}}{1-\chi_{m_i}} \quad \text{[8, 2.3. p.5] and [1, p.97 and 98].}$$

(D_3) $x \in G \setminus G_n \Rightarrow (\forall k \in \mathbb{N}_0) \;\; |D_k(x)| \leq m_n$ [8, 3.61. p.14].

(D_4) $(\forall n \in \mathbb{N}_0) \int\limits_G D_n(x)\, d\mu(x) = 1$ [5, Lemma 2, p.267].

(D_5) $m_k \leq n < m_{k+1} \wedge n = a_k m_k + r \wedge 0 < a_k < p_{k+1} \wedge 0 \leq r < m_k \Rightarrow$

$$D_n = \frac{1-\chi_{m_k}^{a_k}}{1-\chi_{m_k}} D_{m_k} + \chi_{m_k}^{a_k}.D_r \quad \text{[5, Lemma 3, p.267].}$$

2. Results

Theorem 2.

(D_6) $(\forall n \in \mathbb{N}_0)(\forall j \in \{0,1,2,\ldots,p_{n+1}-1\}) \sum\limits_{i=jm_n}^{(j+1)m_n-1} \chi_i = D_{m_n}.\chi_{m_n}^j.$

(D_7) $(\forall n \in \mathbb{N}_0)(\forall a_n \in \{0,1,2,\ldots,p_{n+1}-1\}) D_{a_n.m_n} = D_{m_n}.\frac{1-\chi_{m_n}^{a_n}}{1-\chi_{m_k}}.$

(D_8) $(\forall n \in \mathbb{N}_0) \sum\limits_{i=m_n}^{m_{n+1}-1} \chi_i = D_{m_n}. \sum\limits_{j=1}^{p_{n+1}-1} \chi_{m_n}^j.$

(D_9) $(\forall n \in \mathbb{N}_0) D_{m_{n+1}} = D_{m_n}.\frac{1-\chi_{m_n}^{p_{n+1}}}{1-\chi_{m_n}}$ (Recurrent formulae).

(D_{10})[1] $n=\sum\limits_{i=0}^{N} a_i m_i$, $a_i \in \{0,1,2,\ldots,p_{i+1}-1\} \wedge a_N \neq 0 \Rightarrow$

$$D_n=\sum_{i=0}^{N} D_{m_i}\frac{1-\chi_{m_i}^{a_i}}{1-\chi_{m_i}}\chi_{(a_{i+1}.m_{i+1}+\cdots+a_N.m_N)},$$

with

$$\chi_{(a_{i+1}.m_{i+1}+\cdots+a_N.m_N)}:=\begin{cases}\chi_{m_{i+1}}^{a_{i+1}}\cdots\chi_{m_N}^{a_N}, & i\in\{0,1,\ldots,N-1\};\\ \chi_0, & i=N.\end{cases}$$

(D_{11}) $(\forall n=\sum\limits_{i=0}^{N} a_i m_i,\ a_i\in\{0,1,2,\ldots,p_{i+1}-1\}\wedge a_N\neq 0)$
$(\forall k\in\{1,2,\ldots,N\})(n=a_k m_k+r)\Rightarrow$

$$D_r=\sum_{i\in\{0,1,\ldots,N\}\setminus\{k\}}^{N} D_{m_i}\frac{1-\chi_{m_i}^{a_i}}{1-\chi_{m_i}}\chi_{(a_{i+1}.m_{i+1}+\cdots+a_N.m_N)},$$

and

$$D_n=D_r+D_{m_k}\frac{1-\ \chi_{m_k}^{a_k}}{1-\chi_{m_k}}.\chi_{(a_{k+1}.m_{k+1}+\cdots+a_N.m_N)}.$$

Proof. (D_6)

$$\sum_{i=jm_n}^{(j+1)m_n-1}\chi_i=\chi_{jm_n}+\chi_{jm_n+1}+\cdots+\chi_{jm_n+(m_n-1)}$$
$$=\chi_{m_n}^{j}(\chi_0+\chi_1+\cdots+\chi_{(m_n-1)})=D_{m_n}.\chi_{m_n}^{j}.$$

