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A NOTE ON THE FIXED POINT PROPERTY OF NON-METRIC TREE-LIKE CONTINUA

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Dedicated to Professor Fikret Vajzović on the occasion of his 80th birthday

Abstract. The main purpose of this paper is to study the fixed point property of non-metric tree-like continua. It is proved, using the inverse systems method, that if X is a non-metric tree-like continuum and if $h : X \to X$ is a periodic homeomorphism, then h has the fixed point property (Theorem 2.4). Some theorems concerning the fixed point property of arc-like non-metric continua are also given.

1. Preliminaries

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is denoted by $w(X)$. The cardinality of a set A is denoted by $\text{card}(A)$. We shall use the notion of inverse system as in [3, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$.

Suppose that we have two inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ and $\mathbf{Y} = \{Y_b, q_{bc}, B\}$. A *morphism of the system X into the system* $\mathbf{Y}$ [1, p. 15] is a family $\{\varphi, \{f_b : b \in B\}\}$ consisting of a nondecreasing function $\varphi : B \to A$ such that $\varphi(B)$ is cofinal in A, and of maps $f_b : X_{\varphi(b)} \to Y_b$ defined for all $b \in B$ such that the following

$$\begin{array}{ccc} X_{\varphi(b)} & \xleftarrow{p_{\varphi(b)\varphi(c)}} & X_{\varphi(c)} \\ \downarrow f_b & & \downarrow f_c \\ Y_b & \xleftarrow{q_{bc}} & Y_c \end{array} \tag{1.1}$$

diagram commutes. Any morphism $\{\varphi, \{f_b : b \in B\}\} : \mathbf{X} \to \mathbf{Y}$ induces a map, called the *limit map of the morphism*

$$\lim\{\varphi, \{f_b : b \in B\}\} : \lim \mathbf{X} \to \lim \mathbf{Y}$$

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In the present paper we deal with the inverse systems defined on the same indexing set A. In this case, the map $\varphi : A \to A$ is taken to be the identity and we use the following notation $\{f_a : X_a \to Y_a; a \in A\} : \mathbf{X} \to \mathbf{Y}$.

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *factorizing* [1, p. 17] if for each real-valued mapping $f : \lim \mathbf{X} \to \mathbb{R}$ there exist an $a \in A$ and a mapping $f_a : X_a \to \mathbb{R}$ such that $f = f_a p_a$.

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be σ*-directed* if for each sequence $a_1, a_2, \ldots, a_k, \ldots$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

Lemma 1.1. [1, Corollary 1.3.2, p. 18]. *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a σ-directed inverse system of compact spaces with surjective bonding mappings, then it is factorizing.*

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be τ*-continuous* [1, p. 19] if for each chain B in A with $\text{card}(B) < \tau$ and $\sup B = b$, the diagonal product $\Delta \{p_{ab} : a \in B\}$ maps the space X_b homeomorphically into the space $\lim\{X_a, p_{ab}, B\}$.

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be τ*-system* [1, p. 19] if:

a) $w(X_a) \leq \tau$ for every $a \in A$,
b) The system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is τ-continuous,
c) The indexing set A is τ-complete.

If $\tau = \aleph_0$, then τ-system is called a σ-system. The following theorem is called the *Spectral Theorem* [1, p. 19].

Theorem 1.2. [1, Theorem 1.3.4, p. 19]. *If a τ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ with surjective limit projections is factorizing, then each map of its limit space into the limit space of another τ-system $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ is induced by a morphism of cofinal and τ-closed subsystems. If two factorizing τ-systems with surjective limit projections and the same indexing set have homeomorphic limit spaces, then they contain isomorphic cofinal and τ-closed subsystems.*

Let us remark that the requirement of surjectivity of limit projections of systems in Theorem 1.2 is essential [1, p. 21].

In the sequel we shall use the following result.

Theorem 1.3. [3, Exercise 2.5.D(b), p. 143]. *If for every $s \in S$ an inverse system $\mathbf{X}(s) = \{X_a(s), p_{ab}(s), A\}$ is given, then the family $\Pi\{\mathbf{X}(s) : s \in S\} = \{\Pi\{X_a(s) : s \in S\}, \Pi\{ p_{ab}(s) : s \in S\}, A\}$ is an inverse system and $\lim(\Pi\{\mathbf{X}(s) : s \in S\})$ is homeomorphic to $\Pi\{\lim \mathbf{X}(s) : s \in S\}$.*

A *fixed point of a function* $f : X \to X$ is a point $p \in X$ such that $f(p) = p$. A space X is said to have the *fixed point property* provided that every surjective mapping $f : X \to X$ has a fixed point.

