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THE DIAMETER OF A ZERO-DIVISOR GRAPH FOR FINITE DIRECT PRODUCT OF COMMUTATIVE RINGS

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ABSTRACT. This paper establishes a set of theorems that describe the diameter of a zero-divisor graph for a finite direct product $R_1 \times R_2 \times \cdots \times R_n$ with respect to the diameters of the zero-divisor graphs of $R_1, R_2, \cdots, R_{n-1}$ and $R_n (n > 2)$.

1. INTRODUCTION

All rings in this paper are commutative and not necessary with 1. The concept of zero divisor graph of a commutative ring R was introduced by Beck in [2]. He let all elements of the ring be vertices of the graph and was interested mainly in coloring. In [1], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the non-zero zero-divisors. Among other things, they proved that $\Gamma(R)$ is always connected and its diameter is always less than or equal to 3 [1, Theorem 2.3]. The zero-divisor graph helps us to study the algebraic properties of rings using graph theoretical tools (see, for example, [1], [3], [4]). In [5], J. Warfel describes the diameter of a zero-divisor graph for a direct product $R_1 \times R_2$ with respect to the diameters of the zero-divisor graphs of R_1 and R_2. The main goal in this paper is to generalize some of the results in the paper listed as [5], from $R_1 \times R_2$ to $R_1 \times R_2 \times \cdots \times R_n (n > 2)$ (see section 2).

For the sake of completeness, we state some definitions and notations used throughout. Let R be a commutative ring. We used $Z(R)$ to denote the set of zero-divisors of R; we use $Z^*(R)$ to denote the set of non-zero zero-divisors of R. By the zero-divisor graph of R, denoted $\Gamma(R)$, we mean the graph whose vertices are the non-zero zero-divisors of R, and for distinct $x, y \in Z^*(R)$, there is an edge connecting x and y if and only if $xy = 0$. A graph is said to be connected if there exists a path between any two distinct vertices. For two distinct vertices a and b in the graph $\Gamma(R)$, the

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distance between a and b, denoted $d(a,b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, $d(a,b)=\infty$. The diameter of a connected graph is the supremum of the distances between vertices. We will use the notation $\mathrm{diam}(\Gamma(R))$ to denote the diameter of the graph of $Z^*(R)$. The diameter is zero if the graph consists of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is complete; i.e. each pair of distinct vertices forms an edge. We tacitly assume that R has at least 2 non-zero zero-divisors. Also, though it be an abuse of notation, let $0=(0,0,\cdots,0)$.

2. Finite direct product

In this section, we will investigate the relation between the diameter of a zero-divisor graph of a finite direct product $R_1 \times R_2 \times \cdots \times R_n$ with the diameters of the zero-divisor graphs of $R_1, R_2, \ldots, R_{n-1}$ and R_n. Our starting point is the following lemma:

Lemma 2.1. *Let R be commutative ring with $\mathrm{diam}(\Gamma(R))=1$ and $R=Z(R)$. Then $xy=0$ for all $x,y \in Z(R)$. In particular, $x^2=0$ for every nilpotent element of R.*

Proof. Suppose not. Then there are elements $a,b \in Z(R)$ such that $ab \neq 0$, so by [1, Theorem 2.8], $R \cong Z_2 \times Z_2$; hence $R \neq Z(R)$ which is a contradiction, as required. □

Theorem 2.2. *Let $R_1, R_2, \ldots, R_{n-1}$ and R_n be commutative rings such that $\mathrm{diam}(\Gamma(R_1)) = \cdots = \mathrm{diam}(\Gamma(R_n)) = 1$, and let $R = R_1 \times R_2 \times \cdots \times R_n$ $(n>2)$. Then the following hold:*

(i) $\mathrm{diam}(\Gamma(R))=1$ *if and only if* $R_i = Z(R_i)$ *for every* $i \in \{1,\ldots,n\}$.
(ii) $\mathrm{diam}(\Gamma(R))=2$ *if and only if* $R_i = Z(R_i)$ *and* $R_j \neq Z(R_j)$ *for some* $i,j \in \{1,2,\ldots,n\}$.
(iii) $\mathrm{diam}(\Gamma(R))=3$ *if and only if* $R_i \neq Z(R_i)$ *for every* $i \in \{1,2,\ldots,n\}$.