(D_7) Using (D_6) in third step we have

$$D_{a_n.m_n}=\sum_{i=0}^{a_n m_n-1}\chi_i=\sum_{j=0}^{a_n-1}\left(\sum_{i=jm_n}^{(j+1)m_n-1}\chi_i\right)$$
$$=D_{m_n}.\sum_{j=0}^{a_n-1}\chi_{m_n}^{j}=D_{m_n}.\frac{1-\chi_{m_n}^{a_n}}{1-\chi_{m_k}}.$$

(D_8) $\sum\limits_{i=m_n}^{m_{n+1}-1}\chi_i=\sum\limits_{j=1}^{p_{n+1}-1}\left(\sum\limits_{i=jm_n}^{(j+1)m_n-1}\chi_i\right)=D_{m_n}.\sum\limits_{j=1}^{p_{n+1}-1}\chi_{m_n}^{j}$ (in last step we used (D_6)).

[1]From (D_1), (10) and (11) follows that (D_{10}) is just another form of (D_2). Here we present our proof of this property, which is considerably shorter than the proof given in [1, p.97 and 98].

(D_9) Using (D_8) we have

$$D_{m_{n+1}} - D_{m_n} = \sum_{i=m_n}^{m_{n+1}-1} \chi_i = D_{m_n}.\sum_{j=1}^{p_{n+1}-1} \chi_{m_n}^j$$

$$\Rightarrow D_{m_{n+1}} = D_{m_n}\left(\chi_0 + \sum_{j=1}^{p_{n+1}-1} \chi_{m_n}^j\right) = D_{m_n}\frac{1-\chi_{m_n}^{p_{n+1}}}{1-\chi_{m_n}}.$$

(D_{10}) For the proof we will use mathematical induction. If $n = 1 = 1.m_0$, then $N = 0 \wedge a_0 = 1$, so (D_{10}) obviously is true. Let us assume (D_{10}) holds for arbitrarily chosen $n \in \mathbb{N}$ given by (9), and let us prove that (D_{10}) also holds for $n+1$. Really, if n is such that there exist $s = \min_{i\in\{0,1,\dots,N\}}\{i : a_i < p_{i+1} - 1\}$, then

$$n+1 = (p_1 - 1)m_0 + \cdots + (p_s - 1)m_{s-1} + a_s m_s + \cdots + a_N m_N + 1$$

$$= (a_s + 1)m_s + a_{s+1}m_{s+1} + ldots + a_N m_N + 1 = \sum_{i=0}^{N} b_i m_i,$$

where

$$b_i := \begin{cases} 0, & i < s; \\ a_s + 1, & i = s; \\ a_i, & s < i \le N. \end{cases}$$

Thereby and by $0 \le b_i \le p_{i+1} - 1$ $(\forall i \in \{0, 1, \dots, N\})$, using inductive step we obtain

$$D_{n+1} = \sum_{i=0}^{N} D_{m_i} \frac{1 - \chi_{m_i}^{b_i}}{1 - \chi_{m_i}} \chi_{(b_{i+1}.m_{i+1}+\cdots+b_N.m_N)}.$$

If n is such that does not exist number s, then $a_i = p_{i+1} - 1(\forall i \in \{0, 1, \dots, N\})$, then

$$n+1 = (p_1 - 1)m_0 + \cdots + (p_{N+1} - 1)m_N + 1 = m_{n+1} = \sum_{i=0}^{N+1} c_i m_i,$$

with

$$c_i := \begin{cases} 0, & 0 \le i \le N; \\ 1, & i = N+1. \end{cases}$$

Now

$$\frac{1 - \chi_{m_i}^{c_i}}{1 - \chi_{m_i}} = \begin{cases} 0, & 0 \le i \le N; \\ 1, & i = N+1. \end{cases}$$

then (D_{10}) becomes clearly true equality $D_{m_{N+1}} = D_{m_{N+1}}.1.\chi_0$.

(D_{11}) Let n be given by (9) . Then

$$n = \sum_{i=0}^{N} a_i m_i = a_k m_k + r \Leftrightarrow r = \sum_{i\in\{0,1,2,\dots,N\}\setminus\{k\}} a_i m_i.$$