The key theorem is the following.

Theorem 1.4. [8, Theorem 2, p. 17]. *Let* $\mathbf{X} = \{X_a,\, p_{ab},\, A\}$ *be a* σ*-system of compact spaces with the limit* X *and onto projections* $p_a : X \to X_a$. *Let* $\{f_a : X_a \to X_a\} : \mathbf{X} \to \mathbf{X}$ *be a morphism. Then the induced mapping* $f = \lim\{f_a\} : X \to X$ *has a fixed point if and only if each mapping* $f_a : X_a \to X_a$, $a \in A$, *has a fixed point.*

As an immediate consequence of this theorem and the Spectral theorem 1.2 we have the following result.

Theorem 1.5. [8, Theorem 3, p.17]. *Let a non-metric continuum* X *be the inverse limit of an inverse* σ*-system* $\mathbf{X} = \{X_a,\, p_{ab},\, A\}$ *such that each* X_a *has the fixed point property and each bonding mapping* p_{ab} *is onto. Then* X *has the fixed point property.*

In the sequel we will need some expanding theorems of non-metric compact spaces into σ-directed inverse systems of compact metric spaces.

Theorem 1.6. [7, Theorem 1.6, p. 402]. *If* X *is the Cartesian product* $X = \prod\{X_s : s \in S\}$, *where* $\operatorname{card}(S) > \aleph_0$ *and each* X_s *is compact, then there exists a* σ*-directed inverse system* $\mathbf{X} = \{Y_a, P_{ab}, A\}$ *of the countable products* $Y_a = \prod\{X_\mu : \mu \in a\}$, $\operatorname{card}(a) = \aleph_0$, *such that* X *is homeomorphic to* $\lim \mathbf{X}$.

Corollary 1.7. [7, Corollary 1.7, p. 402]. *For each Tychonoff cube* I^m, $m \geq \aleph_1$, *there exists a* σ*-directed inverse system* $\mathbf{I} = \{I^a, P_{ab}, A\}$ *of the Hilbert cubes* I^a *such that* I^m *is homeomorphic to* $\lim \mathbf{I}$.

Theorem 1.8. [7, Theorem 1.8, p. 403]. *Let* X *be a compact Hausdorff space such that* $w(X) \geq \aleph_1$. *There exists a* σ*-directed inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *of metric compacta* X_a *such that* X *is homeomorphic to* $\lim \mathbf{X}$.

At the end of this section we give an application of Theorem 1.5.

Theorem 1.9. *Let* S *be an infinite set and* $Q = \Pi\{X_s : s \in S\}$ *Cartesian product of compact spaces. If each product* $X_{s_1} \times X_{s_2} \times \cdots \times X_{s_n}$ *of finitely many spaces* X_s *has the fixed point property, then* Q *has the fixed point property.*

Proof. We shall consider the following cases.

Case 1. $\operatorname{card}(S) = \aleph_0$. We may assume that $S = \mathbb{N}$. The proof is a straightforward modification of the proof of [10, Corollary 3.5.3, pp. 106-107]. Let $f : Q \to Q$ be continuous. For every $n \in \mathbb{N}$ define

$$K_n = \{x \in Q : (x_1, \ldots, x_n) = (f(x)_1, \ldots, f(x)_n)\}.$$

It is clear that for every n the set K_n is closed in Q and that $K_{n+1} \subset K_n$. For every $n \in \mathbb{N}$, let o_n be a given point of X_n and $p_n : Q \to X_1 \times \cdots \times X_n$ be the projection. Define continuous function $f_n : X_1 \times \cdots \times X_n \to X_1 \times \cdots \times X_n$ by

$$f_n(x_1, \ldots, x_n) = (p_n f)(x_1, \ldots, x_n, o_{n+1}, o_{n+2}, \ldots).$$

By assumption of Theorem f_n has the fixed point property, say $(x_1, \ldots, x_n)$. It follows that

$$(x_1, \ldots, x_n, o_{n+1}, o_{n+2}, \ldots) \in K_n.$$

We conclude that $\{K_n : n \in \mathbb{N}\}$ is a decreasing collection of nonempty closed subsets of Q. By compactness of Q we have that

$$K = \cap\{K_n : n \in \mathbb{N}\}$$

is nonempty. It is clear that every point in K is a fixed point of f.

