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A FORMULA FOR N-TIMES INTEGRATED SEMIGROUPS ($n \in \mathbb{N}$)

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Dedicated to Professor Fikret Vajzović on the occasion of his 80th birthday

Abstract. Motivated with Hille's first exponential formula for C_0 semigroups, we prove a formula for $n-$times integrated semigroups. At first we prove a formula for twice integrated semigroup, and, later, we generalize this formula for $n-$times integrated semigroups.

1. Introduction

The theory of integrated semigroups of operators on a Banach space was introduced and developed during the last twenty years by Kellermann, Arendt, Thieme, Hieber, Neubrander, and many other mathematicians (for example, see [1, 2, 4, 5, 6, 7]).

Denote by X a Banach space with the norm $\|\cdot\|$; $L(X) = L(X, X)$ is the space of bounded linear operators from X into X.

Definition 1. (Definition 4.1. in [6]). *Let A be a linear operator on a Banach space X, with nonempty resolvent set, and domain $D(A)$. If there exists an $n \in \mathbb{N} \cup \{0\}$, constants M and ω, and a strongly continuous family $(S(t))_{t\geq 0}$ in $L(X)$ with $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ such that resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$ (I-the identity operator) exists and is given by*

$$R(\lambda, A)x = \lambda^n \int_0^\infty e^{-\lambda t} S(t)x \, dt$$

for $x \in X$ and $Re\lambda > \omega$, then A is called the generator of the n-time integrated semigroup $(S(t))_{t\geq 0}$.

Theorem 2. (Lemma 5.1. in [6]). *Let A be the generator of the n-times integrated semigroup $(S(t))_{t\geq 0}$. If $n = 0$, then $(S(t))_{t\geq 0}$ is a strongly continuous semigroup. If $n \in \mathbb{N}$, then, the following hold:*

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a) *$S(t)x = 0$ for all $t \geq 0$ implies $x = 0$, and $S(0) = 0$.*
b) *For every $x \in D(A)$ we have $S(t)x \in D(A)$, $AS(t)x = S(t)Ax$, and*

$$S(t)x = \int_0^t S(s)Ax\,ds + \frac{t^n}{n!}\,x.$$

c) *For every $x \in X$ we have $\int_0^t S(s)xds \in D(A)$ and*

$$A\int_0^t S(s)x\,ds = S(t)x - \frac{t^n}{n!}\,x.$$

d) *For all $x \in X$ and $t, s \geq 0$,*

$$S(t)S(s)x = \frac{1}{(n-1)!}\left[\int_0^{t+s}(t+s-r)^{n-1}S(r)x\,dr - \int_0^t (t+s-r)^{n-1}S(r)x\,dr - \int_0^s (t+s-r)^{n-1}S(r)x\,dr\right].$$

The motivation for this paper is the well known Hille's first exponential formula for strongly continuous semigroups, and a result for once integrated semigroups.

Theorem 3. (Theorem 1.2.2. in [3]). *Suppose $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on a Banach space X, with infinitesimal generator A. Then, for each $f \in X$ and each $t \geq 0$*

$$T(t)f = \lim_{\tau\to 0+} e^{tA_\tau} f,$$

the limit existing uniformly with respect to t in any finite interval $[0, b]$.

In this theorem, the operator A_τ is defined as

$$A_\tau := \frac{T(\tau) - I}{\tau}$$

so that

$$T(t)f = \lim_{\tau\to 0+} e^{\frac{t}{\tau}[T(\tau)-I]} f = \lim_{\tau\to 0+} e^{-\frac{t}{\tau}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{t}{\tau}\right)^k [T(\tau)]^k f$$
$$= \lim_{\tau\to 0+} e^{-\frac{t}{\tau}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{t}{\tau}\right)^k T(k\tau)f.$$

These facts were inspiration for the next result for once integrated semigroups.

Theorem 4. (Theorem 3.2. in [8]). *Let $(S(t))_{t\geq 0}$ be a once integrated exponentially bounded semigroup on a Banach space X, with generator A. Then for all $x \in X$ and $t \geq 0$ we have*

$$S(t)x = \lim_{h\to 0+} \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{t}{h}\right)^n A^{n-1} S^n(h)x,$$

with the limit existing uniformly with respect to t in any finite interval $[0,T]$.

2. A formula for twice integrated semigroups

Theorem 5. *Let $(S(t))_{t\geq 0}$ be a twice integrated exponentially bounded semigroup on a Banach space X, with generator A. Then, for all $x \in D(A)$ and $t \geq 0$ we have:*

$$S(t)x = \lim_{h\to 0+} \left[\frac{t}{h} S(h)x + \sum_{n=2}^{\infty} \frac{1}{n!} \left(\frac{t}{h}\right)^n A^{n-2} \left(AS(h) + hI\right)^n x\right] \qquad (1)$$

with the limit existing uniformly with respect to t in any finite interval $[0,T]$.