Then we have

$$\begin{aligned} D_n &= \sum_{i=0}^{n-1} \chi_i \\ &= \sum_{i=0}^{a_0m_0+\cdots+a_{k-1}m_{k-1}-1} \chi_i + \sum_{i=a_0m_0+\ldots+a_{k-1}m_{k-1}}^{a_0m_0+\cdots+a_km_k-1} \chi_i + \sum_{i=a_0m_0+\cdots+a_km_k}^{a_0m_0+\ldots+a_Nm_N-1} \chi_i \\ &= S_1 + S_2 + S_3, \end{aligned}$$

otherwise

$$D_n = D_r + S_2 \wedge D_r = S_1 + S_3. \tag{16}$$

$$\begin{aligned} S_1 &= \sum_{i=0}^{a_0m_0+\cdots+a_{k-1}m_{k-1}-1} \chi_i \\ &= \sum_{i=a_1m_1+\cdots+a_Nm_N}^{a_0m_0+\cdots+a_Nm_N-1} \chi_i + \sum_{i=a_2m_2+\cdots+a_Nm_N}^{a_1m_1+\cdots+a_Nm_N-1} \chi_i + \cdots + \sum_{i=a_km_k+\cdots+a_Nm_N}^{a_{k-1}m_{k-1}+\cdots+a_Nm_N-1} \chi_i \\ &= \sum_{j=1}^{k} \sum_{i=L_j}^{L_{j-1}-1} \chi_i, \end{aligned}$$

where we put

$$L_j := \begin{cases} a_jm_j + \cdots + a_Nm_N, & 0 \le j \le N; \\ 0, & j = N+1. \end{cases} \tag{17}$$

Because

$$\begin{aligned} \sum_{i=L_j}^{L_{j-1}-1} \chi_i = \sum_{i=L_j}^{L_j+a_{j-1}m_{j-1}-1} \chi_i &= \sum_{i=0}^{a_{j-1}m_{j-1}-1} \chi_{L_j+i} \\ &= \chi_{L_j} \sum_{i=0}^{a_{j-1}m_{j-1}-1} \chi_i = \chi_{L_j} \sum_{s=0}^{a_{j-1}-1} \sum_{i=sm_{j-1}}^{(s+1)m_{j-1}-1} \chi_i \\ &= \chi_{L_j}.D_{m_{j-1}} \sum_{s=0}^{a_{j-1}-1} \chi_{m_{j-1}}^s = \chi_{L_j}.D_{m_{j-1}} \frac{1-\chi_{m_{j-1}}^{a_{j-1}}}{1-\chi_{m_{j-1}}}, \end{aligned}$$

we have

$$S_1 = \sum_{j=1}^{k} \chi_{L_j}.D_{m_{j-1}} \frac{1-\chi_{m_{j-1}}^{a_{j-1}}}{1-\chi_{m_{j-1}}} = \sum_{j=0}^{k-1} D_{m_j} \frac{1-\chi_{m_j}^{a_j}}{1-\chi_{m_j}} \chi_{(a_{j+1}.m_{j+1}+\cdots+a_N.m_N)}. \tag{18}$$

$$S_3 = \sum_{i=a_0m_0+\cdots+a_km_k}^{a_0m_0+\cdots+a_Nm_N-1} \chi_i = \sum_{i=a_{k+2}m_{k+2}+\cdots+a_Nm_N}^{a_{k+1}m_{k+1}+\ldots+a_Nm_N-1} \chi_i$$

$$+ \sum_{i=a_{k+3}m_{k+3}+\cdots+a_Nm_N}^{a_{k+2}m_{k+2}+\ldots+a_Nm_N-1} \chi_i + \ldots + \sum_{i=a_Nm_N}^{a_{N-1}m_{N-1}+a_Nm_N-1} \chi_i + \sum_{i=0}^{a_Nm_N-1} \chi_i$$

$$= \sum_{j=k+2}^{N+1} \sum_{i=L_j}^{L_{j-1}-1} \chi_i = \sum_{j=k+2}^{N+1} D_{m_{j-1}} \frac{1-\chi_{m_{j-1}}^{a_{j-1}}}{1-\chi_{m_{j-1}}} \chi_{L_j}$$

$$= \sum_{j=k+1}^{N} D_{m_j} \frac{1-\chi_{m_j}^{a_j}}{1-\chi_{m_j}} \chi_{L_{j+1}}.$$