Case 2. $\operatorname{card}(A) \geq \aleph_1$. By Theorem 1.6 there exists a σ-directed inverse system $\mathbf{X} = \{Y_a, P_{ab}, A\}$ of the countable products $Y_a = \prod\{X_\mu : \mu \in a\}$, $\operatorname{card}(a) = \aleph_0$, such that Q is homeomorphic to $\lim \mathbf{X}$. By Case 1 each Y_a has the fixed point property. Finally, by Theorem 1.5 we infer that Q has the fixed point property. □

2. The fixed point property of the inverse limit space of tree-like continua

A continuum X with precisely two non-separating points is called a *generalized arc.*

A *simple n-od* is the union of n generalized arcs $A_1O, A_2O, \ldots, A_\alpha O$, each two of which have only the point O in common. The point O is called the *vertex* or the *top* of the n-od.

By a *branch point* of a compact space X we mean a point p of X which is the vertex of a simple triod lying in X. A point $x \in X$ is said to be *end point* of X if for each neighborhood U of x there exists a neighborhood V of x such that $V \subset U$ and $\operatorname{card}(Bd(V)) = 1$.

Let S be the set of all end points and of all branch points of a continuum X. An arc pq in X is called a *free arc* in X if $pq \cap S = \{p, q\}$.

A continuum is a *graph* if it is the union of a finite number of metric free arcs. A *tree* is an acyclic graph.

A continuum X is *tree-like* if for each open cover $\mathcal{U}$ of X, there is a tree $X_{\mathcal{U}}$ and a $\mathcal{U}$-mapping $f_{\mathcal{U}} : X \to X_{\mathcal{U}}$ (the inverse image of each point is contained in a member of $\mathcal{U}$).

Every tree-like continuum is hereditarily unicoherent. An expanding theorem of tree-like continua into inverse σ-systems of metric tree-like continua is proven in [8, Theorem 4, p. 19].

Theorem 2.1. *If X is a tree-like non-metric continuum, then there exists a σ-system $\mathbf{X}_\sigma = \{X_\Delta, P_{\Delta\Gamma}, A_\sigma\}$ of metric tree-like continua X_Δ and onto mappings $P_{\Delta\Gamma}$ such that X is homeomorphic to* $\lim \mathbf{X}_\sigma$.

From [9, Theorem 1], [9, Theorem 2] and [9, Corollary 1] we obtain the following well known result [11, Theorem 2.13, p. 24].

Theorem 2.2. *Each metrizable tree-like continuum is homeomorphic with the inverse limit of an inverse sequence of trees.*

Now we shall investigate the fixed point property of non-metric tree-like continua.

Let $h : X \to X$ be a homeomorphism. We will denote by $h^i : X \to X$ the i-th iterations of h and we will suppose that $h^0 = id$. A homeomorphism $h : X \to X$ is called *periodic* if $h^n = id$ for some integer $n > 1$.

In [4, Theorem 1.5] Fugate and McLean proved the following result.

Theorem 2.3. *Tree-like metric continua have the fixed point property for periodic homeomorphisms.*

Now we shall prove that this is true in non-metrizable settings.

Theorem 2.4. *Every non-metrizable tree-like continuum X has the fixed point property for periodic homeomorphisms $h : X \to X$.*

Proof. The proof is broken into several steps.

Step 1. There exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric tree-like continua X_a and surjective bonding mappings p_{ab} such that X is homeomorphic to $\lim \mathbf{X}$. This follows from 2.1.

Step 2. There exists a set B cofinal in A such that for each $b \in B$ there is a homeomorphism $h_b : X_b \to X_b$ such that the diagrams

$$\begin{array}{lcl} X_b & \xleftarrow{p_{bc}} & X_c \\ \downarrow h_b & & \downarrow h_c \\ X_b & \xleftarrow{p_{bc}} & X_c \end{array} \quad \text{and} \quad \begin{array}{lcr} X_b & \xleftarrow{p_b} & \lim \mathbf{X} \\ \downarrow h_b & & \downarrow h \\ X_b & \xleftarrow{p_b} & \lim \mathbf{X} \end{array} \tag{2.1}$$

commute and a collection $\{h_b : b \in B\}$ induces a homeomorphism h.