Proof. (i) Assume that $R_i = Z(R_i)$ for every $i=1,2,\ldots,n$ and let $a=(a_1,\ldots,a_n)$, $b=(b_1,\ldots,b_n)$ be elements of $Z^*(R)$. By Lemma 2.1, $a_ib_i=0$ for all i, so $ab=0$; hence $\mathrm{diam}(\Gamma(R))=1$. Conversely, assume that $R_j \neq Z(R_j)$ for some $j \in \{1,2,\ldots,n\}$. Then, for some $x_j, y_j \in R_j$, $x_jy_j \neq 0$. Set $x=(0,\ldots,x_j,0,\ldots,0)$, $y=(0,\ldots,y_j,0,\ldots,0)$, and let $0 \neq a_i \in R_i$ where $i \neq j$. Since $x(0,\ldots,a_i,0,\ldots,0)=0$, $y(0,\ldots,a_i,0,\ldots,0)=0$ and $xy \neq 0$, we must have $\mathrm{diam}(\Gamma(R))>1$ which is a contradiction.

(ii) If $R_i = Z(R_i)$ and $R_j \neq Z(R_j)$ for some $i,j \in \{1,2,\ldots,n\}$, then by (i), the fact that $R_j \neq Z(R_j)$ implies that $\mathrm{diam}(\Gamma(R))>1$. Then there exist $r=(r_1,\ldots,r_n) \in Z^*(R))$ and $s=(s_1,\ldots,s_n) \in Z^*(R))$ such that $d(r,s) \neq 1$, so $rs \neq 0$. Since $R_i = Z(R_i)$, there must exist $t_i \in R_i$ such that

$t_iu_i = 0$ for all $u_i \in R_i$ by Lemma 2.1. Set $t = (0, 0, \ldots, t_i, 0, \ldots, 0)$. Then $r - t - s$ is a path. Therefore, a path of length two can be found between any two vertices of $\Gamma(R)$ by way of t. Thus $\operatorname{diam}(\Gamma(R)) = 2$. Conversely, assume that $\operatorname{diam}(\Gamma(R)) = 2$. If $R_i = Z(R_i)$ for every $i = 1, 2, \ldots, n$, then by (i), $\operatorname{diam}(\Gamma(R)) = 1$ which is a contradiction. So, let for each i, $R_i \neq Z(R_i)$. Then there must exist $x_i \in R_i - Z(R_i)$ for every $i \in \{1, 2, \ldots, n\}$. Let for each i, $z_i \in Z^*(R_i)$. So there is an element z_i' of $Z^*(R_i)$ such that $z_iz_i' = 0$ for all i. If $a = (z_1, x_2, \ldots, x_n)$ and $b = (x_1, z_2, x_3 \ldots, x_n)$, then $a(z_1', 0, \ldots, 0) = 0$ and $b(0, z_2', 0, \ldots, 0) = 0$, so $a, b \in Z^*(R)$. As $ab \neq 0$, the distance between the vertices is greater than one. Since $\operatorname{diam}(\Gamma(R)) = 2$, there must be some $c = (c_1, \ldots, c_n) \in Z^*(R)$ such that $ac = bc = 0$. Then $c = 0$, which is not an element of $Z^*(R)$. But this is a contradiction. Thus $R_i = Z(R_i)$ and $R_j \neq Z(R_j)$ for some $i, j \in \{1, 2, \ldots, n\}$.

(iii) This follows from (i) and (ii). □

We will need the following lemma from [5, Lemma 3.1].

Lemma 2.3. *Let R be a commutative ring such that* $\operatorname{diam}(\Gamma(R)) = 2$ *and* $R = Z(R)$. *Then for all $x, y \in R$, there exists an element z of $Z^*(R)$ such that $xz = yz = 0$.*

Theorem 2.4. *Let $R_1, R_2, \ldots, R_{n-1}$ and R_n be commutative rings such that* $\operatorname{diam}(\Gamma(R_1)) = \cdots = \operatorname{diam}(\Gamma(R_n)) = 2$, *and let $R = R_1 \times R_2 \times \cdots \times R_n$ $(n > 2)$. Then the following hold:*

(i) $\operatorname{diam}(\Gamma(R)) \neq 1$.
(ii) $\operatorname{diam}(\Gamma(R)) = 2$ *if and only if* $R_i = Z(R_i)$ *for some* $i \in \{1, 2, \ldots, n\}$.
(iii) $\operatorname{diam}(\Gamma(R)) = 3$ *if and only if* $R_i \neq Z(R_i)$ *for every* $i \in \{1, 2, \ldots, n\}$

Proof. (i) Is clear.