So,

$$S_3 = \sum_{j=k+1}^{N} D_{m_j} \frac{1-\chi_{m_j}^{a_j}}{1-\chi_{m_j}} \chi_{(a_{j+1}.m_{j+1}+\cdots+a_N.m_N)}. \tag{19}$$

$$S_2 = \sum_{i=a_0m_0+\cdots+a_{k-1}m_{k-1}}^{a_0m_0+\cdots+a_km_k-1} \chi_i = \sum_{i=a_{k+1}m_{k+1}+\cdots+a_Nm_N}^{a_km_k+\cdots+a_Nm_N-1} \chi_i$$

$$= \sum_{i=L_{k+1}}^{L_k-1} \chi_i = \sum_{i=L_{k+1}}^{L_{k+1}+a_km_k-1} \chi_i = \chi_{L_{k+1}} \sum_{i=0}^{a_km_k-1} \chi_i.$$

So,

$$S_2 = D_{m_k} \frac{1-\chi_{m_k}^{a_k}}{1-\chi_{m_k}} \chi_{(a_{k+1}.m_{k+1}+\cdots+a_N.m_N)}. \tag{20}$$

(D_{11}) follows form $(16), (18), (19)$ and (20).

□

Remark 1.

a) (D_{11}) can be written in the following form

$$D_{n-a_km_k} = \sum_{i\in\{0,1,2,\ldots,N\}\setminus\{k\}} D_{m_i} \frac{1-\chi_{m_i}^{a_i}}{1-\chi_{m_i}} \chi_{(a_{i+1}.m_{i+1}+\cdots+a_N.m_N)}.$$

b) Putting $k = N$, in (D_{11}), we obtain (D_5), i.e.

$$D_n = \frac{1-\ \chi_{m_N}^{a_N}}{1-\ \chi_{m_N}} D_{m_N} + \chi_{m_N}^{a_N}.D_r.$$

Corollary 1. (i) $x \in G_n \setminus G_{n+1} \Rightarrow D_{m_k}(x) = \begin{cases} m_k, & 0 \le k \le n; \\ 0, & k > n. \end{cases}$

(ii) $D_{m_n}(x) = m_n.\zeta_{G_n}(x)$ (see (D_1)).

(iii) $x \in G \setminus G_n \Rightarrow \sum\limits_{i=m_n}^{\infty} \chi_i(x) = 0.$

(iv) $(\forall x \in G \setminus \{0\})(\exists_1 n = n(x) \in \mathbb{N}_0)\ D_{m_k}(x) = \begin{cases} m_k, & 0 \le k \le n; \\ 0, & k > n. \end{cases}$

(v) $(\forall x \in G \setminus \{0\}) \sum\limits_{n=0}^{\infty} \chi_n(x) = 0.$

Proof. (i) $x \in G_n \Rightarrow x \in G_k (\forall k \le n)$ (from (1)). Therefore and from (11) using fact that is $G_k^{\perp} \subseteq G_n^{\perp} (\forall k \le n)$ we obtain

$$D_{m_k}(x) = m_k.$$

By $x \notin G_{n+1} \Rightarrow x \notin G_k (\forall k \ge n+1)$. Therefore, using (5), follows

$$D_{m_k}(x) = \prod_{i=0}^{k-1} \frac{1 - \chi_{m_i}^{p_{i+1}}(x)}{1 - \chi_{m_i}(x)} = 0,$$

because this product contains the factor $\frac{1-\chi_{m_n}^{p_{n+1}}(x)}{1-\chi_{m_n}(x)} = \frac{1-\varepsilon^{p_{n+1}}}{1-\varepsilon} = 0$, where $\varepsilon = \chi_{m_n}(x)$ is $p_{n+1} - th$ root of unit.

(ii) Follows directly from (i).

(iii) $x \notin G_n \Rightarrow x \notin G_k (\forall k \ge n)$. Hence, by (ii), $D_{m_k}(x) = 0 (\forall k \ge n)$. Therefore we have

$$\sum_{i=m_n}^{\infty} \chi_i(x) = \sum_{k=n}^{\infty} (D_{m_{k+1}}(x) - D_{m_k}(x)) = 0.$$

(iv) By $G \setminus \{0\} = \bigcup\limits_{n=0}^{\infty} (G_n \setminus G_{n+1})$ and fact that is $\{G_n \setminus G_{n+1}\}_{n \in \mathbb{N}_0}$ tasselation of set $G \setminus \{0\}$, we conclude that an arbitrarily chosen $x \in G \setminus \{0\}$ there exists one (and only one) $n = n(x) \in \mathbb{N}_0$ such that $x \in G_n \setminus G_{n+1}$. Hence and by (i) follows (iv).