Apply Theorem 1.2.

Step 3. For each $b \in B$ and each $n \in \mathbb{N}$ the diagrams

$$\begin{array}{lcl} X_b & \xleftarrow{p_{bc}} & X_c \\ \downarrow h_b^n & & \downarrow h_c^n \\ X_b & \xleftarrow{p_{bc}} & X_c \end{array} \quad \text{and} \quad \begin{array}{lcr} X_b & \xleftarrow{p_b} & \lim \mathbf{X} \\ \downarrow h_b^n & & \downarrow h^n \\ X_b & \xleftarrow{p_b} & \lim \mathbf{X} \end{array} \tag{2.2}$$

commute and a collection $\{h_b^n : b \in B\}$ induces a homeomorphism h^n.

By Step 2 we infer that the statement is true for $n = 1$. Suppose that it is true for $n - 1$. Let us prove that it is true for n. We have $h_b^n p_{bc} = h_b(h_b^{n-1} p_{bc}) = h_b(p_{bc} h_c^{n-1}) = (h_b p_{bc}) h_c^{n-1} = (p_{bc} h_c) h_c^{n-1} = p_{bc} h_c^n$. Similarly, $h_b^n p_b = h_b(h_b^{n-1} p_b) = h_b(p_b h^{n-1}) = (h_b p_b) h^{n-1} = (p_b h) h^{n-1} = p_b h^n$. Hence, the diagrams 2.2 commute. This means that $\{h_b^n : b \in B\}$ induces a homeomorphism h^n.

Step 4. If a homeomorphism h is periodic then each homeomorphism h_b is periodic. Let $x_b \in X_b$ be any point and let n be a period of h, i.e., $h^n(x) = x$ for every $x \in X$. There exists a point $x \in X$ such that $p_b(x) = x_b$. From $h_b^n p_b = p_b h^n$ and $h^n(x) = x$ it follows that $h_b^n p_b(x) = p_b h^n(x)$ and $h_b^n(x_b) = p_b(x) = x_b$.

Final Step. Now we are ready to finish the proof. From Steps 1-4 and Theorem 2.3 it follows that inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ from Step 1 satisfies the assumptions of Theorem 1.5. The proof is completed. □

3. The fixed point property of the inverse limit space of arc-like continua

A continuum X is *arc-like* if for each open cover $\mathcal{U}$ of X, there is an arc $X_{\mathcal{U}}$ and a $\mathcal{U}$-mapping $f_{\mathcal{U}} : X \to X_{\mathcal{U}}$ (the inverse image of each point is contained in a member of $\mathcal{U}$). Every arc-like continuum is tree-like since an arc is a tree. Similarly as Theorem 2.2 we have the following theorem.

Theorem 3.1. *If X is a arc-like non-metric continuum, then there exists a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric arc-like continua X_a and onto mappings p_{ab} such that X is homeomorphic to* $\lim \mathbf{X}$.

Now we investigate the fixed point property of non-metrizable arc-like continua. In [2, Theorem 1, p. 663] showed the following result.

Theorem 3.2. *Suppose that M is the Cartesian product of n compact arc-like metric continua $X_1, X_2, \ldots, X_n$ and f is a continuous mapping of M into itself. Then there is a point $x \in M$ such that $x = f(x)$.*

For $n = 2$ we have the following result.

Theorem 3.3. [6, p. 199, Exercise 22.26]. *If X and Y are metric arc-like continua, then $X \times Y$ has the fixed point property.*

If $n = 1$, then we have the following result.

Theorem 3.4. [11, p. 253, Corollary 12.30],[5]. *If X is an arc-like metric continuum, then X has the fixed point property.*

Theorems 1.5 and 3.4 imply the following theorem.

Theorem 3.5. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse σ-system of metric arc-like continua X_a and surjective bonding mappings p_{ab}, then $X = \lim \mathbf{X}$ has the fixed point property.*

Now we shall prove that each arc-like continuum has the fixed point property.