(ii) Let $R_i = Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$. By (i), there are elements $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of $Z^*(R)$ such that $x \neq y$ and $xy \neq 0$. Since $x_i, y_i \in R_i$, Lemma 2.3 gives $x_iz_i = 0 = y_iz_i$ for some non-zero element z_i of $Z(R_i)$. Let $z = (0, \ldots, z_i, 0, \ldots, 0)$. Since $xz = 0 = yz$, we must have $x - z - y$ is a path; hence a path of length two can be found between any two vertices of $\Gamma(R)$ by way of z. So, $\operatorname{diam}(\Gamma(R)) = 2$. Conversely, assume that $\operatorname{diam}(\Gamma(R)) = 2$ and let $R_i \neq Z(R_i)$ for each $i \in \{1, 2, \ldots, n\}$. Let for each i, $e_i \in Z^*(R_i)$ and $m_i \in R_i - Z(R_i)$. So there is an element e_i' of $Z^*(R_i)$ such that $e_ie_i' = 0$ for all i. If $a = (e_1, m_2, \ldots, m_n)$ and $b = (m_1, e_2, m_3, \ldots, m_n)$, then $a(e_1', 0, \ldots, 0) = 0$ and $b(0, e_2', 0, \ldots, 0) = 0$, so $a, b \in Z^*(R)$. As $ab \neq 0$, the distance between the vertices is greater than one. Since $\operatorname{diam}(\Gamma(R)) = 2$, there must be some $c = (c_1, \ldots, c_n) \in Z^*(R)$ such that $ac = 0 = bc$. Then $c = 0$, which is a contradiction. Thus $R_i \neq Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$.

(iii) This follows from (i) and (ii). □

Theorem 2.5. *Let $R_1, R_2, \ldots, R_{n-1}$ and R_n be commutative rings such that* $\mathrm{diam}(\Gamma(R_1)) = \cdots = \mathrm{diam}(\Gamma(R_n)) = 3$*, and let $R = R_1 \times R_2 \times \cdots \times R_n$ $(n > 2)$. Then* $\mathrm{diam}(\Gamma(R)) = 3$.

Proof. Since for each $i \in \{1, 2, \ldots, n\}$, $\mathrm{diam}(\Gamma(R_i)) = 3$, there exist $x_i, y_i \in Z^*(R_i)$ with $x_i \neq y_i, x_i y_i \neq 0$ such that there is no $z_i \in Z^*(R_i)$ with $x_i z_i = 0 = y_i z_i$. Consider $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. For each $i \in \{1, 2, \ldots, n\}$, there are elements $x'_i, y'_i \in Z^*(R_i)$ such that $x_i x'_i = 0$ and $y_i y'_i = 0$, so $x, y \in Z^*(R)$. As $xy \neq 0$, we must have $\mathrm{diam}(\Gamma(R)) \neq 1$. If $\mathrm{diam}(\Gamma(R)) = 2$, then $\mathrm{d}(x, y) \neq 1$ implies there is an element $a = (a_1, \ldots, a_n) \in Z^*(R)$ with $xa = 0 = ya$; hence $a = 0$ by our assumption which is a contradiction, so $\mathrm{diam}(\Gamma(R)) = 3$ must hold. □

Theorem 2.6. *Let $R_1, R_2, \ldots, R_{n-1}$ and R_n be commutative rings such that* $\mathrm{diam}(\Gamma(R_i)) = 1$, $\mathrm{diam}(\Gamma(R_j)) = 2$ *for some $i, j \in \{1, 2, \ldots, n\}$ and there is no $k \in \{1, 2, \ldots, n\}$ with* $\mathrm{diam}(\Gamma(R_k)) = 3$*, and let $R = R_1 \times R_2 \times \cdots \times R_n$ $(n > 2)$. Then the following hold:*

(i) $\mathrm{diam}(\Gamma(R)) \neq 1$.
(ii) $\mathrm{diam}(\Gamma(R)) = 2$ *if and only if $R_i = Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$.*
(iii) $\mathrm{diam}(\Gamma(R)) = 3$ *if and only if $R_i \neq Z(R_i)$ for every $i \in \{1, 2, \ldots, n\}$.*

Proof. (i) Is clear.