Proof. Let $x \in D(A)$ be fixed. Then, $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$, for all $t \geq 0$. Also, for all $t, s \geq 0$,

$$S(t)S(s)x = \int_0^{t+s} (t+s-r)S(r)x\,dr - \int_0^t (t+s-r)S(r)x\,dr - \int_0^s (t+s-r)S(r)x\,dr. \qquad (2)$$

For every $a \geq 0$, using integration by parts, the fact that operator A is closed, and $A\int_0^r S(u)xdu = S(r)x - \frac{r^2}{2}x$, we have

$$\begin{aligned} A\int_0^a (t+s-r)S(r)x\,dr &= (t+s-r)\left(S(r)x - \frac{r^2}{2}x\right)\Big|_0^a + \int_0^a \left(S(r)x - \frac{r^2}{2}x\right) dr \\ &= (t+s-a)\left(S(a)x - \frac{a^2}{2}x\right) - \frac{a^3}{6}x + \int_0^a S(r)x\,dr. \end{aligned} \qquad (3)$$

From (2) and (3) we obtain

$$AS(t)S(s)x = \int_0^{t+s} S(r)x\,dr - \int_0^t S(r)x\,dr - \int_0^s S(r)xdr - sS(t)x - tS(s)x \qquad (4)$$

and

$$A^2S(t)S(s)x = S(t+s)x - S(t)x - S(s)x - sAS(t)x - tAS(s)x - stx. \tag{5}$$

From (5) with $t = s = h$ we have:

$$A^2S^2(h)x = S(2h)x - 2S(h)x - 2hAS(h)x - h^2x.$$

Because of $AS(t)x = S(t)Ax$ (with $x \in D(A)$), we obtain

$$(AS(h) + hI)^2 x = S(2h)x - 2S(h)x. \tag{6}$$

We want to prove by induction that for every $n \in \mathbb{N}$, $n \geq 2$,

$$A^{n-2}(AS(h) + hI)^n x = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} S[(n-k)h]\, x. \tag{7}$$

From (6) we see that (7) holds for $n = 2$. Assume that (7) holds for some fixed $n \in \mathbb{N}$, $n \geq 2$.

Then for $n+1$ we have

$$\begin{aligned} A^{n-1}(AS(h) + hI)^{n+1} x &= A\left[A^{n-2}(AS(h) + hI)^n\right](AS(h) + hI)\, x \\ &= \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} AS[(n-k)h]\,(AS(h) + hI)\, x \\ &= \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \big\{A^2S[(n-k)h]\, S(h)x \\ &\qquad + hAS[(n-k)h]\, x\big\}. \end{aligned}$$

Now, using (5) we obtain

$$\begin{aligned} A^{n-1}(AS(h) + hI)^{n+1} x &= \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \big\{S[(n+1-k)h]\, x \\ &- S[(n-k)h]\, x - S(h)x - (n-k)hAS(h)x - (n-k)h^2x\big\}. \end{aligned}$$

Because of: $\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k) = 0$ and $\sum_{k=0}^{n-1}(-1)^k\binom{n}{k} = (-1)^{n+1}$, we have further that

$$\begin{aligned} A^{n-1}(AS(h) + hI)^{n+1} x &= \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \big\{S[(n+1-k)h]\, x - S[(n-k)h]\, x\big\} \\ &+ (-1)^n S(h)x = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k} S[(n+1-k)h]\, x. \end{aligned}$$

Hence, (7) holds for $n+1$ too, and, by induction, for every $n \in \mathbb{N}$, $n \geq 2$.

Therefore,

$$\frac{t}{h}S(h)x+\sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{t}{h}\right)^{n}A^{n-2}\left(AS(h)+hI\right)^{n}x=\frac{t}{h}S(h)x+\frac{1}{2!}\left(\frac{t}{h}\right)^{2}$$

$$[S(2h)x-2S(h)x]+\frac{1}{3!}\left(\frac{t}{h}\right)^{3}[S(3h)x-3S(2h)x+3S(h)x]$$

$$+\frac{1}{4!}\left(\frac{t}{h}\right)^{4}[S(4h)x-4S(3h)x+6S(2h)x+4S(h)x]+\cdots$$

$$=\underbrace{\left[1-\frac{t}{h}+\frac{1}{2!}\left(\frac{t}{h}\right)^{2}-\cdots\right]}_{=e^{-\frac{t}{h}}}\cdot\left[\frac{t}{h}S(h)x+\frac{1}{2!}\left(\frac{t}{h}\right)^{2}S(2h)x\right.$$

$$\left.+\frac{1}{3!}\left(\frac{t}{h}\right)^{3}S(3h)x+\cdots\right]=e^{-\frac{t}{h}}\cdot\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{t}{h}\right)^{n}S(nh)x.$$

Hence, we need to prove that

$$S(t)x=\lim_{h\to 0+}e^{-\frac{t}{h}}\cdot\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{t}{h}\right)^{n}S(nh)x. \tag{8}$$

This formula one can prove in the same manner as in proof of theorem 1.2.2. in [3], or in proof of theorem 3.2. in [8]. □

3. Generalization to n-times integrated semigroups

In this section we give and prove the general case of the formula we are considering. In [6] Neubrander proved: If $(S(t))_{t\geq 0}$ is an exponentially bounded k-times integrated semigroup on the Banach space X, with generator A, and the sets $C^m\subset X$ ($m\in\mathbf{N}$) are defined as follows:

$$C^m:=\left\{x\in X : t\to S(t)x\in C^m(R^+,X)\right\},$$

then

$$\ldots D(A^{3k})\subset C^{3k}\subset D(A^{2k})\subset C^{2k}\subset D(A^k)\subset C^k,$$

and $\left(S^{(k)}(t)\right)_{t\geq 0}$ is a C_0 semigroup on C^k.