Theorem 3.6. *Every arc-like continuum X has the fixed point property.*

Proof. If X is metrizable, then apply Theorem 3.4. If X is non-metrizable, then there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric arc-like continuum and each p_{ab} is a weakly confluent surjection. This means that the system $\mathbf{X} = \{X_a, p_{ab}, A\}$ satisfies the assumption of Theorem 1.5. We infer that $\lim \mathbf{X}$ has the fixed point property. Hence X has the fixed point property since X is homeomorphic to $\lim \mathbf{X}$. □

Using Theorem 1.9 Dyer [2, Corollary, p.665] showed the following general result.

Theorem 3.7. *The Cartesian product of the elements of any collection of arc-like metric continua has the fixed point property.*

We will show that Theorem 3.7 is true for non-metrizable arc-like continua.

Theorem 3.8. *The Cartesian product of the elements of any collection of arc-like continua of the same weight has the fixed point property.*

Proof. If for every $s \in S$ we have an arc-like non-metrizable continuum $X(s)$, then, for every $s \in S$, there exists an inverse system $\mathbf{X}(s) = \{X_a(s), p_{ab}(s), A(s)\}$ such that $X(s)$ is homeomorphic to $\lim \mathbf{X}(s)$ and every $X_a(s)$ is a metric arc-like continuum (Theorem 3.1). If $\mathrm{w}(X(s_1)) = \mathrm{w}(X(s_2)), s_1, s_2 \in S$, then $A(s_1) = A(s_2)$ and we may suppose that $A(s) = A$ for every $s \in S$. By Theorem 1.3 the family $\Pi\{\mathbf{X}(s) : s \in S\} = \{\Pi\{X_a(s) : s \in S\}, \Pi\{p_{ab}(s) : s \in S\}, A\}$ is an inverse system and $\lim(\Pi\{\mathbf{X}(s) : s \in S\})$ is homeomorphic to $\Pi\{\lim \mathbf{X}(s) : s \in S\}$. From Theorem 3.7 it follows that each $\Pi\{X_a(s) : s \in S\}$ has the fixed point property. Finally, from Theorem 1.5 it follows that $\Pi\{X(s) : s \in S\}$ has the fixed point property. □

Question. Is it true that the assumption "*of the same weight*" in Theorem 3.8 can be omitted?

As an immediate application we give the following generalization of Brouwer Fixed-Point Theorem. Let L be a non-metric arc. The space X is said to be a *generalized n-cell* if it is homeomorphic to $L^n = L \times L \times \cdots \times L$ (n factors).

Theorem 3.9. *Every mapping $f : L^n \to L^n$ has a fixed point, i.e., L^n has the fixed point property.*

Now we can to prove the following result.

Theorem 3.10. *If $L_1, \ldots, L_n$ are arcs (metric or non-metric), then $L_1 \times L_2 \times \cdots \times L_n$ has the fixed point property.*

Proof. **Step 1**. If M is a subarc of the arc L, then there exists a retraction $r : L \to M$. Let a, b, c, d be end points of L and M such that $a \leq c < d \leq b$. We define $r : L \to M$ as follows:

$$r(x) = \begin{cases} c & \text{if} \quad a \leq x \leq c, \\ x & \text{if} \quad c \leq x \leq d, \\ d & \text{if} \quad d \leq x \leq b. \end{cases}$$

Step 2. If $L_1, L_2, \ldots, L_n$ is a finite collection of arcs, then there is an arc L such that $L_1, L_2, \ldots, L_n$ are subarcs of L. For each $i \in \{1, 2, \ldots, n\}$ let a_i, b_i be a pair of end points of L_i such that $a_i < b_i$. If we identify the pair of points $\{b_1, a_2\}, \{b_2, a_3\}, \ldots, \{b_{n-1}, a_n\}$ we obtain an arc L such that $L_i \subset L$ for each $i \in \{1, 2, \ldots, n\}$.