(ii) First, assume that $R_i = Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$; we show that $\mathrm{diam}(\Gamma(R)) = 2$. By hypothesis, we divided the proof into two cases.
Case 1. $\mathrm{diam}(\Gamma(R_i)) = 1$. It then follows from Lemma 2.2 that $xy = 0$ for all $x, y \in Z(R_i)$. By (i), there must exist $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in Z^*(R)$ with $xy \neq 0$. If $z_i \in Z^*(R_i)$, then $x(0, \ldots, z_i, \ldots, 0) = 0$, so $z = (0, \ldots, z_i, \ldots, 0)$ is an element of $Z^*(R)$. Clearly, $x - z - y$ is a path. Hence, a path of length two can be found between any two vertices of $\Gamma(R)$ by way of z. So, $\mathrm{diam}(\Gamma(R)) = 2$.
Case 2. $\mathrm{diam}(\Gamma(R_i)) = 2$. By (i), there must exist $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in Z^*(R)$ with $xy \neq 0$. By Lemma 2.3, there is an element z_i of $Z^*(R_i)$ such that $x_i z_i = y_i z_i = 0$. Set $z = (0, \ldots, z_i, 0, \ldots, 0)$. Then $x - z - y$ is a path, and hence a path of length two can be found between any two vertices of $\Gamma(R)$ by way of z. So, $\mathrm{diam}(\Gamma(R)) = 2$.

Next assume that $\mathrm{diam}(\Gamma(R)) = 2$; we show that $R_i = Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$. Suppose that for each $i \in \{1, 2, \ldots, n\}$, $R_i \neq Z(R_i)$. Let for each i, $x_i \in Z^*(R_i)$ and $m_i \in R_i - Z(R_i)$. So there is an element x'_i of $Z^*(R_i)$ such that $x_i x'_i = 0$ for all i. If $a = (x_1, m_2, \ldots, m_n)$ and $b = (m_1, x_2, m_3, \ldots, m_n)$, then $a(x'_1, 0, \ldots, 0) = 0$ and $b(0, x'_2, 0, \ldots, 0) = 0$, so $a, b \in Z^*(R)$. As $ab \neq 0$, the distance between the vertices is greater than one. Since $\mathrm{diam}(\Gamma(R)) = 2$, there must be some $c = (c_1, \ldots, c_n) \in Z^*(R)$

such that $ac = 0 = bc$. Then $c = 0$, which is a contradiction. Thus $R_i \neq Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$.

(iii) This follows from (i) and (ii). □

Theorem 2.7. *Let $R_1, R_2, \ldots, R_{n-1}$ and R_n be commutative rings such that* $\mathrm{diam}(\Gamma(R_i)) = 1$, $\mathrm{diam}(\Gamma(R_j)) = 3$ *for some $i, j \in \{1, 2, \ldots, n\}$ and there is no $k \in \{1, 2, \ldots, n\}$ with* $\mathrm{diam}(\Gamma(R_k)) = 2$, *and let $R = R_1 \times R_2 \cdots \times R_n$ ($n > 2$). Then the following hold:*

(i) $\mathrm{diam}(\Gamma(R)) \neq 1$.

(ii) $\mathrm{diam}(\Gamma(R)) = 2$ *if and only if* $\mathrm{diam}(\Gamma(R_i)) = 1$ *and $R_i = Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$.*

(iii) $\mathrm{diam}(\Gamma(R)) = 3$ *if and only if there is no $i \in \{1, 2, \ldots, n\}$ with* $\mathrm{diam}(\Gamma(R_i)) = 1$ *and $R_i = Z(R_i)$.*

Proof. (i) Is clear.