We need the following lemma.

Lemma 6. *Let $(S(t))_{t\geq 0}$ be a k-times integrated semigroup on Banach space X, with generator A. Then, for every $x\in C^k$, and every $n\geq k$ ($n\in\mathbf{N}$),*

$$A^{n-k}\left(S^{(k-1)}(t)\right)^{n}x=\sum_{i=0}^{n-1}(-1)^{i}\binom{n}{i}S\left((n-i)t\right)x.$$

Proof. We can assume that $0 \in \rho(A)$ (the resolvent set of A), i.e. the operator A^{-1} exists, because, if A is generator of integrated semigroup $(S(t))_{t\geq 0}$, then $A_1 = A-\lambda$ is generator of some other integrated semigroup $(S_\lambda(t))_{t\geq 0}$, so that $0 \in \rho(A_1)$. The family $\left(S^{(k)}(t)\right)_{t\geq 0}$ is a C_0 semigroup on C^k. Therefore, for $x \in C^k$,

$$\left[S^{(k)}(t) - I\right]^n x = \sum_{i=0}^{n}(-1)^i \binom{n}{i} \left(S^{(k)}(t)\right)^{n-i} x$$
$$= \sum_{i=0}^{n}(-1)^i \binom{n}{i} S^{(k)}((n-i)t)x.$$

For $x \in C^k$, $\left[S^{(k)}(t) - I\right] x = AS^{(k-1)}(t)x$. Hence,

$$(AS^{(k-1)}(t))^n x = \sum_{i=0}^{n}(-1)^i \binom{n}{i} \left[AS^{(k-1)}((n-i)t) + I\right] x$$
$$= \sum_{i=0}^{n}(-1)^i \binom{n}{i} AS^{(k-1)}((n-i)t)x,$$

because $\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0$. Multiplying the relation above from the left with A^{-1} we obtain

$$A^{n-1}\left(S^{(k-1)}(t)\right)^n x = \sum_{i=0}^{n}(-1)^i \binom{n}{i} S^{(k-1)}((n-i)t)x.$$

Because of $S^{(k-1)}(t)x = AS^{(k-2)}(t)x + tx$, we have:

$$A^{n-1}\left(S^{(k-1)}(t)\right)^n x = \sum_{i=0}^{n}(-1)^i \binom{n}{i} \left[AS^{(k-2)}((n-i)t)x + (n-i)tx\right].$$

Because $\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i) = 0$, multiplication of the relation above from the left with A^{-1} gives

$$A^{n-2}\left(S^{(k-1)}(t)\right)^n x = \sum_{i=0}^{n}(-1)^i \binom{n}{i} S^{(k-2)}((n-i)t)x.$$

If we continue to repeat this procedure $k-2$ times, we obtain

$$A^{n-k}\left(S^{(k-1)}(t)\right)^n x = \sum_{i=0}^{n}(-1)^i \binom{n}{i} S((n-i)t)x = \sum_{i=0}^{n-1}(-1)^i \binom{n}{i} S((n-i)t)x,$$

because $S(0) = 0$. □

Theorem 7. *Let $(S(t))_{t\geq 0}$ be a k- times integrated exponentially bounded semigroup on a Banach space X, with generator A ($k \in \mathbf{N}$). Then, for all $x \in C^k$ and $t \geq 0$ we have:*

$$S(t)x = \lim_{h\to 0+}\Bigg[\sum_{j=1}^{k}\frac{1}{(j-1)!}\left(\frac{t}{h}\right)^{j-1}\sum_{i=0}^{j-1}(-1)^i\binom{j-1}{i}S\left((j-i-1)h\right)x + \sum_{n=k}^{\infty}\frac{1}{n!}\left(\frac{t}{h}\right)^{n}A^{n-k}\left(S^{(k-1)}(h)\right)^{n}x\Bigg],$$

with the limit existing uniformly with respect to t in any finite interval $[0,T]$.

Proof. Note that we have proved the assertion of theorem for once and twice integrated semigroups. In the general case, for k-times integrated semigroup ($k \in N$), using the previous lemma we obtain:

$$\begin{aligned} S(t)x &= \lim_{h\to 0+}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{h}\right)^{n}\sum_{i=0}^{n}(-1)^i\binom{n}{i}S\left((n-i)h\right)x \\ &= \lim_{h\to 0+}e^{-\frac{t}{h}}\cdot\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{t}{h}\right)^{n}S(nh)x. \end{aligned}$$

The rest of proof is the same as in [8], using the last relationship. □

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