(v) Let $n = n(x)$ from (iv). Then we have

$$\begin{aligned} \sum_{n=0}^{\infty} \chi_n(x) &= \sum_{i=0}^{m_n - 1} \chi_i(x) + \sum_{i=m_n}^{\infty} \chi_i(x) \\ &= D_{m_n}(x) + \sum_{k=n}^{\infty} (D_{m_{k+1}}(x) - D_{m_k}(x)) \\ &= m_n + (0 - m_n) = 0. \end{aligned}$$

□

Theorem 3. *For all* $n = \sum_{i=0}^{N} a_i m_i$, $a_N \neq 0$ *and for all* $k \in \mathbb{N}_0$ *and all* $x \in G_k \setminus G_{k+1}$

$$D_n(x) = \begin{cases} n, & k > N; \\ D_{n,k} \cdot \chi_{(a_{k+1} \cdot m_{k+1} + \cdots + a_N \cdot m_N)}(x), & k \leq N, \end{cases}$$

where

$$D_{n,k} := \varepsilon^{a_k} . \left(\sum_{i=0}^{k-1} a_i m_i \right) + m_k \frac{1 - \varepsilon^{a_k}}{1 - \varepsilon} (\forall k \leq N),$$

$\varepsilon = \chi_{m_k}(x)$ *is some primitive* p_{k+1}*-th root of unit.*

Proof. a) $k > N \Rightarrow G_k^{\perp} \supseteq G_i^{\perp} (\forall i \leq N)$. Hence and from (D_{10}) we have

$$D_n(x) = \sum_{i=0}^{N} D_{m_i}(x) \frac{1 - \chi_{m_i}^{a_i}(x)}{1 - \chi_{m_i}(x)} \chi_{m_{i+1}}^{a_{i+1}}(x) \ldots \chi_{m_N}^{a_N}(x) = \sum_{i=0}^{N} a_i m_i = n.$$

b) $k \leq N \Rightarrow (G_k^{\perp} \supseteq G_i^{\perp} (\forall i \leq k)) \wedge \chi_{m_i} \notin G_k^{\perp} (\forall i \geq k) \wedge x \notin G_i (\forall i > k) \wedge D_{m_i}(x) = 0 \ (\forall i \geq k) \wedge \chi_{m_k}(x) = \varepsilon$ (some primitive p_{k+1}-th root of unit). Hence and from (D_{10}) we have

$$\begin{aligned} D_n(x) &= \left(\sum_{i=0}^{k-1} a_i m_i \right) . \chi_{m_k}^{a_k}(x) \ldots \chi_{m_N}^{a_N}(x) \\ &\quad + m_k \frac{1 - \chi_{m_k}^{a_k}(x)}{1 - \chi_{m_k}(x)} \chi_{m_{k+1}}^{a_{k+1}}(x) \ldots \chi_{m_N}^{a_N}(x) \\ &= D_{n,k} \cdot \chi_{m_{k+1}}^{a_{k+1}}(x) \ldots \chi_{m_N}^{a_N}(x). \end{aligned}$$

□

Corollary 2. *For all* $n = \sum_{i=0}^{N} a_i m_i$, $a_N \neq 0$ *and for all* $k \in \mathbb{N}_0$ *and all* $x \in G_k \setminus G_{k+1}$

$$|D_n(x)| = \begin{cases} n, & k > N; \\ |D_{n,k}| \leq \sum_{i=0}^{k} a_i m_i, & k \leq N. \end{cases}$$

Theorem 4. a) $\int_{G_k} D_n(x) \, d\mu(x) = \begin{cases} \frac{n}{m_k}, & n < m_k; \\ 1, & n \geq m_k. \end{cases}$

b) $\int_{G_k \setminus G_{k+1}} D_n(x) \, d\mu(x) = \begin{cases} \frac{n(p_{k+1}-1)}{m_{k+1}}, & n \leq m_k; \\ 1 - \frac{n}{m_{k+1}}, & m_k < n < m_{k+1}; \\ 0, & n \geq m_{k+1}. \end{cases}$

Proof. a) By (11) we have

$$\int_{G_k} D_n(x)\,d\mu(x) = \int_{G_k} \sum_{i=0}^{n-1} \chi_i(x)\,d\mu(x) = n\int_{G_k} d\mu(x) = \frac{n}{m_k} \quad (\text{if } n < m_k).$$