(ii) Let i be such that $\mathrm{diam}(\Gamma(R_i)) = 1$ and $R_i = Z(R_i)$; we show that $\mathrm{diam}(\Gamma(R)) = 2$. It follows from [1, Theorem 2.8] that $a_i b_i = 0$ for every $a_i, b_i \in Z(R_i)$. By (i), there must exist $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in Z^*(R)$ such that $xy \neq 0$. Assume that $a_i \in Z^*(R_i)$ and set $a = (0, 0, \ldots, a_i, 0, , 0)$. Then $ax = 0 = ay$, so $a \in Z^*(R)$. Therefore, $x - a - y$ is a path, and hence a path of length two can be found between any two vertices of $\Gamma(R)$ by way of a. So $\mathrm{diam}(\Gamma(R)) = 2$. Conversely, assume that $\mathrm{diam}(\Gamma(R)) = 2$; we show that $\mathrm{diam}(\Gamma(R_i)) = 1$ and $R_i = Z(R_i)$ for some i. Suppose not. Let $i_1, \ldots, i_k$ be such that $\mathrm{diam}(\Gamma(R_{i_r})) = 1$ $(1 \leq r \leq k)$, and let $j_1, \ldots, j_t$ be such that $\mathrm{diam}(\Gamma(R_{j_s})) = 3$ $(1 \leq s \leq t)$. Since for each s $(1 \leq s \leq t)$, $\mathrm{diam}(\Gamma(R_{j_s})) = 3$, there exist $x_{j_s}, y_{j_s} \in Z^*(R_{j_s})$ with $x_{j_s} \neq y_{j_s}, x_{j_s} y_{j_s} = 0$ such that there is no $z_{j_s} \in Z^*(R_{j_s})$ with $x_{j_s} z_{j_s} = y_{j_s} z_{j_s} = 0$. Moreover, for each s $(1 \leq s \leq t)$, there must exist $x'_{j_s}, y'_{j_s} \in Z^*(R_{j_s})$ with $x_{j_s} x'_{j_s} = 0$ and $y_{j_s} y'_{j_s} = 0$. Now for each r $(1 \leq r \leq k)$, let $m_{i_r} \in R_{i_r} - Z(R_{i_r})$. Set $c = (m_{i_1}, \ldots, x_{j_1}, \ldots, x_{j_t}, \ldots)$ and $d = (m_{i_1}, \ldots, y_{j_1}, \ldots, y_{j_t}, \ldots)$. Then $c(0, \ldots, x'_{j_1}, 0, \ldots, 0) = 0$, so $c \in Z^*(R)$. Similarly , $d \in Z^*(R)$. As $cd \neq 0$ and $\mathrm{diam}(\Gamma(R)) = 2$, there must be some $e = (e_1, \ldots, e_n) \in Z^*(R)$ such that $ce = de = 0$. Then $e = 0$, which is a contradiction. Thus $\mathrm{diam}(\Gamma(R_i)) = 1$ and $R_i = Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$.

(iii) Since $\Gamma(R)$ is connected and $\mathrm{diam}(\Gamma(R)) \leq 3$, we must have the diameter of $\Gamma(R)$ is either 2 or 3 by (i). If $\mathrm{diam}(\Gamma(R)) = 2$, then by (ii), $\mathrm{diam}(\Gamma(R_i)) = 1$ and $R_i = Z(R_i)$ for some $i \in \{1, 2, , n\}$ which is a contradiction. Thus $\mathrm{diam}(\Gamma(R)) = 3$. The proof of the other implication is clear. □

Theorem 2.8. *Let $R_1, R_2, \ldots, R_{n-1}$ and R_n be commutative rings such that* $\mathrm{diam}(\Gamma(R_i)) = 2$, $\mathrm{diam}(\Gamma(R_j)) = 3$ *for some $i, j \in \{1, 2, \ldots, n\}$ and there is* *no* $k \in \{1, 2, \ldots, n\}$ *with* $\operatorname{diam}(\Gamma(R_k)) = 1$*, and let* $R = R_1 \times R_2 \cdots \times R_n$ *(*$n > 2$*). Then the following hold:*

(i) $\operatorname{diam}(\Gamma(R)) \neq 1$.