Step 3. $L_1 \times L_2 \times \cdots \times L_n$ is a retract of L^n. Let L and $L_1, L_2, \ldots, L_n$ be as in Step 2. Let $r_i : L \to L_i, i \in \{1, 2, \ldots, n\}$ be a retraction defined in Step 1. Let us prove that $r = r_1 \times r_2 \times \cdots \times r_n$ is a retraction of L^n onto $L_1, L_2, \ldots, L_n$. If $(y_1, y_2, \ldots, y_n) \in L^n$, then we have: $r_1 \times r_2 \times \cdots \times r_n(y_1, y_2, \ldots, y_n) = (r_1(y_1), r_2(y_2), \ldots, r_n(y_n)) \in L_1 \times L_2 \times \cdots \times L_n$ since $r_i(y_i) \in L_i$. If $(x_1, x_2, \ldots, x_n) \in L_1 \times L_2 \times \cdots \times L_n$, then $r_1 \times r_2 \times \cdots \times r_n(x_1, x_2, \ldots, x_n) = (r_1(x_1), r_2(x_2), \ldots, r_n(x_n)) = (x_1, x_2, \ldots, x_n) \in L_1 \times L_2 \times \cdots \times L_n$ since $r_i(x_i) \in x_i$.

Step 4. The product $L_1 \times L_2 \times \cdots \times L_n$ has the fixed point property since it is retract of the product L^n which has the fixed point property (Theorem 3.9). The proof is completed. □

Theorem 3.11. *If $L = \Pi\{L_s : s \in S\}$ is a Cartesian product of arcs L_s, then L has the fixed point property.*

Proof. Apply Theorems 3.10 and 1.9. □

For Cartesian product of two arc-like continua the assumption concerning the weight in Theorem 3.8 can be omitted.

Theorem 3.12. *If X and Y are non-metrizable arc-like continua, then $X \times Y$ has the fixed point property.*

Proof. First we shall prove that if X is any arc-like continuum and if Y is a metric arc-like continuum, then $X \times Y$ has the fixed point property. By Theorem 3.1 there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric arc-like continuum and X is homeomorphic to $\lim \mathbf{X}$. It is clear that $\mathbf{X} \times Y = \{X_a \times Y, p_{ab} \times id, A\}$ is a σ-directed inverse system whose limit is homeomorphic to $X \times Y$. Every $X_a \times Y$ has the fixed point property since it is the product of metric arc-like continua (Theorem 3.3). Applying Theorem 1.5 we infer that $X \times Y$ has the fixed point property.

Suppose now that X and Y are non-metric arc-like continua. Using again Theorem 3.1 we obtain a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric arc-like continuum and X is homeomorphic to $\lim \mathbf{X}$. It is clear that $\mathbf{X}{\times}Y = \{X_a \times Y, p_{ab} \times id, A\}$ is a σ-representation of $X \times Y$. From the first part of this proof it follows that every $X_a \times Y$ has the fixed point property since it is the product of metric arc-like continuum X_a and an arc-like continuum Y. Applying Theorem 1.5 we infer that $X \times Y$ has the fixed point property. □

We close this section with the fixed point property for multifunctions on arc-like continua.

A multifunction, $F : X \to Y$, from a space X to a space Y is a point-to-set correspondence such that, for each $x \in X$, $F(x)$ is a subset of Y. For any $y \in Y$, we write $F^{-1}(y) = \{x \in X : y \in F(x)\}$. If $A \subset X$ and $B \subset Y$, then $F(A) = \cup\{F(x) : x \in A\}$ and $F^{-1}(B) = \cup\{F^{-1}(y) : y \in B\}$.

A multifunction $F : X \to Y$ is said to be continuous if and only if (i) $F(x)$ is closed for each $x \in X$, (ii) $F^{-1}(B)$ is closed for each closed set B in Y, (iii) $F^{-1}(V)$ is open for each open set V in Y.

A topological space X is said to have F.p.p (fixed point property for multi-valued functions) if for every multi-valued continuous function $F : X \to X$ there exists a point $x \in X$ such that $x \in F(x)$. It follows that X has F.p.p if for every single-valued continuous function $F : X \to 2^X$ there exists a point $x \in X$ such that $x \in F(x)$.

Theorem 3.13. [12]. *If X is any metric arc-like continuum, then X has the F.p.p.*

Now we shall prove the following result.