If $n \geq m_k$, then

$$\begin{aligned}\int_{G_k} D_n(x)\,d\mu(x) &= \int_{G_k} \sum_{i=0}^{n-1} \chi_i(x)\,d\mu(x)\\ &= \int_{G_k} \Big(\sum_{i=0}^{m_k-1} \chi_i(x) + \sum_{i=m_k}^{n-1} \chi_i(x) \Big)\,d\mu(x)\\ &= m_k \int_{G_k} d\mu(x) = 1 \quad (\text{by } (15)).\end{aligned}$$

b)

$$\begin{aligned}\int_{G_k \setminus G_{k+1}} D_n(x)\,d\mu(x) &= \int_{\bigcup_{j=1}^{p_{k+1}-1}(jx_k+G_{k+1})} \sum_{i=0}^{n-1} \chi_i(x)\,d\mu(x)\\ &= \sum_{j=1}^{p_{k+1}-1} \int_{j.G_k+G_{k+1}} \Big(\sum_{i=0}^{n-1} \chi_i(x) \Big)\,d\mu(x)\\ &= \sum_{j=1}^{p_{k+1}-1} \sum_{i=0}^{n-1} \int_{j.G_k+G_{k+1}} \chi_i(x)\,d\mu(x)\\ &= \sum_{j=1}^{p_{k+1}-1} \sum_{i=0}^{n-1} \int_{G_{k+1}} \chi_i(j.x_k+x)\,d\mu(j.x_k+x)\\ &= \sum_{j=1}^{p_{k+1}-1} \sum_{i=0}^{n-1} \chi_i^j(x_k).\mu(G_{k+1}).\zeta_{G_{k+1}}^{\perp}(\chi_i).\end{aligned}$$

So,

$$\int_{G_k \setminus G_{k+1}} D_n(x)\,d\mu(x) = \mu(G_{k+1}). \sum_{j=1}^{p_{k+1}-1} \sum_{i=0}^{n-1} \chi_i^j(x_k).\zeta_{G_{k+1}^{\perp}}(\chi_i). \tag{21}$$

Now we discuss separately each of the three (only) possible cases:

$b_1)$ $n \leq m_k$; $b_2)$ $m_k < n < m_{k+1}$; $b_3)$ $n \geq m_{k+1}$.

In the case b_1) we have $\chi_i(x_k) = 1\ \ (\forall i < m_k)$, then

$$\int\limits_{G_k \setminus G_{k+1}} D_n(x)\, d\mu(x) = \sum_{j=1}^{p_{k+1}-1} \sum_{i=0}^{n-1} 1. \frac{1}{m_{k+1}}.1 = \frac{n(p_{k+1}-1)}{m_{k+1}}.$$

In the case b_2) we have

$$\int\limits_{G_k \setminus G_{k+1}} D_n(x)\, d\mu(x) = \sum_{j=1}^{p_{k+1}-1} \left(\sum_{i=0}^{m_k-1} 1^j + \sum_{i=m_k}^{n-1} \chi_i^j(x_k) \right) . \frac{1}{m_{k+1}}.1$$

$$= \frac{1}{m_{k+1}}. \sum_{j=1}^{p_{k+1}-1} \left(m_k + \sum_{i=m_k}^{n-1} \left(\varepsilon_i^j \right) \right) = 1 - \frac{n}{m_{k+1}},$$

because $\sum_{j=1}^{p_{k+1}-1} \varepsilon_i^j = -1$, for each fixed $i \in \{m_k, m_k+1, \ldots, n-1\}$.

In the case b_3) we have

$$\int\limits_{G_k \setminus G_{k+1}} D_n(x)\, d\mu(x) = \frac{1}{m_{k+1}} \sum_{j=1}^{p_{k+1}-1} \left(\sum_{i=0}^{m_k-1} 1^j + \sum_{i=m_k}^{m_{k+1}-1} \chi_i^j(x_k) \right)$$

$$= \frac{m_k(p_{k+1}-1)}{m_{k+1}} + \frac{1}{m_{k+1}} \sum_{i=m_k}^{m_{k+1}-1} \sum_{j=1}^{p_{k+1}-1} \varepsilon_i^j = 0,$$

because ε_i is some primitive p_{k+1}-th root of unit (we take in account that $\zeta_{G_{k+1}}^{\perp}(\chi_i) = 0$, for $i \geq m_{k+1}$).

□

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(Received: January 10, 2008)
(Revised: April 22, 2008)

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