(ii) $\operatorname{diam}(\Gamma(R)) = 2$ *if and only if* $\operatorname{diam}(\Gamma(R_i)) = 2$ *and* $R_i = Z(R_i)$ *for some* $i \in \{1, 2, \ldots, n\}$.

(iii) $\operatorname{diam}(\Gamma(R)) = 3$ *if and only if there is no* $i \in \{1, 2, \ldots, n\}$ *with* $\operatorname{diam}(\Gamma(R_i)) = 2$ *and* $R_i = Z(R_i)$.

Proof. (i) Is clear.

(ii) Let i be such that $\operatorname{diam}(\Gamma(R_i)) = 2$ and $R_i = Z(R_i)$; we show that $\operatorname{diam}(\Gamma(R)) = 2$. Then by Lemma 2.3, $a_i b_i = 0$ for every $a_i, b_i \in Z(R_i) = R_i$. By (i), there must exist $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in Z^*(R)$ such that $xy \neq 0$. Assume that $a_i \in Z^*(R_i)$ and set $a = (0, 0, \ldots, a_i, 0, \ldots, 0)$. Then $ax = 0 = ay$, so $a \in Z^*(R)$. Therefore $x - a - y$ is a path, and hence a path of length two can be found between any two vertices of $\Gamma(R)$ by way of a. So, $\operatorname{diam}(\Gamma(R)) = 2$. Conversely, assume that $\operatorname{diam}(\Gamma(R)) = 2$; we show that $\operatorname{diam}(\Gamma(R_i)) = 2$ and $R_i = Z(R_i)$ for some i. Suppose that for each i ($1 \leq i \leq n$), if $\operatorname{diam}(\Gamma(R_i)) = 2$, then $R_i \neq Z(R_i)$. Let $i_1, \ldots, i_k$ be such that $\operatorname{diam}(\Gamma(R_{i_r})) = 2$ ($1 \leq i \leq k$), and let $j_1, \ldots, j_t$ be such that $\operatorname{diam}(\Gamma(R_{j_s})) = 3$ ($1 \leq s \leq t$). By assumption, for each r ($1 \leq r \leq k$), $R_{i_r} \neq Z(R_{i_r})$. For each r ($1 \leq r \leq k$), let $m_{i_r} \in R_{i_r} - Z(R_{i_r})$. Since for each s ($1 \leq s \leq t$), $\operatorname{diam}(\Gamma(R_{j_s})) = 2$, there exist $x_{j_s}, y_{j_s} \in Z^*(R_{j_s})$ with $x_{j_s} \neq y_{j_s}, x_{j_s} y_{j_s} = 0$ such that there is no $z_{j_s} \in Z^*(R_{j_s})$ with $x_{j_s} z_{j_s} = 0 = y_{j_s} z_{j_s}$. Moreover, for each s ($1 \leq s \leq t$), there must exist $x'_{j_s}, y'_{j_s} \in Z^*(R_{j_s})$ with $x_{j_s} x'_{j_s} = 0$ and $y_{j_s} y'_{j_s} = 0$. Set $c = (m_{i_1}, \ldots, x_{j_1}, \ldots, x_{j_t}, \ldots)$ and $cd = (m_{i_1}, \ldots, y_{j_1}, \ldots, y_{j_t}, \ldots)$. Then $c(0, \ldots, x'_{j_1}, 0, \ldots, 0) = 0$, so $c \in Z^*(R)$. Similarly, $d \in Z^*(R)$. As $cd \neq 0$ and $\operatorname{diam}(\Gamma(R)) = 2$, there must be some $e = (e_1, \ldots, e_n) \in Z^*(R)$ such that $ce = 0 = de$. Then $e = 0$, which is a contradiction. Thus $\operatorname{diam}(\Gamma(R_i)) = 2$ and $R_i = Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$.

(iii) This follow from (i) and (ii). □

Theorem 2.9. *Let* $R_1, R_2, \ldots, R_{n-1}$ *and* R_n *be commutative rings such that* $\operatorname{diam}(\Gamma(R_i)) = 1$, $\operatorname{diam}(\Gamma(R_j)) = 2$ *and* $\operatorname{diam}(\Gamma(R_k)) = 3$ *for some elements* i, j *and* k *of the set* $\{1, 2, \ldots, n\}$*, and let* $R = R_1 \times R_2 \cdots \times R_n$ *(*$n > 2$*). Then the following hold:*

(i) $\operatorname{diam}(\Gamma(R)) \neq 1$.