Theorem 3.14. *Each arc-like continuum X has the F.p.p.*

Proof. If an arc-like continuum is metrizable, then it has F.p.p (Theorem 3.13). Suppose that arc-like continuum X is non-metrizable. By virtue of Theorem 3.1 there exists a σ-system $\mathbf{X}_\sigma = \{X_\Delta, P_{\Delta\Gamma}, A_\sigma\}$ of metric arc-like continua X_Δ and onto mappings $P_{\Delta\Gamma}$ such that X is homeomorphic to $\lim \mathbf{X}_\sigma$. Moreover, we have the inverse system $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$ whose limit is 2^X. Let $F : X \to 2^X$ be a continuous mapping. From Theorem 1.2 it follows that there exists a subset B cofinal in A such that for every $b \in B$ there exists a continuous mapping $F_b : X_b \to 2^{X_b}$ with the property that $\{F_b : b \in B\}$ is a morphism which induce F. Theorem 3.13 implies that the set $Y_b \subset X_b, b \in B$, of fixed points of F_b is non-empty. Let us prove that Y_b is a closed subset of X_b. We shall prove that $U_b = X_b \diagdown Y_b$ is open. Let $x_b \in U_b$. This means that x_b and $F_b(x_b)$ are disjoint closed subsets of X_b. By the normality of X_b there exists a pair of open sets U, V such that $x \in U$ and $Y_b \subset V$. From the upper semi-continuity of F_b it follows that there exists an open set $W \subset U$ such that for every $x \in W$ we have $f(x) \subset V$. Hence,

U_b is open and, consequently, Y_b is closed. Now, we shall prove that the collection $\{Y_b, p_{bc}|Y_c, B\}$ is an inverse system. To do this we have to prove that if $c > b$, then $p_{bc}(Y_c) \subset Y_b$. Let x_c be a point of Y_c. This means that $x_c \in f_c(x_c)$. Hence, $p_{bc}(x_c) \in p_{bc}(F_c(x_c)) = F_b p_{bc}(x_c)$. We conclude that the point $x_b = p_{bc}(x_c)$ has the property $x_b \in f_b(x_b)$, i.e., $x_b = p_{bc}(x_c) \in Y_b$. Finally, $p_{bc}(Y_c) \subset Y_b$. and $\{Y_b, p_{bc}|Y_c, B\}$ is an inverse system with non-empty limit. Let $Y = \lim \{Y_b, p_{bc}|Y_c, B\}$. In order to complete the proof we shall prove that for every $x \in Y$ we have $x \in F(x)$. Now we have $p_b(x) \in Y_b$, i.e., $p_b(x) \in F_b(p_b(x)) = p_b F(x)$, for every $b \in B$. It follows that $x \in F(x)$ since $x \notin F(x)$ implies that there is a $b \in B$ such that $p_b(x) \notin p_b F(x)$. We conclude that F has the fixed point property. □

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References

[1] A. Chigogidze, *Inverse spectra*, Elsevier, 1996.
[2] E. Dyer, *A fixed point theorem*, Proc. Am. Math. Soc., 7 (1956), 662–672.
[3] R. Engelking, *General Topology*, PWN, Warszawa, 1977.
[4] J. B. Fugate and T. B. McLean, *Compact groups of homeomorphisms on tree-like continua*, Trans. Am. Math. Soc., 267 (1981), 609–620.
[5] O. H. Hamilton, *A fixed point property for pseudo-arcs and certain other metric continua*, Proc. Am. Math. Soc., 2 (1951), 173–174.
[6] A. Illanes and S. B. Nadler, Jr., *Hyperspaces : Fundamentals and Recent Advances*, Marcel Dekker, Inc., New York and Basel, 1999.
[7] I. Lončar, *A fan X admits a Whitney map for C(X) iff it is metrizable*, Glas. Mat. Ser. III, 38 (58) (2003), 395–411.
[8] I. Lončar, *The fixed point property for arc component preserving mappings of non-metric tree-like continua*, Math. Commun., 10 (2005), 15–21.
[9] S. Mardešić, *Chainable continua and inverse limits*, Glas. Mat. Fiz. i Astr., 14 (1959), 219–232.
[10] J. van Mill, *Infinite-Dimensional Topology*, Elsevier Sci. Pub., 1989.
[11] S. B. Nadler, Jr., *Continuum Theory*, Marcel Dekker, New York, 1992.
[12] L. E. Ward, *A fixed point theorem*, Amer. Math. Montly, 65 (1958), 271–272.

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