(ii) $\operatorname{diam}(\Gamma(R)) = 2$ *if and only if* $\operatorname{diam}(\Gamma(R_i)) \leq 2$ *and* $R_i = Z(R_i)$ *for some* $i \in \{1, 2, \ldots, n\}$.

(iii) $\operatorname{diam}(\Gamma(R)) = 3$ *if and only if there is no* $i \in \{1, 2, \ldots, n\}$ *with* $\operatorname{diam}(\Gamma(R_i)) \leq 2$ *and* $R_i = Z(R_i)$.

Proof. (i) Is clear.

(ii) Let $\text{diam}(\Gamma(R_i)) \leq 2$ and $R_i = Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$; we show that $\text{diam}(\Gamma(R)) = 2$. We divided the proof into two cases.

Case 1. $\text{diam}(\Gamma(R_i)) = 1$ and $R_i = Z(R_i)$ for some i. By a similar argument as in Theorem 2.7 (ii), we get $\text{diam}(\Gamma(R)) = 2$.

Case 2. $\text{diam}(\Gamma(R_i)) = 2$ and $R_i = Z(R_i)$ for some i. By a similar argument as in Theorem 2.8 (ii), we get $\text{diam}(\Gamma(R)) = 2$. Conversely, suppose that $\text{diam}(\Gamma(R)) = 2$. It is easy to see from Theorem 2.8 (ii) that $\text{diam}(\Gamma(R_i)) \leq 2$ and $R_i = Z(R_i)$ for some i.

(iii) This follow from (i) and (ii). □

Corollary 2.10. *Let $R_1, R_2, \ldots, R_{n-1}$ and R_n be commutative rings with identity, and let $R = R_1 \times R_2 \cdots \times R_n$ $(n > 2)$. Then* $\text{diam}(\Gamma(R)) = 3$.

Proof. For each $i \in \{1, 2, \ldots, n\}$, $R_i \neq Z(R_i)$ since $1_{R_i} \notin Z(R_i)$. Now the assertion follows from Theorem 2.2, Theorem 2.4 and Theorem 2.6 (for an alternative proof see [3, 2.6 (4)]). □

Example 1. (i) Assume that R is a commutative ring (not necessary with 1) and let $S = \text{Mat}(R)$ be the set of all 2×2 matrices of the form

$$A = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$$

where $a \in R$. It is easy to see that if A, B are non-zero elements of S, then $AB = 0$; hence $Z(S) = S$ and $\text{diam}(\Gamma(S)) = 1$.

(ii) Let Z_{25} denote the ring of integers modulo 25. Then $Z^*(Z_{25}) = \{5, 10, 15, 20\}$, $Z_{25} \neq Z(Z_{25})$ and $\text{diam}(\Gamma(Z_{25})) = 1$. Clearly, $Z(Z_2 \times Z_4) \neq Z_2 \times Z_4$ and $\text{diam}(\Gamma(Z_2 \times Z_4)) = 3$.

(iii) If $R_1 = R_2 = \cdots = R_n = S$ and $R = R_1 \times \cdots \times R_n$, then $\text{diam}(\Gamma(R)) = 1$ by Theorem 2.2 (i).

(iv) If $R_1 = Z_{25}$, $R_2 = \cdots = R_n = S$ and $R = R_1 \times \cdots \times R_n$, then $\text{diam}(\Gamma(R)) = 2$ by Theorem 2.2 (ii).

(v) If $R_1 = Z_{25} = R_2 = \cdots = R_n$ and $R = R_1 \times \cdots \times R_n$, then $\text{diam}(\Gamma(R)) = 3$ by Theorem 2.2 (iii).

(vi) If $R_1 = Z_2 \times Z_4 = R_2 = \cdots = R_n$ and $R = R_1 \times \cdots \times R_n$, then $\text{diam}(\Gamma(R)) = 3$ by Theorem 2.5 (or Corollary 2.10).

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